THE UNIVERSITY OF CHICAGO
Department of Economics
Econ 304: Math Camp
Selected practice problems, lectures 1-4

4. *If* $\{s_n\}_{n=1}^{\infty}$ *is a sequence of real numbers with limit* L, *prove that this limit is unique.*

Assume L and M are two limits of the sequence $\{s_n\}$. Take $\varepsilon = \frac{1}{2}|L-M|$. Then $\exists n_L \in \mathbb{N}$ such that $\forall n > n_L$ we have $|L - s_n| < \varepsilon$. Also, $\exists n_M \in \mathbb{N}$ such that $\forall n > n_M$ $|M - s_n| < \varepsilon$. Therefore, $|L-M| = \varepsilon + \varepsilon > |L-s_n| + |M-s_n| = |L-s_n| + |s_n - M| > |L-M|$. So, we obtain the false result $|L-M| > |L-M|$, and this means our assumption was wrong.

5. *If* $\{s_n\}_{n=1}^{\infty}$ *is a sequence of real numbers such that* $s_n \leq M$, *for all* n and $\lim_{n\to\infty} s_n = L$, *prove that* $L \leq M$.

We prove this by contradiction. Assume $L > M$ and take $\varepsilon = L - M$. By definition of L, $\exists N \in \mathbb{N}$ such that $\forall n > N$ we have $|L - s_n| < \varepsilon = L - M$. Therefore, $L - s_n < L - M$, which implies $s_n > M$. But this contradicts out initial assumption.

10.(a) *Decide whether the sequence* $\left\{\frac{n^2}{n+5}\right\}_{n=1}^{\infty}$ *has a limit and find it if it does.*

The sequence diverges. To show this we need to find $\varepsilon > 0$ such that $\forall N \in \mathbb{N}, L \in \mathbb{R}$ there exists $n > N$ such that $|\frac{n^2}{n+5} - L| > \varepsilon$. This condition will be satisfied in particular if $\frac{n^2}{n+5} - L > \varepsilon$. For convenience let us do the following decomposition:

$$\frac{n^2}{n+2} = \frac{n^2 + 5n - 5n - 25 + 25}{n+5} = n - 5 + \frac{25}{n+5}.$$

We will now look for n that satisfies a condition, $n - 5 - L > \varepsilon$, which is even stronger than the previous one, since $\frac{25}{n+5} > 0$. Let M smallest integer that is greater than $\varepsilon + L + 5$. Clearly, $n = \max\{N, M\}$ satisfies satisfies $|\frac{n^2}{n+5} - L| > \varepsilon$, and so we have shown that the sequence $\left\{\frac{n^2}{n+5}\right\}$ diverges.

(b) *Decide whether the sequence* $\left\{\frac{3n}{n+7n^2}\right\}_{n=1}^{\infty}$ *has a limit and find it if it does.*

The sequence converges to 0. To see this we divide both numerator and denominator by n^2 and use properties of the limits.

11. *Prove that if* $\{|s_n|\}_{n=1}^{\infty}$ *converges, then* $\{s_n\}_{n=1}^{\infty}$ *is bounded. Must* $\{s_n\}_{n=1}^{\infty}$ *also converge?*

Let L be the limit of $\{|s_n|\}_{n=1}^{\infty}$ and take $\varepsilon = 1$. Since $\{|s_n|\}_{n=1}^{\infty}$ converges, $\exists N \in \mathbb{N}$ such that $||s_n| - L| < 1 \; \forall n > N$. This implies $|s_n| - L < 1 \; \forall n > N$, or $|s_n| < 1 + L \; \forall n > N$. Therefore $\{s_n\}_{n=1}^{\infty}$ is bounded by $\max\{L+1, \max_{i=1,..,N}\{s_i\}\}$. Note that $\{s_n\}_{n=1}^{\infty}$ need not converge in general. Consider, for example $\{(-1)^n\}_{n=1}^{\infty}$.

