# Perception, Utility and Evolution

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**Abstract**

This paper presents a model of the evolution of the hedonic utility function in which perception is imperfect. Netzer (2009) considers a model with perfect perception and finds that the optimal utility function allocates marginal utility where decisions are made frequently. This paper shows that it is also beneficial to allocate marginal utility away from more perceptible events. The introduction of perceptual errors can lead to qualitatively different utility functions, such as discontinuous functions with flat regions rather than continuous and strictly increasing functions.

*JEL:* D81, D83.

*Keywords:* Evolution; Hedonic Utility; Perception.

## 1 Introduction

Based on the hypothesis that individual decisions are guided by hedonic utility, several recent studies have found an evolutionary foundation for $S$-shaped utility functions. Though they use different mechanisms, Netzer (2009) and Rayo and Becker (2007) are two notable examples.[1] Netzer (2009) derives a utility function which is steep in regions where decisions

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[1] See also Robson (2001). Robson and Samuelson (2010) survey the literature.

have to be made frequently, and where wrong decisions would lead to large losses. Rayo and Becker (2007) consider the case of an individual with local information, bounded utility and limited perception to derive $S$-shaped utility functions that spread incentives over the output domain.

Besides advancing our understanding of happiness as a tool for decision making, these recent contributions are also linked to risk preferences. Because of this link, and the possibility of an evolutionary foundation of the $S$-shape, the debate is followed by both proponents and detractors of prospect theory.[2]

This paper builds on the framework developed in Netzer (2009) by allowing limited perception to have a prominent role. In a sense, the paper confirms the robustness of Netzer's result when a different model is considered. In particular, Netzer's result is still optimal in the limit. However, in this general setting, Netzer's intuition can be overturned because nature faces a trade off between perception and robustness. Indeed, nature may not allocate marginal utility in regions where decisions have to be made frequently. Thus, this paper shows what is lost by approximating limited perception with perfect perception.

## 2 The environment

The environment is based on Netzer (2009), though the original setup can be traced back at least to Robson (2001). Suppose that an agent makes repeated choices between two options, $x$ and $y$. These options are independently drawn from a normalized set, $[0, 1]$, according to a distribution function $F$ with full support. The options are identified with fitness levels: option $x$ yields fitness $x$, where fitness could simply be thought of as the number of offspring.

The agent has a hedonic utility function $U(x)$, which is shaped by natural selection. Naturally, the agent will choose the option that makes her happier. Because evolution, through natural selection, optimizes the utility

[2]See Kahneman and Tversky (1979). For a recent review on prospect theory, see Barberis (2013).

function, I will use the common metaphor of a gene that maximizes offspring. Thus, the "selfish gene" will design the utility function that promises the largest expected fitness.[3]

However, not all utility functions are admissible. Besides focusing on measurable utility functions, there will be two constraints. First, the human brain has a limited size. Thus, happiness must be bounded. Second, the human senses are imperfect. Thus, happiness cannot be perceived without errors.[4] More formally, I will make the following assumptions.

**Assumption 1** (Bounded happiness)**.** Without loss of generality, the utility function must be bounded between 0 and 1.

**Assumption 2** (Limited perception)**.** The agent will perceive the happiness of $x$ as $U(x) + \varepsilon$, where the error $\varepsilon$ is a random variable.[5]

These assumptions are are based in Rayo and Becker (2007) with some technical differences, the most important of which is the additive noise. Rayo and Becker (2007) model limited perception as an agent that cannot distinguish outcomes within intervals—similar to Netzer (2009).

## 2.1 The difference with Netzer (2009)

Assumption 2 deserves further discussion because it is the main variation from Netzer (2009). In his paper, Netzer models limited perception "by assuming that utility can take only discrete, albeit extremely numerous, values. [...] The set of admissible utility functions is thus restricted to the set of increasing step functions with $N \in \mathbb{N}$ jumps, each corresponding to a utility increment of size $1/N$. As a result, the agent cannot distinguish two alternatives located on the same step of the utility function. Any choice between such alternatives will have to be random, and a mistake can occur."

[3]See Dawkins (1976). I assume that the utility function values offspring. See Rayo and Robson (2014) for a theory on the evolution of utility functions over intermediate goods such as food, shelter, etc.

[4]See Rayo and Becker (2007) or Netzer (2009) for further motivation.

[5]Note that limited perception of happiness is different from limited perception of the alternatives. If the $x$ is perceived with a noise, then evolution will operate on the senses needed to distinguish one alternative from another. In this model, however, we are interested in the consequences of perceiving utility $U(x)$ with noise.

To derive his main result, Netzer takes the limit when $N \to \infty$, so perception becomes perfect. Netzer takes this elegant approach because, with finite $N$, the optimization problem becomes intractable. In this way, Netzer finds a solution with a clear interpretation; the perfect perception solution is the limit of a series of solutions with limited perception. In fact, Corollary 3 below can be interpreted as a proof of the robustness of Netzer's result.

However, by maintaining limited perception we can focus on the issues that we miss by taking the limit. The upshot is that in theory the omissions may be considerable. On the other hand, what we lose in the approximation we gain in tractability. As we shall see, with limited perception the existence of an analytic solution is not guaranteed.

# 3 The gene's problem

With limited perception, the agent gets confused. She will choose $x$ over $y$ if and only if $U(x) + \varepsilon_0 \geq U(y) + \varepsilon_1$, where the errors are iid and share the spirit of a discrete choice model (McFadden, 1974).[6] With a slight abuse of notation, let $\varepsilon \equiv \varepsilon_0 - \varepsilon_1$ and call $G$ the distribution of $\varepsilon$. Note that $\varepsilon$ is symmetric by construction.[7] Then, the agent chooses $x$ over $y$ if and only if $U(x) - U(y) \geq \varepsilon$ with $x, y \sim F$ and $\varepsilon \sim G$, all independent. Assume that the distributions can be represented by bounded densities $f$ and $g$ with finitely many discontinuities. In Section 8, I explore the possibility of multiplicative errors and its connection to Weber's Law.

The agent chooses $x$ with probability $P[U(x) > U(y) + \varepsilon] = P[\varepsilon < U(x) - U(y)] = G[U(x) - U(y)]$. Let $V$ represent the "expected fitness"

[6] Notice also a similarity with the drift-diffusion model of decision making (Fehr and Rangel, 2011).

