

THE UNIVERSITY OF CHICAGO
Department of Economics
Econ 304: Math Camp
Selected practice problems, lectures 9-10

1. Find the derivatives of the following functions:

(a) $f(x) = 11x^5 - 6x^3 + 8$

$$f'(x) = 55x^4 - 18x^2$$

(b) $f(x) = 1$

$$f'(x) = 0$$

(c) $f(x) = (x^2 + 1)^2$

$$f'(x) = 4x(x^2 + 1)$$

(d) $f(x) = \frac{x^3}{1-x}$

$$f'(x) = \frac{3x^2 - 2x^3}{(1-x)^2}$$

(e) $f(x) = (x^2 - 1)(x - 3)$

$$f'(x) = 3x^2 - 6x - 1$$

(f) $f(x) = \frac{ax+b}{cx+d}$

$$f'(x) = \frac{ad-bc}{(cx+d)^2}$$

4. Use the inverse formula for a derivative to find the derivative of \sqrt{x} at $x = 2$.

$$\begin{aligned} y &= f(x) = \sqrt{x} \Rightarrow x = f^{-1}(y) = y^2 \\ f'(x) &= \left[\frac{df^{-1}(y)}{dy} \right]^{-1} = [2y]^{-1} = [2\sqrt{x}]^{-1} = \frac{1}{2\sqrt{x}} \end{aligned}$$

7. Find two positive numbers whose sum is 40 and whose product is maximized.

The problem we need to solve here is

$$\max_{x \in [0,40]} (40 - x)x$$

The first-order condition for this problem is

$$\frac{d(40-x)x}{dx} = 40 - 2x = 0$$

And the second condition for the maximum is satisfied, $\frac{d(40-2x)}{dx} = -2 < 0$.
So the two numbers are $x_1 = 20$ and $x_2 = 20$.

12.(a) Use integration by parts to evaluate the following integral:

$$\int_0^1 \frac{x}{(x+1)^2} e^x dx$$

As I said many times on the TA sessions, the trick that works in most problems that are not straightforward is adding and subtracting the same thing:

$$\int_0^1 \frac{x}{(x+1)^2} e^x dx = \int_0^1 \frac{x+1-1}{(x+1)^2} e^x dx = \int_0^1 \frac{e^x}{x+1} dx - \int_0^1 \frac{e^x}{(x+1)^2} dx$$

$$\int_0^1 \frac{e^x}{(x+1)^2} dx = \frac{-e^x}{x+1} \Big|_0^1 - \int_0^1 \frac{-e^x}{x+1} dx$$

$$\begin{aligned} \int_0^1 \frac{x}{(x+1)^2} e^x dx &= \int_0^1 \frac{e^x}{x+1} dx + \frac{e^x}{x+1} \Big|_0^1 - \int_0^1 \frac{e^x}{x+1} dx \\ &= -\frac{e^x}{x+1} \Big|_0^1 = -\frac{e}{2} + 1 \end{aligned}$$

13. Use Taylor's theorem to prove that for all $x \in \mathbb{R}$,

(a) $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$

The Taylor's expansion around the point x_0 takes the form

$$f(x) = \sum_{i=0}^{\infty} \frac{f^{(i)}(x_0)(x-x_0)^i}{i!}$$

We will use $x_0 = 0$ for all functions in this problem. Note that

$$\frac{d^{(i)} e^x}{dx^i} \Big|_{x=0} = e^x \Big|_{x=0} = 1.$$

So we finally get

$$e^x = \sum_{i=0}^{\infty} \frac{x^i}{i!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$$

(b) $\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots$

$$\left. \frac{d^{(i)} \sin x}{dx^i} \right|_{x=0} = \begin{cases} 0 & , \text{ if } n \text{ is even} \\ 1 & n = 1 + 4i, i = 1, 2, \dots \\ -1 & n = 3 + 4i, i = 1, 2, \dots \end{cases}$$

Therefore,

$$\sin(x) = \sum_{i=0}^{\infty} \left[\frac{x^{1+4i}}{(1+4i)!} - \frac{x^{3+4i}}{(3+4i)!} \right] = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots$$

(c) $\cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots$

$$\left. \frac{d^{(i)} \cos x}{dx^i} \right|_{x=0} = \begin{cases} 0 & , \text{ if } n \text{ is odd} \\ 1 & n = 4i, i = 1, 2, \dots \\ -1 & n = 2 + 4i, i = 1, 2, \dots \end{cases}$$

Therefore,

$$\cos(x) = \sum_{i=0}^{\infty} \left[\frac{x^{4i}}{(4i)!} - \frac{x^{2+4i}}{(2+4i)!} \right] = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots$$

- 16.** Use the separating hyperplane theorem to prove that if the linear system of n equations in m unknowns,

$$A_{n \times m} x_{m \times 1} = b_{n \times 1},$$

has no solution, then there exists $p \in \mathbb{R}^n$, nonzero, such that $p^T A = 0$, and $p^T b \neq 0$. (That is, some linear combination of the equations cancels all the variables on the left-hand side, yet produces non-zero number on the right-hand side.)

Consider the non-empty closed and convex set (both properties are easy to check) $C = \{y \in \mathbb{R}^n : y = Ax, \text{ for some } x \in \mathbb{R}^m\}$ and note that $b \notin C$. You can check easily that if $y \in C$, then $\alpha y \in C$, for any $\alpha \in \mathbb{R}$. Now we use the separating hyperplane theorem to argue that there exists $p \in \mathbb{R}^n$ and $c \in \mathbb{R}$ such that $p^T y > c > p^T b$, for all $y \in C$. Note that it also means that $\alpha p^T y \geq c$, for all $\alpha \in \mathbb{R}$. But this is only possible if $p^T y = 0$ for all $y \in C$. Therefore, $p^T Ax = 0$, for all $x \in \mathbb{R}^m$. This can only be true if $p^T A = \mathbf{0}$. And we know that $p^T b \neq 0$ because $p^T b < c < p^T y = 0$.