14. *Let* $s_n = \frac{1+2+...+n}{n^2}$. *Show that* $\{s_n\}_{n=1}^{\infty}$ *is monotone and bounded and that* $\lim_{n\to\infty} s_n = \frac{1}{2}$.

First, note that using the expression for the sum of arithmetic progression we can write $s_n = \frac{1+2+...+n}{n^2} = \frac{n(n+1)/2}{n^2} = \frac{n+1}{2n} = \frac{1}{2} + \frac{1}{n}$. Clearly $\{s_n\}_{n=1}^{\infty}$ is decreasing and bounded from below by $\frac{1}{2}$. Also, using properties of the limit we can get $\lim_{n\to\infty}\left(\frac{1}{2} + \frac{1}{n}\right) = \lim_{n\to\infty}\frac{1}{2} + \lim_{n\to\infty}\frac{1}{n} = \frac{1}{2} + 0 = \frac{1}{2}$.

17. *If* $\{s_n\}_{n=1}^{\infty}$ *is a Cauchy sequence of real numbers having a subsequence that converges to* L, *prove that* $\{s_n\}_{n=1}^{\infty}$ *itself converges to* L.

Let $\{s_{n_i}\}_{i=1}^{\infty}$ be the convergent subsequence of $\{s_n\}_{n=1}^{\infty}$ and let L be its limit. Let us take an arbitrary $\varepsilon > 0$.. Define $\varepsilon' \equiv \frac{\varepsilon}{2}$. We know that $\exists K \in \mathbb{N}$ such that $|s_{n_i} - L| < \varepsilon' \; \forall i > K$. We also know that $\exists M \in \mathbb{N}$ such that $|s_n - s_m| < \varepsilon'$ $\forall n, m > M$. Define $N = \max\{M, n_K\}$ and for an arbitrary $n > N$ let us find $n_i > n$. Since $\{s_n\}_{n=1}^{\infty}$ is Cauchy, we must have $|s_n - s_{n_i}| < \varepsilon'$. Also, since $\{s_{n_i}\}_{i=1}^{\infty}$ converges to L, we must have $|s_{n_i} - L| < \varepsilon'$. But then

$$\begin{aligned} |s_n - L| &= |s_n - s_{n_i} + s_{n_i} - L| \\ &\leq |s_n - s_{n_i}| + |s_{n_i} - L| \\ &< \varepsilon' + \varepsilon' = \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \end{aligned}$$

Therefore, $s_n \to L$.

21. *Prove that if* $\lim_{x\to a^+} f(x) = \lim_{x\to a^-} f(x) = L$, *then* $\lim_{x\to a} f(x) = L$.

Recall the definitions: $\lim_{x\to a^+} f(x) = L$ if $\forall\varepsilon > 0 \; \exists\delta_1 > 0$ such that $|f(x) - L| < \varepsilon$ whenever $a < x < a + \delta_1$; similarly, $\lim_{x\to a^-} f(x) = L$ if $\forall\varepsilon > 0 \; \exists\delta_2 > 0$ such that $|f(x) - L| < \varepsilon$ whenever $a - \delta_2 < x < a$. Take $\delta \equiv \min\{\delta_1, \delta_2\}$. Note that conditions $a - \delta < x < a$ and $a < x < a + \delta$ together imply $|x - a| < \delta$. Therefore, we have $|f(x) - L| < \varepsilon$ whenever $|x - a| < \delta$, which means $\lim_{x\to a} f(x) = L$.

23. *Show that if* ρ *and* σ *are metrics for* M, *then so is* $\rho + \sigma$.

To verity that $\mu \equiv \rho + \sigma$ is a metric we just verify that if satisfies all properties of a metric whenever ρ and σ satisfy them.

1. $\rho(x,x) = 0$, $\sigma(x,x) = 0, . \forall x \in M$
   $\Rightarrow \rho(x,x) + \sigma(x,x) = 0, . \forall x \in M$
   $\Rightarrow \mu(x,x) = 0, . \forall x \in M$.