[7] If $Z$ and $Z'$ are iid, $Z^s \equiv Z - Z' \sim Z' - Z = -(Z - Z') = -Z^s$, which implies that $Z^s$ is a symmetric random variable.

given a utility function $U$. Thus,

$$V[U] = \int_0^1 \int_0^1 xf(x)f(y)G\Big(U(x) - U(y)\Big)dxdy + \int_0^1 \int_0^1 yf(x)f(y)\left[1 - G\Big(U(x) - U(y)\Big)\right] dxdy.$$

$$\therefore V[U] = \mathbb{E}[X] + \int_0^1 \int_0^1 (x-y)f(x)f(y)G\Big(U(x) - U(y)\Big)dxdy$$

Therefore, the gene's problem becomes

$$\max_{U \in \mathcal{U}} \quad V[U] \quad \text{subject to} \quad 0 \leq U(x) \leq 1, \tag{1}$$

where $\mathcal{U}$ is the set of admissible functions, which I assume is the set of measurable functions bounded between 0 and 1.

**Remark.** The function $U$ appears twice inside the integral with different running variables. Intuitively, any change of $U$ at any point must be weighted versus the whole function. Thus, this problem is not local.

In a sense, this framework is a generalization of Netzer's. Netzer (2009) assumes that $\varepsilon_0 = 0$ and restricts the set of admissible utility functions to a specific set of step functions.

Yet in another sense, both this paper and Netzer (2009) are particular cases of a more general framework, namely, where the errors $\varepsilon_0(x)$ are functions of outcomes. In this sense, a step function $U_{step}$, such as those admissible in Netzer (2009), can be understood as a utility function with an error that depends on the outcome: $U_{step}(x) = U(x) + \varepsilon_0(x)$. This paper, then, "unpacks" the $U_{step}$ by allowing for a stochastic shock $\varepsilon_0$, which doesn't depend on $x$, and *any* measurable $U(x)$.

# 4 A simple example

Before proceeding with the formal analysis, this section introduces a simple example that overturns the intuition of allocating more marginal utility where decisions have to be made more frequently.

Suppose that errors are uniform $U[-1, 1]$. That is: $G(\varepsilon) = (1 + \varepsilon)/2$.

Outcomes $X$ have a pdf denoted by $f$. In this case:

$$V[U] = \mathbb{E}[X] + \frac{1}{2}\int_0^1 \int_0^1 (x-y)\, f(x)f(y)\big[1 + U(x) - U(y)\big]\,dxdy.$$

After simplifications,

$$V[U] = \mathbb{E}[X] + \int_0^1 \big(x - \mathbb{E}[X]\big) f(x)U(x)dx.$$

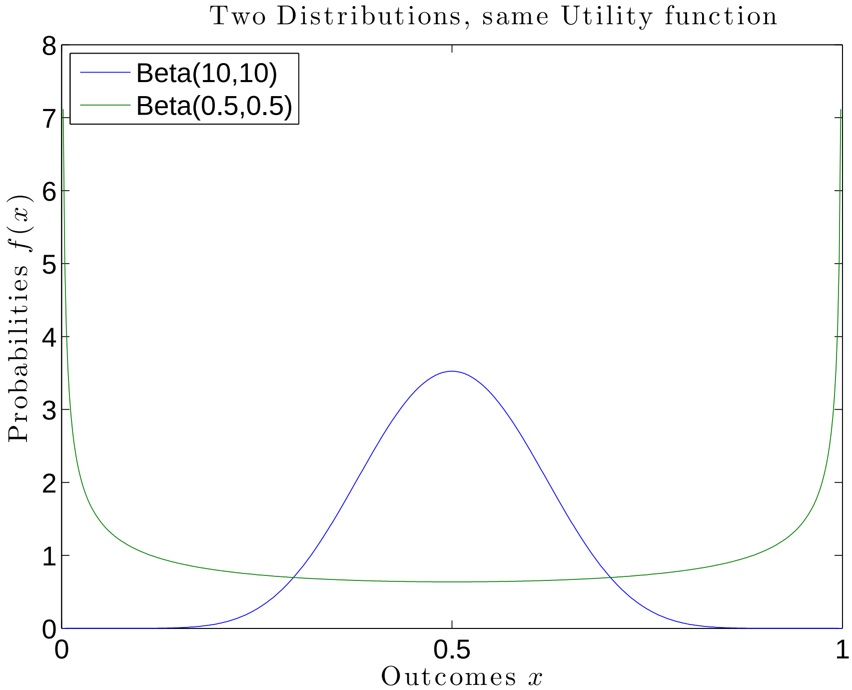
This implies that the optimal utility function $U^*$ should be an indicator function for $x \geq \mathbb{E}[X]$:

$$U^*(x) = \begin{cases} 1 & x \geq \mathbb{E}[X] \\ 0 & x < \mathbb{E}[X] \end{cases}$$

Note that this is optimal *regardless* of $f$. In fact, the optimal utility allocates all the marginal utility at $\mathbb{E}[X]$ even if $f(\mathbb{E}[X])$ should be low. Figure 1 represents this case: two different distribution functions have the same optimal utility function, which is a step function. However, one pdf allocates more mass at the extremes while the other one at the center.

Figure 2 compares the optimal utility function and Netzer (2009)'s result when the pdf is a Beta$(0.5, 0.5)$, which allocates more mass at the extremes. The optimal utility function is a step function, essentially flat almost everywhere. However, even though $f(1/2)$ is the minimum of $f$, $U^*$ has infinite marginal utility at $x = 1/2$. The intuition of Netzer is overturned because, once we allow for errors, the optimization problem becomes non-local. One has to weight the local marginal utility against the marginal utility evaluated at every point. In other words, mistakes are not only made for alternatives with similar fitness, but also when comparing alternatives with very different fitness and therefore the optimal solution is not only determined by the local density, but by the global probability function.

Granted, this example is extreme because the errors are big. However, as we will see, this intuition holds true for any size of the errors. Indeed, as long as there is limited perception, there will be flat regions in the utility




*Note:* The two distributions are Beta$(10, 10)$ and Beta$(0.5, 0.5)$, which implies that $\mathbb{E}[X] = 1/2$ for both. For both distributions, the optimal utility function is $U^*(x) = \mathbb{1}\{x \geq 1/2\}$.

Figure 1: Two different distributions with the same optimal utility function

function *regardless* of $f$.

