THE UNIVERSITY OF CHICAGO
Department of Economics
Econ 304: Math Camp
Selected practice problems, lectures 1-4

2. *For each of the following differential equations, (1) draw a phase diagram for the given differential equation; (2) use the phase diagram to describe how solutions behave for large t, and how this behavior depends on the initial condition; (3) by solving the equation, find the value of a where the transition from one type of behavior to another occurs. (These are good practice for evaluating integrals, too.)*

(a) $\dot{y} - \frac{1}{2}y = 2\cos(t)$.

Recall that the solution for a differential equation $\dot{y} + ay = x(t)$ is of the form $y(t) = e^{-at}\int e^{at}x(t)dt$. In our case $a = -\frac{1}{2}, x(t) = 2\cos(t)$. We obtain the expression for $\int e^{-\frac{1}{2}t}2\cos(t)dt$ by twice integrating by parts:

$$\begin{aligned} 2\int e^{-\frac{t}{2}}\cos(t)dt &= 2e^{-\frac{t}{2}}\sin(t)dt + \int e^{-\frac{t}{2}}\sin(t)dt \\ &= 2e^{-\frac{t}{2}}\sin(t)dt - e^{-\frac{t}{2}}\cos(t)dt - \frac{1}{2}\int e^{-\frac{t}{2}}\cos(t)dt \\ &= \frac{4}{5}\left[2\sin(t)dt - \cos(t)dt\right]e^{-\frac{t}{2}} + b. \end{aligned}$$

This gives us $y(t) = \frac{4}{5}\left[2\sin(t)dt - \cos(t)dt\right] + be^{\frac{t}{2}}$. Since $y(0) = -\frac{4}{5} + b = a$, it must be that $b = a + \frac{4}{5}$. So, finally,

$$y(t) = \frac{4}{5}\left[2\sin(t)dt - \cos(t)dt\right] + \left[a + \frac{4}{5}\right]e^{\frac{t}{2}}.$$

Depending on a the expression goes to $+\infty$ (if $a > \frac{4}{5}$), to $-\infty$ (if $a < \frac{4}{5}$), or oscillates (if $a = \frac{4}{5}$).

(b) $2\dot{y} - y = e^{t/3}$.

Again,

$$\begin{aligned} y(t) &= e^{-at}\int e^{at}x(t)dt = e^{\frac{t}{2}}\int e^{-\frac{t}{2}}\frac{1}{2}e^{\frac{t}{3}}dt = -3e^{t/3} + be^{t/2}. \\ y(0) &= -3 + b = a \quad \Rightarrow \quad b = a + 3. \\ y(t) &= -3e^{t/3} + [a+3]e^{t/2}. \end{aligned}$$

If $a > -3$, then $y(t)$ goes to $+\infty$, otherwise it goes to $-\infty$.

(c) $3\dot{y} - 2y = e^{-\pi t/2}$.

$$
\begin{aligned}
y(t) &= e^{-at}\int e^{at}x(t)dt = e^{\frac{2t}{3}}\int e^{-\frac{2t}{3}}\frac{1}{3}e^{-\frac{\pi t}{2}}dt = -\frac{2}{2\pi+4}e^{-\frac{\pi t}{2}} + be^{\frac{2t}{3}}.\\
y(0) &= -\frac{2}{2\pi+4} + b = a \quad\Rightarrow\quad b = a + \frac{2}{2\pi+4}.\\
y(t) &= -\frac{2}{2\pi+4}e^{-\frac{\pi t}{2}} + \left[a + \frac{2}{2\pi+4}\right]e^{\frac{2t}{3}}.
\end{aligned}
$$

We see that $y(t)$ goes to $+\infty$ (if $a > -\frac{2}{2\pi+4}$), to $-\infty$ (if $a < -\frac{2}{2\pi+4}$), or to 0 (if $a = -\frac{2}{2\pi+4}$).

3.(b) *For the following system of differential equations, (1) draw the phase diagram; (2) find the general analytic solution of the system; and (3) explain the relationship between the analytic solution and the phase diagram.*

$$
\dot{y} = \begin{bmatrix} -3 & \sqrt{2} \\ \sqrt{2} & -3 \end{bmatrix} y.
$$

The eigenvalues and corresponding eigenvectors of the matrix are $\lambda_1 = -3+\sqrt{2}$, $e_1 = (1,1)'$ and $\lambda_1 = -3-\sqrt{2}$, $e_1 = (-1,1)'$. So the solution is $y(t) = c_1 e^{-\left(3+\sqrt{2}\right)t}\cdot(1,1)' + c_2 e^{-\left(3-\sqrt{2}\right)t}\cdot(1,-1)'$. The system is stable.

4. *Solve the following system and describe how its behavior depends on initial conditions:*

$$
\dot{y} = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} y.
$$

We have seen in class that the eigenvalues of the matrix are $\lambda_1 = 2$, $\lambda_2 = -1$. The eigenvector corresponding to $\lambda_1 = 2$ is $e_1' = (1,1,1)$. The eigenvectors corresponding to $\lambda_2 = -1$ are $e_2' = (1,0,-1)$ and $e_3' = (0,1,-1)$. Therefore, the solution to the system is

$$
y(t) = c_1 e^{2t}\cdot(1,1,1)' + c_2 e^{-t}\cdot(1,0,-1)' + c_3 e^{-t}\cdot(0,1,-1)'.
$$

Note that $y(t) \to (0,0,0)'$ if $c_1 = 0$.

5. A very similar problem was discussed on the last lecture. Also, this problem is solved in Barro and Sala-i-Martin.