2. $\rho(x,y) > 0$, $\sigma(x,y) > 0, .\forall x, y \in M$, $x \neq y$
   $\Rightarrow \rho(x,y) + \sigma(x,y) > 0, .\forall x, y \in M$, $x \neq y$
   $\Rightarrow \mu(x,y) > 0, .\forall x, y \in M$, $x \neq y$.

3. $\rho(x,y) = \rho(y,x)$, $\sigma(x,y) = \sigma(y,x), .\forall x \in M$
   $\Rightarrow \rho(x,y) + \sigma(x,y) = \rho(y,x) + \sigma(y,x), .\forall x \in M$
   $\Rightarrow \mu(x,y) = \mu(y,x), .\forall x \in M$.

4. $\rho(x,y) + \rho(y,z) \geq \rho(x,z)$, $\sigma(x,y) + \sigma(y,z) \geq \sigma(x,z)$
   $\Rightarrow \rho(x,y) + \rho(y,z) + \sigma(x,y) + \sigma(y,z) \geq \rho(x,z) + \sigma(x,z)$
   $\Rightarrow \mu(x,y) + \mu(y,z) \geq \mu(x,z)$.

32. *Let f and g be continuous real-valued functions on a metric space M. Prove that $A = \{x \in M : f(x) < g(x)\}$ is open.*

Let us take an arbitrary $x_0 \in A$ and define $\varepsilon \equiv g(x_0) - f(x_0)$. By continuity of f there exist $\delta_f > 0$ such that $|f(x) - f(x_0)| < \frac{\varepsilon}{2}$ whenever $|x - x_0| < \delta_f$. Similarly, there exist $\delta_g > 0$ such that $|g(x) - g(x_0)| < \frac{\varepsilon}{2}$ whenever $|x - x_0| < 0$. Define $\delta \equiv \min\{\delta_f, \delta_g\}$ and notice that $|f(x) - f(x_0)| + |g(x) - g(x_0)| < \varepsilon$. This implies that $|f(x) - f(x_0) - g(x) + g(x_0)| < \varepsilon = g(x_0) - f(x_0)$. This, in turn, implies $f(x) - f(x_0) - g(x) + g(x_0) < g(x_0) - f(x_0)$, or, $f(x) - g(x) < 0$. Therefore, $f(x) < g(x)$ whenever $|x - x_0| < \delta$.

36. *Prove the any finite subset of a metric space is closed.*

Let $F \subseteq M$ be a finite set and let L be the limit of a convergent sequence $\{s_n\}_{n=1}^{\infty}$ on F. We must prove $L \in F$. Take $\varepsilon \equiv \min_{x,y \in F, x \neq y} \rho(x,y)$. Since $\{s_n\}_{n=1}^{\infty}$ is convergent, it must also be Cauchy by Theorem 1.13 from the lectures. Therefore there must exist $N \in \mathbb{N}$ such that $\rho(s_n, s_m) < \varepsilon \ \forall n, m > N$. But then it must be that $s_n = s_m \ \forall n, m > M$ because $s_n, s_m \in F$ and $\rho(s_n, s_m) < \min_{x,y \in F, x \neq y} \rho(x,y)$. Therefore $s_n = L$, $\forall n > N$ and since $s_n \in F$, we conclude that $L \in F$.

40. *The discrete metric assigns distance 1 to any pair of distinct points in $\mathbb{R}$. Prove that $[0,1]$ is not connected as a subset of $\mathbb{R}$ when the metric is the discrete metric.*

The way to prove the claim is to show that any set is closed under a discrete metric. Consider an arbitrary set A. Let δ be the discrete metric described above and let $s_n \to L$, $s_n \in A$. This implies that $\exists N \in \mathbb{N}$ such that $s_n = L \ \forall n > N$ (consider $\varepsilon < 1$ to demonstrate this). Therefore $L \in A$ whenever $s_n \in A \ \forall n$, and so A must be closed. But then $[0,1]$ can be covered by two disjoint closed sets, e.g., $[0, 1/2]$ and $(1/2, 1]$, and therefore $[0,1]$ is not connected.