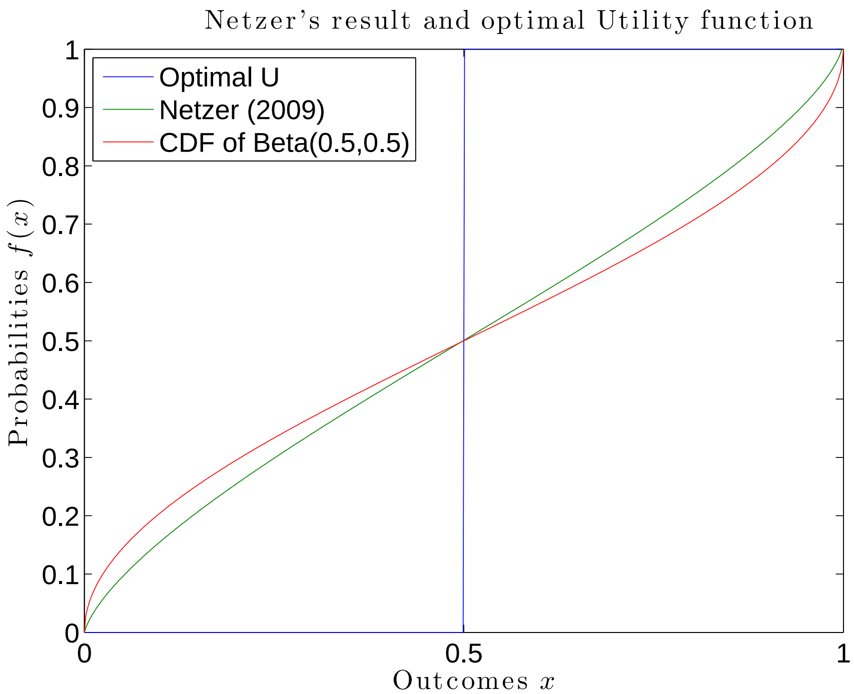
# 5 Existence and uniqueness

The first step to tackle this problem is to assure existence and uniqueness of a solution. As is, we can not apply an existence theorem for the set $\mathcal{U}$ because it is not compact. But we can prove the theorem for a subset of $\mathcal{U}$: the subset of monotone functions.

In Appendix A, Lemma 2 shows that if a solution exists, it must be monotone, non-decreasing. Therefore, we can restrict our search. Let $\mathcal{M} \subset \mathcal{U}$ denote the set of admissible monotone functions:

$$\mathcal{M} \equiv \Big\{ U : [0,1] \mapsto [0,1] \;\Big|\; U \text{ is monotone, non-decreasing} \Big\}.$$

And note that $\mathcal{M}$ is closed, bounded and convex. From here, we can




*Note:* The outcomes $X \sim \text{Beta}(0.5, 0.5)$, which implies that $\mathbb{E}[X] = 1/2$. The optimal utility function is $U^*(x) = \mathbb{1}\{x \geq 1/2\}$. The utility function in Netzer (2009) equals $U(x) = c \int_0^x f(x)^{2/3} dx$, where $f$ is the pdf of $X$ and $c$ is a normalizing constant.

Figure 2: Comparison of the optimal utility function and Netzer's result

establish existence and uniqueness.

Since the optimal utility function is not defined if $f(x) = 0$, I will formally assume full support, except perhaps at the extremes.

**Assumption 3.** Suppose that $f(x) > 0$ for all $x \in (0, 1)$. Moreover, without loss of generality, if $f(0) = 0$ then set $U(0) = 0$ and if $f(1) = 0$ then set $U(1) = 1$.

**Proposition 1.** A solution to $\max_{U \in \mathcal{M}} V[U]$ exists. Moreover, if $G$ is strictly concave in $[0, \infty)$, and hence strictly convex in $(-\infty, 0]$, then the solution is unique.

*Proof.* I will prove that the maximum is attained. Note that $\mathcal{M}$ is sequentially compact (Helly's First Theorem (Montesinos, Zizler and Zizler, 2015)). Also note that $V$ is sequentially continuous, ie, if $U_n \to U$, then $V[U_n] \to V[U]$. To see this, note that $G \leq 1$ so that $V$ is bounded and, by

Lebesgue's Dominated Convergence Theorem, $V[U_n] \to V[U]$. Therefore[8], $V$ attains a maximum in $\mathcal{M}$.

To prove uniqueness, note that $V[U]$ can be written as

$$V[U] = \mathbb{E}[X] + \int_0^1 \int_y^1 (y-x)f(x)f(y)G[U(y)-U(x)]dxdy + \int_0^1 \int_0^y (y-x)f(x)f(y)G[U(y)-U(x)]dxdy.$$

Here, we can see that $(y-x)G(U(y)-U(x))$ is concave when $G$ is concave in $[0,\infty)$ and convex in $(-\infty,0]$. Then, $V$ is concave and the maximum is unique. □

# 6 The optimal utility function

Since the optimization problem is non-local, general properties of the solution are hard to characterize. Indeed, if we take the approach of calculus of variations we end up with an integral equation, which in general does not have analytic solutions. However, we can prove a few properties. In particular, there will be flat regions in the utility function.

**Proposition 2.** Let $U$ be the utility function that solves $\max_{U\in\mathcal{M}} V[U]$. Then, $U$ has the following form:

$$U(x) = \begin{cases} 0 & x \leq a \\ u(x) & x \in (a,b) \\ 1 & x \geq b, \end{cases}$$

for $0 < a \leq b < 1$. In particular, if $a < b$, then $u : (a,b) \mapsto (0,1)$ is continuous, $\lim_{x\to a^+} u(x) = 0$ and $\lim_{x\to b^-} u(x) = 1$. That is, if $a < b$,

[8] It is a well known result that if a space $X$ is sequentially compact, and if a function $f$ is sequentially continuous, then $f$ achieves a maximum in $X$. For completeness, here is a proof: Let $\alpha \equiv \sup_{x\in X} f(x)$. Then, $\alpha \in \mathbb{R}\cup\{\infty\}$. Then, $\exists\{x\}_{n=0}^{\infty} \in X$ such that $f(x_n) \to \alpha$. By sequential compactness of $X$, $\exists x^* \in X$ and $\exists$ a subsequence $\{x_{n_k}\}$ such that $x_{n_k} \to x^*$. By sequential continuity, $f(x_{n_k}) \to f(x^*)$. Because $\{f(x_{n_k})\}$ is a subsequence of $\{f(x_n)\}$, then $f(x_{n_k}) \to \alpha \Rightarrow \alpha = f(x^*)$.

then $U$ is continuous. Moreover, if $G$ is strictly concave in $[0, \infty)$, then $u$ is differentiable, and hence $U$ is differentiable almost everywhere.

*Proof.* Let us begin by writing the Lagrangian:

$$\mathcal{L}[U, \lambda_0, \lambda_1] = V + \int_0^1 \Big[\lambda_0(x)U(x) - \lambda_1(x)[U(x) - 1]\Big] dx. \tag{2}$$

The Karush–Kuhn–Tucker conditions are:

$$\begin{aligned} -U(x) \leq 0, \quad \lambda_0(x) \geq 0, \quad \lambda_0(x)U(x) = 0; \\ U(x) - 1 \leq 0, \quad \lambda_1(x) \geq 0, \quad \lambda_1(x)[U(x) - 1] = 0. \end{aligned} \tag{3}$$

In the jargon of Calculus of Variations[9], for $U^*$ to solve (1), we need that the variation of $\mathcal{L}$ equals zero along $U^*$, in other words, the First Order Condition is $\delta\mathcal{L} = 0$. The variation $\delta\mathcal{L}$ is defined as:

$$\delta\mathcal{L} \equiv \frac{\partial}{\partial\alpha} \mathcal{L}\left[U(x) + \alpha\delta U(x)\right]\Big|_{\alpha=0}$$

where $\delta U \in \mathcal{U}$ is some admissible perturbation, close enough to $U$ such that $U + \alpha\delta U \in \mathcal{U}$ for all $\alpha$ sufficiently close to 0.

I will start with $\delta V$ and then continue with the rest of $\delta\mathcal{L}$.

$$\begin{aligned} \delta V &\equiv \frac{\partial}{\partial\alpha} V\left[U(x) + \alpha\delta U(x)\right]\Big|_{\alpha=0} \\ &= \int_0^1 \int_0^1 (y - x)f(x)f(y)g[U(y) - U(x)][\delta U(y) - \delta U(x)]dxdy \\ &= \int_0^1 \int_0^1 (y - x)f(x)f(y)g[U(y) - U(x)]\delta U(y)dxdy \\ &\quad - \underbrace{\int_0^1 \int_0^1 (y - x)f(x)f(y)g[U(y) - U(x)]\delta U(x)dxdy}_{\text{Switch the variables here}}. \end{aligned}$$

Because $x$ and $y$ are dummy variables, rename the variables in the underbraced part of this equation; where it says $x$, write $y$ and where it says $y$, write $x$. Then, because $U$ is measurable, apply Fubini's Theorem to interchange the order of integration. Finally, since $g$ is symmetric, we

[9] See Elsgolc (2007).

get the following expression:

$$\delta V = \int_0^1 \delta U(y) \underbrace{\int_0^1 2(y-x)f(x)f(y)g\big(U(y)-U(x)\big)dx}_{\equiv I(y)}\, dy.$$

Define the interior integral as $I(y)$, which is a function of $y$. Then,

$$\delta V = \int_0^1 \delta U(y) I(y) dy.$$

Now, we focus on the second part of $\delta\mathcal{L}$. Using $\delta\mathcal{L} = \frac{\partial}{\partial\alpha}\mathcal{L}[U+\alpha\delta U]\big|_{\alpha=0}$, we get that:

$$\begin{aligned}\delta\mathcal{L} &= \delta V + \int_0^1 \Big[\lambda_0(x)\delta U(x) - \lambda_1(x)\delta U(x)\Big]dx \\ &= \int_0^1 \delta U(x)\Big[I(x) + \lambda_0(x) - \lambda_1(x)\Big]dx.\end{aligned}$$

Since, $\delta\mathcal{L} = 0$ should hold for any arbitrary (but close) deviation $\delta U$, subject to some regularity conditions, we can apply the Fundamental Theorem of Calculus of Variations to conclude that the First Order Condition[10] becomes

$$I(x) + \lambda_0(x) - \lambda_1(x) = 0 \qquad \forall x \in [0,1].$$

Therefore,

$$I(x) \begin{cases} < 0 & \text{if} \quad U(x) = 0 \\ = 0 & \text{if} \quad U(x) \in (0,1) \\ > 0 & \text{if} \quad U(x) = 1. \end{cases}$$

The following lemma establishes that $I$ is in fact continuous. Its proof is in Appendix B.

**Lemma 1.** $I(y)$ is continuous for all $y \in [0,1]$ and for all $U \in \mathcal{M}$.

[10]We already discussed existence and uniqueness but see Appendix C for the Second Order Conditions.

Also note that

$$I(0) = -2f(0)\int_0^1 yf(y)g\Big(U(0) - U(y)\Big)dy.$$

It follows that for any $U \in \mathcal{M}$, if $f(0) > 0$, then $I(0) < 0$ and if $f(0) = 0$, then $I(0) = 0$ and, by assumption, $U(0) = 0$. But in both cases, for a small enough $a > 0$, we conclude that $I(a) < 0$. Therefore, $U(a) = 0$. By monotonicity, $U(x) = 0$ for all $x \in [0, a]$.

The case of $I(1) = 2f(1)\int_0^1 (1-y)f(y)g\big(U(1) - U(y)\big)dy > 0$ is analogous so that $I(b) > 0$ for $b$ close enough to 1. Again, by monotonicity, $U(x) = 1$ for all $x \in [b, 1]$.

Define $u \equiv U$ for $x \in (a, b)$.

To prove that $u$ is continuous, consider $u(c)$ for an arbitrary $c \in (a, b)$ and let $u(c^-)$ be its limit from the left and $u(c^+)$ from the right. By assumption, $I(c) = 0$ must hold, then

$$I(c) = 0 = \lim_{x \to c^-} I(x) = \int_0^1 2(c-y)f(c)f(y)g\left(u(c^-) - U(y)\right)dy,$$
$$I(c) = 0 = \lim_{x \to c^+} I(x) = \int_0^1 2(c-y)f(c)f(y)g\left(u(c^+) - U(y)\right)dy,$$

which implies that $u(c^-) = u(c^+)$ (when $g$ is not flat). Thus, $u$ is continuous.

To prove that $U$ is continuous when $a < b$, define $u(a) \equiv \lim_{x \to a^+} u(x)$. Because $I(x)$ is continuous,

$$I(a) = 0 = \lim_{x \to a^-} I(x) = \int_0^1 2(a-y)f(a)f(y)g\left(u(a) - U(y)\right)dy,$$
$$I(a) = 0 = \lim_{x \to a^+} I(x) = \int_0^1 2(a-y)f(a)f(y)g\left(0 - U(y)\right)dy,$$

which implies that $u(a) = 0$. The case for $u(b)$ is analogous.

Finally, to prove that $u$ is differentiable, fix a $c \in (a, b)$ and consider the following function:

$$I(y, u) \equiv \int_0^1 2(y-x)f(y)f(x)g\left(u - U(x)\right)dx.$$

$$\Rightarrow \frac{\partial I(y,u)}{\partial u} = \int_0^1 2(y-x)f(y)f(x)g'\left(u - U(x)\right)dx.$$

Note that $\frac{\partial I(c,u(c))}{\partial u} = \int_0^1 2(c-x)f(c)f(x)g'\left(u(c) - U(x)\right)dx < 0$. To see this, note that for $x > c$, $g' < 0$ because $G$ is concave in $[0,\infty]$, while for $x < c$, $g' > 0$ because $G$ is convex in $[-\infty, 0]$. Thus, by the Implicit Function Theorem, $u'(c)$ exists. □

The proof offers little intuition but consider $I(0)$, which represents what happens if, at the optimum, $U(0)$ changes. $I(0)$ has an integral because $U(0)$ must be compared against the whole function $U(y)$. Thus, the optimal $U^*$ saves marginal utility at the edges, where it makes little difference, to have a larger slope at the middle.