41. *Prove the any finite subset of a metric space is compact.*

Let $F \subseteq M$ be a finite set and let $\{s_n\}_{n=1}^{\infty}$ be an arbitrary sequence on F. Since $\{s_n\}_{n=1}^{\infty}$ is an infinite sequence, it has to take some value, $x_0 \in F$ infinitely many times (very easy to prove by contradiction). Let $\{s_{n_i}\}_{i=1}^{\infty}$ be a subsequence of $\{s_n\}_{n=1}^{\infty}$ such that $s_{n_i} = x_0$, $\forall i$. Clearly, $s_{n_i} \to x_0$. And since $\{s_n\}_{n=1}^{\infty}$ was an arbitrary sequence, F must be compact.

42,43. *Prove that every closed and bounded subset of $\mathbb{R}^n$ is compact. Prove that $A \times B$ is a compact subset of $\mathbb{R}^2$ whenever A and B are compact subsets of $\mathbb{R}$.*

Let us first prove the second part of the claim. Let us take an arbitrary sequence $\{s_n\} = \{(s_{1n}, s_{2n})\}$ in $A \times B$ and consider a sequence $\{s_{1n}\}$ in A constituted by first components of every s_n. Since A is compact, this sequence has a converging subsequence, $\{s_{1n_i}\}$. Now, let us consider $\{t_n\} \equiv \{s_{n_i}\}$, a subsequence of $\{s_n\}$ with the property that its first components converge. For the same reason as previously, $\{t_n\}$ must have a subsequence $\{t_{n_i}\}$ such that the sequence of its second components converges. But if $\{t_{1n_i}\}$ and $\{t_{2n_i}\}$ converge, so must $\{t_{n_i}\}$ (easy to prove). Now let us prove that every closed and bounded subset of $\mathbb{R}^n$ is compact. The proof is very similar. Take a closed and bounded set $C \in \mathbb{R}^n$ and an arbitrary sequence $\{s_i\}$, $s_i \in C, \forall i$. Since $\{s_i\}$ is bounded so must be $\{s_{Ki}\}$, $K = 1, .., n$ (easy to show). Consider the sequence $\{s_{1i}\}$. It must have a convergent subsequence $\{s_{1i_m}\}$. Now, as before we go back to the original n-dimensional sequence and take a subsequence $\{t_i\} \equiv \{s_{i_m}\}$. The first coordinates of t_i's converge. By repeating the same procedure for all other coordinates one by one we will obtain a convergent subsequence of $\{s_i\}$, which was just an arbitrary sequence in $C \in \mathbb{R}^n$. Therefore C must be compact.

47. *Prove that the maximization problem*

$$\max_{\{x_1, x_2, \ldots\}} \sum_{n=1}^{\infty} \frac{1}{2^n} (\sin x_n)\, e^{-x_{n+1}}$$

subject to $x_n \in \mathbb{R}$ and $|x_n| < 2$ for every n, has a solution. (Hint: consider the space M of real sequences $\{x_n\}_{n=1}^{\infty}$ such that $|x_n| < 2$ for every n, and consider metric $\rho(x, y) = \sum_{n=1}^{\infty} \frac{1}{2^n} |x_n - y_n|$ for $x = \{x_n\}_{n=1}^{\infty}$ and $y = \{y_n\}_{n=1}^{\infty}$ in $< M$.)

To apply the corollary of Theorem 1.31 we just have to convince ourselves that the constraint set $\{\{x_n\}_{n=1}^{\infty}$ *such that* $x_n \in \mathbb{R}$ *and* $|x_n| < 2$ *for every* $n\}$ is compact and the function $f : M \to M$, such that $f_n(x) = (\sin x_n)\, e^{-x_{n+1}}$, is continuous.