What is special about the edges? Intuitively, since the utility at the extremes is extreme, the event of a high $x$ versus a low $y$ is easy to decide for the agent. However, the case of high $x$ versus medium $y$ is more difficult. To solve this, the gene builds perception in the case of high $x$ versus medium $y$ by compromising perception in the case of high $x$ versus high $y$. This results in a robust strategy: when errors are big, optimize behavior by having extreme utility in regions where losses and rewards are high.

The following corollary gives us a sense of what happens in the limit case of perfect perception, in which case, Netzer's result is optimal and the utility function has no flat regions.

**Corollary 3.** Suppose that the errors $\varepsilon \sim \mathcal{N}(0, \sigma^2)$. Then, when $\sigma^2 \to 0$, there are no flat regions in the optimal utility function. In particular, Netzer's result is optimal, as is any strictly increasing function.

*Proof.* Given $\sigma^2$, choose a small enough $a > 0$ such that $I(a) < 0$. In this case:

$$I(a) = 2f(a)\int_0^1 (a-y)f(y)\frac{1}{\sigma\sqrt{2\pi}}e^{-\frac{[U(a)-U(y)]^2}{2\sigma^2}}dy \to 0,$$

when $\sigma^2 \to 0$. The case for $I(b) > 0$ is analogous. □

# 7 Properties of the optimal utility function

As mentioned before, the non-locality of the integral equation makes any further analysis complicated. In particular, the integral equation may not be solved by Picard iterations (cf. Montesinos, Zizler and Zizler (2015)).

However, a few results can be proven in the next set of corollaries. Proofs can be found in Appendix D.

First, with normal errors, as the variance of the shocks increase, the flat areas of the utility function expand. The simple example of Section 4 shows the extreme case of large errors. Intuitively, as errors become large, the utility function must also become extreme to compensate the loss in perception.

**Corollary 4.** Suppose that the error $\varepsilon \sim \mathcal{N}(0, \sigma^2)$. Then, as $\sigma$ increases, $a$ increases and $b$ decreases. That is, as errors become large, the flat areas of the utility function expand.

Figure 3 presents a numerical exercise. It shows how the optimal utility function behaves when the size of the error changes. As $\sigma^2$ decreases, the flat regions shrink.[11]

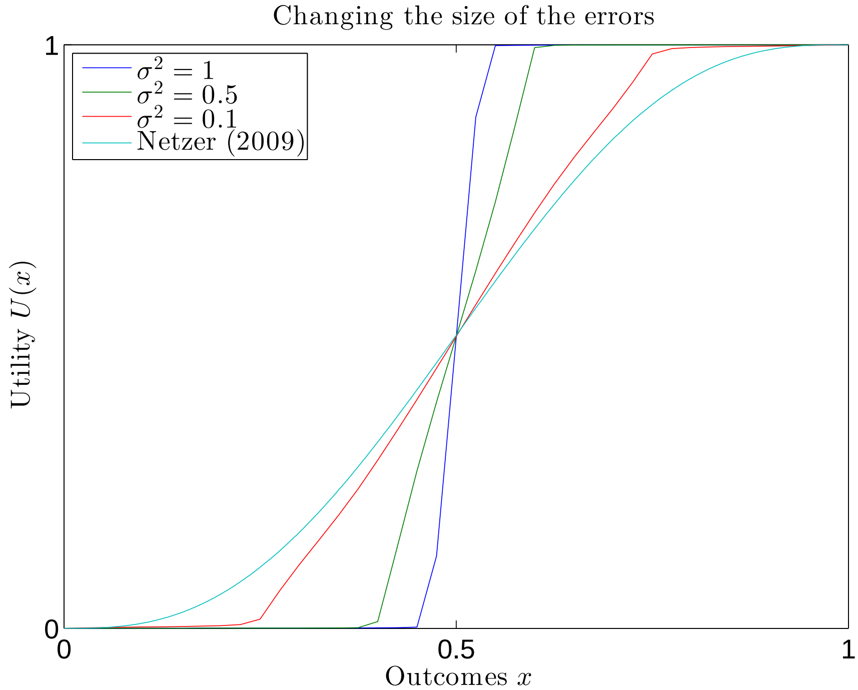
Second, when the variance of the outcomes $x$ increases, the flat areas of the utility function expand. Similarly to larger errors, with larger volatility, the utilty function must also become volatile to compensate.

**Corollary 5.** Suppose that the error $\varepsilon$ has a pdf $g$ and that outcomes $y \sim \text{Beta}(\alpha, \alpha)$. Then, as $\alpha$ increases, $a$ increases and $b$ decreases. That is, as the variance of the outcomes decreases, the flat areas of the utility function expand.

Finally, as the expected outcome $x$ increases, the utility function shifts to the right to accommodate a better environment.

**Corollary 6.** Suppose that the error $\varepsilon$ has a pdf $g$ and that outcomes $y \sim \text{Beta}(\alpha, \beta)$. Then, as $\alpha$ increases, both $a$ and $b$ increase. That is, as

[11] I used Knitro, via Matlab, to solve a discrete version of the optimization problem. The codes are available upon request or at home.uchicago.edu/~jtudon.




*Note:* The errors $\varepsilon \sim \mathcal{N}(0, \sigma^2)$ are compared for $\sigma^2 = 1$, 0.5, 0.1. The outcomes are $X \sim \text{Beta}(4, 4)$, which has a nice bell curve. Netzer (2009) corresponds to the case where $\sigma^2 \to 0$. To put it in perspective, $\sigma^2 = 0.1$ means that 5% of mistakes are going to be larger than 0.2 in absolute value.

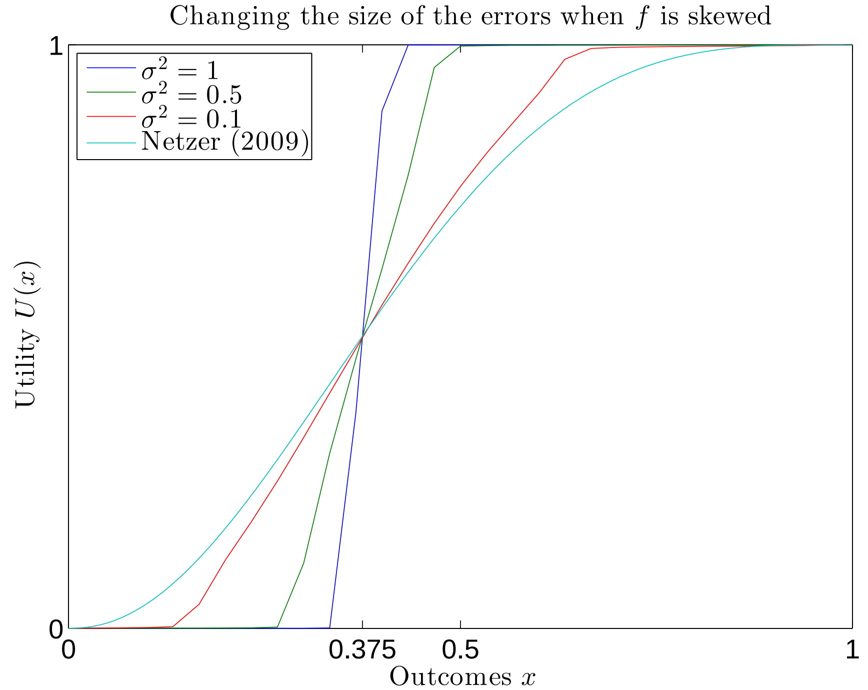
Figure 3: Optimal $U^*$ for different sizes of errors

the expected outcome increases, the non-extreme part of the utility function shifts to the right.

Figure 4 repeats the exercise of Figure 3 but with a skewed distribution for the outcomes. The resulting utility functions are equally skewed, showing loss aversion and diminishing sensitivity around the mean, $\mathbb{E}[X]$.

# 8 Concluding remarks

The numerical and analytical results suggest that, when errors are large, the marginal utility gets concentrated around the mean, as in Figures 3 and 4. On the other hand, the intuition for robustness holds: with limited perception, the robust approach pushes to distinguish relatively bad outcomes from good ones.




*Note:* The errors $\varepsilon \sim \mathcal{N}(0, \sigma^2)$ are compared for $\sigma^2 = 1$, 0.5, 0.1. The outcomes are $X \sim \text{Beta}(3, 5)$, which has a curve skewed to the left. Netzer (2009) corresponds to the case where $\sigma^2 \to 0$. The utility functions exhibit loss aversion and diminishing sensitivity around $x = .375 = \mathbb{E}[X]$.

Figure 4: Optimal $U^*$ for different sizes of errors and skewed $f$

On the multiplicative errors, note the connection with the Psychophysical Law of Weber and Fechner, which states that noticeable differences between two stimuli are proportional to the magnitude of the stimuli.[12] In fact, "[Fechner] argued that the sensation function is logarithmic since equal relative changes in the stimuli brought about equal absolute changes in sensation" (Sinn, 2003).

Back to this paper, instead of additive errors, suppose that the utility function $\widehat{U}$ is shocked with a proportional (multiplicative) error so that $x$ is selected over $y$ iff $\widehat{U}(x)\varepsilon_x \geq \widehat{U}(y)\varepsilon_y$. Then, taking logs and renaming $U \equiv \ln \widehat{U}$, we get that $x$ is chosen iff $U(x) - U(y) \geq \varepsilon \equiv \ln(\varepsilon_y/\varepsilon_x)$. Assuming that these expressions make (technical) sense, it can be shown that the results hold with minor rescaling. Because this is a model of cardinal

[12] In other words, between two weights, the agent notices only differences of, say, 10% or more. See Sinn (2003).

utility functions, Weber's Law is mathematically equivalent to the present model.

As a final thought, let us think about extrapolating the resulting utility function to an environment with risk where the agent faces lotteries: for example, $p[x] + q[y]$ would be the lottery of getting $x$ with probability $p$ and getting $y$ with probability $q$. There is a problem.

Netzer (2009) and I solved for the utility function over outcomes $x$ but not over lotteries $p[x] + q[y]$. In the risky environment, the gene would solve for an optimal *value* function that takes into account outcomes and probabilities—as in prospect theory. But even if the optimal value function can be separated into a utility function and a probability weighting function, such utility function will likely be different from the optimal utility function of the environment without risk. Herold and Netzer (2015) deal precisely with this problem as they explore the possibility that the gene chooses a probability weighting function given that the utility function is $S$-shaped. In this sense, prospect theory functions evolved as a series of kludges (Ely, 2011).[13] All in all, we have to be cautious of extrapolating our result to a risky environment.

Having said that, this paper makes us aware of the costs and benefits of assuming limited perception. To be sure, Rayo and Becker (2007) already derive $S$-shaped utility functions. However they do so by imbuing the agent with local information that the principal would find relevant. By contrast, in this paper, the $S$-shape is sustained only with bounded utility and limited information. Moreover, the $S$-shape will have flat regions precisely because information is limited.

# A Lemma 2

**Lemma 2.** For any $U \in \mathcal{U}$ exists a monotone, non-decreasing function $U^* \in \mathcal{U}$ such that $V[U] \leq V[U^*]$.

[13] Steiner and Stewart (2016) take a different approach: they consider noise in the information processing of a prospect versus a safe option, which leads to a probability weighting function.

*Proof.* This proof is not as trivial as it seems, since changing $U$ at any point will change the whole integral. The strategy will be to consider simple functions first, and then apply the dominated convergence theorem to prove the general case.

The objective function can be written as:

$$V[U] = K + 2\int_0^1 \int_y^1 (x-y)f(x)f(y)G\big(U(x)-U(y)\big)dxdy$$

where $K = \mathbb{E}[X] + \int_0^1 \int_y^1 (y-x)f(x)f(y)dxdy$.

Suppose that $U_n$ is a simple function that is constant on the intervals $\left(\frac{s-1}{n}, \frac{s}{n}\right]$, $s = 1, \dots, n$. Each simple function of this form can be written as the step function $U_n(x) = \sum_{s=1}^n u_s \mathbb{1}\left\{\left(\frac{s-1}{n}, \frac{s}{n}\right]\right\}(x)$. Now we can define the $n$-vector $u$ as the vector with values $u_s$, which is the value that $U_n(x)$ takes on the $s$th interval. From this definition we see that each $n$-vector corresponds to a step function.

We will now define the sorting operator $S$ acting on the vector $u$ as follows. Let $l$ be an integer in $1, \dots, n$ such that $u_l > u_m$ for some $l < m$. If $l$ exists, set $S(u)$ to be the $n$-vector with $u_m$ on the $l$th position and $u_l$ on the $m$th position, and all other elements equal to the corresponding elements of $u$. If such an $l$ does not exist, set $S(u) = u$. Then, $S$ sorts two elements of $u$ in increasing order.[14]

Similarly, let $x_n$ and $f_n$ be two simple functions; the idea is that $x_n \to x$ and $f_n \to f$. With a slight abuse of notation, also let $x_n$ and $f_n$ stand for the $n$-vectors with $x_s$ and $f_s$ in the $s$th position. Therefore, we can approximate $V[U_n]$ with

$$\begin{aligned} V_n[U_n] &\equiv K + 2\int_0^1 \int_y^1 (x_n(x) - x_n(y))f_n(x)f_n(y)G\big(U_n(x)-U_n(y)\big)dxdy \\ &= K + \frac{2}{n^2}\sum_{j=1}^n \sum_{i=j}^n (x_i - x_j)f_i f_j G(u_i - u_j). \end{aligned}$$

In Appendix B, I prove the following lemma.

**Lemma 3.** *$V_n[U_n] \le V_n[S(U_n)]$.*

[14] For example: if $u = (5\ 2\ 3) \Rightarrow S(u) = (2\ 5\ 3)$.

Therefore, if we apply the sorting operator $S$ sufficiently many times to $U_n$, to a maximum of $n$ times, we find the rearranged vector $U_n^*$: a vector completely sorted in ascending order. Every application of the sorting operator $S$ improves $V_n$, and since we apply it a finite number of times $V_n[U_n^*] \equiv V_n[S \circ S \circ \cdots \circ SU_n] \geq V_n[U_n]$.

Now that this holds for simple functions, we go for the general case. Since $f$, $U$ and the identity are measurable functions, there are sequences of bounded simple functions $\{f_n\}$, $\{U_n\}$ and $\{x_n\}$ converging almost everywhere to $f$, $U$ and $x$. Define the increasing rearrangement of $U$ as

$$U^*(x) \equiv \inf \left\{ q \in \mathbb{R} \ : \ x \leq \int_0^1 \mathbb{1}\{U(z) \leq q\} dz \right\}.$$

This rearrangement always exists; it is like sorting the values of $U$ in increasing order. This is a quantile function.

From this definition, $U_n \to U$ a.e. implies that the rearrangements also converge a.e. to the rearrangement: $U_n^* \to U^*$ a.e. (see Chernozhukov, Fernández-Val and Galichon (2009) and the references therein). Moreover, $V_n[U_n] \to V[U]$ and, since the inequality works for every $n$, by Lebesgue's dominated convergence theorem, $V[U] \leq V[U^*]$. □

# B Proofs

*Proof of Lemma 3.* Consider the following expression:

$$\frac{V_n[U_n] - V_n[S(U_n)]}{2} = \frac{1}{n^2} \sum_{j=1}^{n} \sum_{i=j}^{n} (x_i - x_j) f_i f_j \left[ G(u_i - u_j) - G(Su_i - Su_j) \right]$$

First, observe that $G(u_i - u_j) - G(Su_i - Su_j) = 0$ for all $i, j \neq l, m$. Next, for $i = l$, there are $j = 1, \ldots, l$ summands and for $i = m$, there are $j = 1, \ldots, m$ summands. Similarly, for $j = l$, there are $i = l, \ldots, n$ summands and for $j = m$, there are $i = m, \ldots, n$ summands. However, we are double-counting 3 summands, but 2 are zero: The non-zero summand corresponds to $j = l$ and $i = m$. Finally, note that $G(u_m - u_j) - G(Su_m -$

$Su_j) = -[G(u_j - u_m) - G(Su_j - Su_m)]$. Thus,

$$
\begin{aligned}
& n^2 \frac{V_n[U_n] - V_n[S(U_n)]}{2} \\
&= \sum_{j=1}^{n} \sum_{i=j}^{n} (x_i - x_j) f_i f_j \left[ G(u_i - u_j) - G(Su_i - Su_j) \right] \\
&= \sum_{i=l}^{n} (x_i - x_l) f_j f_l \left[ G(u_i - u_l) - G(Su_i - Su_l) \right] \\
&+ \sum_{i=m}^{n} (x_i - x_m) f_j f_m \left[ G(u_i - u_m) - G(Su_i - Su_m) \right] \\
&+ \sum_{j=1}^{l} (x_l - x_j) f_l f_j \left[ G(u_l - u_j) - G(Su_l - Su_j) \right] \\
&+ \sum_{j=1}^{m} (x_m - x_j) f_m f_j \left[ G(u_m - u_j) - G(Su_m - Su_j) \right]
\end{aligned}
$$

$$
\begin{aligned}
&= \sum_{i=1}^{n} (x_i - x_l) f_i f_l \left[ G(u_i - u_l) - G(Su_i - Su_l) \right] \\
&\quad + \sum_{i=1}^{n} (x_i - x_m) f_i f_m \left[ G(u_i - u_m) - G(Su_i - Su_m) \right] \\
&\quad - (x_m - x_l) f_m f_l \left[ G(u_m - u_l) - G(u_l - u_m) \right].
\end{aligned}
$$

At this point, use the definition of $S$ and conclude that the preceding sum equals:

$$
\begin{aligned}
&= \sum_{i=1, i \neq l,m}^{n} (x_i - x_l) f_i f_l \left[ G(u_i - u_l) - G(u_i - u_m) \right] \\
&\quad + \sum_{i=1, i \neq l,m}^{n} (x_i - x_m) f_i f_m \left[ G(u_i - u_m) - G(u_i - u_l) \right] \\
&\quad + (x_m - x_l) f_m f_l \left[ G(u_m - u_l) - G(u_l - u_m) \right].
\end{aligned}
$$

$$
= \sum_{i=1}^{n} \left[ (x_i - x_l) f_l - (x_i - x_m) f_m \right] f_i \left[ G(u_i - u_l) - G(u_i - u_m) \right].
$$

Now we have to show that the preceding expression is non-positive.

By assumption $u_l > u_m$. Moreover, $G$ is a non-decreasing cdf and $x_l < x_m$. Therefore, $G(u_i - u_l) - G(u_i - u_m) \leq 0$.

On the other hand, $(x_i - x_l)f_l - (x_i - x_m)f_m \geq 0$: Suppose that $f_l > f_m$, then $(x_i - x_l)f_l > (x_i - x_m)f_l > (x_i - x_m)f_m$. Now suppose that $f_l \leq f_m$, then $f_m(x_m - x_i) > f_m(x_l - x_i) \geq f_l(x_l - x_i)$, which implies that $(x_i - x_l)f_l - (x_i - x_m)f_m \geq 0$.

Therefore we conclude that $V_n[U_n] - V_n[S(U_n)] \leq 0$. □

*Proof of Lemma 1.* Define $h_y(x) \equiv 2(y - x)f(x)f(y)g\left(U(y) - U(x)\right)$ so that $I(y) = \int_0^1 h_y(x)dx$. Since $U \in \mathcal{M}$, $h_y$ is (Lebesgue) integrable $\forall y \in [0,1]$. Since $f$ and $g$ are bounded, there exists a function $B$ such that $|h_y(x)| \leq B$ almost everywhere. Now, fix $y$ and consider a sequence $\{y_n\}_{n=1}^{\infty} \in \mathbb{R}$ converging to $y$. Then $h_{y_n} \to h_y$ a.e. and $|h_{y_n}(x)| \leq B$ a.e. Therefore,

$$\lim_{y_n \to y} I(y_n) = \lim_{y_n \to y} \int_0^1 h_{y_n}(x)dx = \int_0^1 \lim_{y_n \to y} h_{y_n}(x)dx = \int_0^1 h_y(x)dx = I(y).$$

□

# C Second order conditions

For the sake of completeness, note that $\left.\frac{\partial^2}{\partial^2 \alpha} V[U + \alpha \delta U]\right|_{\alpha=0}$ is equal to

$$\int_0^1 \int_0^1 (x - y)f(x)f(y)g'\big(U(x) - U(y)\big)[\delta U(x) - \delta U(y)]^2 dxdy \leq 0,$$

since $U \in \mathcal{M}$.

## D Corollaries

*Proof of Corollary 4.* Consider $x \in [0, a]$ and recall that $I(x) \leq 0$ and $U(x) = 0$ for such $x$. In particular $I(a) = 0$ and $U(a) = 0$. In other words,

$$I(a) = \int_0^1 (a - y) f(y) e^{-\frac{U(y)^2}{2\sigma^2}} dy = 0.$$

The function $w(y, \sigma) \equiv e^{-\frac{U(y)^2}{2\sigma^2}}$ is a positive weight. Since $U(y)$ is monotone, $w$ is increasing in $\sigma$ and decreasing in $y$. In particular $w(y, \sigma) = 1$ for $y \in [0, a)$ and for any $\sigma$, and $w(y, \sigma) < 1$ for $y \in [a, 1]$. Moreover, $(a - y) > 0$ for $y \in [0, a)$ and $(a - y) \leq 0$ for $y \in [a, 1]$. Since $f(y)$ is fixed, it follows that increasing $\sigma$ increases the weight of the negative part of the integral, relative to the positive part. To maintain $I(a) = 0$, the positive part must increase in size and thus, the interval $[0, a)$ must expand. Then, $a$ must increase.

The case of $b$ is analogous. □

*Proof of Corollary 5.* As in Corollary 4, consider

$$I(a) = \int_0^1 (a - y) \frac{y^{\alpha - 1}(1 - y)^{\alpha - 1}}{\mathrm{B}(\alpha, \alpha)} g\left(U(y)\right) dy = 0,$$

where $\mathrm{B}(\alpha, \alpha)$ is the beta function, which becomes irrelevant. As $\alpha$ increases, the weight shifts towards $1/2$ and away from the interval $[0, a)$, which is the positive part of the integral. Since the equality $I(a) = 0$ must be maintained, the interval must expand. Then, $a$ must increase.

The case of $b$ is analogous. □

*Proof of Corollary 6.* As in Corollary 5, consider

$$I(a) = \int_0^1 (a - y) \frac{y^{\alpha - 1}(1 - y)^{\beta - 1}}{\mathrm{B}(\alpha, \beta)} g\left(U(y)\right) dy = 0,$$

where $\mathrm{B}(\alpha, \beta)$ is the beta function, which becomes irrelevant. As $\alpha$ increases, the weight shifts away from the interval $[0, a)$, which is the positive part of the integral. Since the equality $I(a) = 0$ must be maintained, the interval must expand. Then, $a$ must increase.

For $b$, consider

$$I(b) = \int_0^1 (b-y)\frac{y^{\alpha-1}(1-y)^{\beta-1}}{\mathrm{B}(\alpha,\beta)} g\left(U(y)-1\right) dy = 0.$$

As $\alpha$ increases, the weight shifts towards the interval $(b, 1]$, which is the negative part of the integral. Since $I(b) = 0$ must hold, the interval $(b, 1]$ must shrink. Then, $b$ must increase. □

# References


**Barberis, Nicholas C.** 2013. "Thirty Years of Prospect Theory in Economics: A Review and Assessment." *Journal of Economic Perspectives*, 27(1): 173–196.

**Chernozhukov, V., I. Fernández-Val, and A. Galichon.** 2009. "Improving point and interval estimators of monotone functions by rearrangement." *Biometrika*, 96(3): 559–575.

**Dawkins, Richard.** 1976. *The Selfish Gene.* . 2006 ed., New York:Oxford University Press.

**Elsgolc, Lev D.** 2007. *Calculus of Variations.* . Dover ed., Mineola, New York:Dover Publications.

**Ely, Jeffrey C.** 2011. "Kludged." *American Economic Journal: Microeconomics*, 3(August 2011): 210–231.

**Fehr, Ernst, and Antonio Rangel.** 2011. "Neuroeconomic Foundations of Economic Choice—Recent Advances." *Journal of Economic Perspectives*, 25(4): 3–30.

**Herold, Florian, and Nick Netzer.** 2015. "Second-best Probability Weighting."

**Kahneman, Daniel, and Amos Tversky.** 1979. "Prospect Theory: An Analysis of Decision under Risk." *Econometrica*, 47(2): 263–292.

**McFadden, Daniel.** 1974. "Conditional logit analysis of qualitative choice behavior." In *Frontiers in Econometrics.* , ed. P. Zarembka, Chapter 4, 105–142. New York:Academic Press.

**Montesinos, Vicente, Peter Zizler, and Václav Zizler.** 2015. *An Introduction to Modern Analysis.* . 1st ed., Springer International Publishing.

**Netzer, Nick.** 2009. "Evolution of Time Preferences and Attitudes toward Risk." *The American Economic Review*, 99(3): 937–955.

**Rayo, Luis, and Arthur J. Robson.** 2014. "Biology and the Arguments of Utility."

**Rayo, Luis, and Gary S. Becker.** 2007. "Evolutionary Efficiency and Happiness." *Journal of Political Economy*, 115(2): 302–337.

**Robson, Arthur J.** 2001. "The Biological Basis of Economic Behavior." *Journal of Economic Literature*, 39(1): 11–33.

**Robson, Arthur J., and Larry Samuelson.** 2010. "The Evolutionary Foundations of Preferences." In *Handbook of Social Economics.*

**Sinn, Hans-Werner.** 2003. "Weber's Law and the Biological Evolution of Risk Preferences: The Selective Dominance of the Logarithmic Utility Function, 2002 Geneva Risk Lecture." *The Geneva Papers on Risk and Insurance Theory*, 28(2): 87–100.

**Steiner, Jakub, and Colin Stewart.** 2016. "Perceiving prospects properly." *American Economic Review*, 106(7): 1601–1631.