Homogenized unftf varieties and algebraic frame completion

Extended Abstract: ISSAC’ 18 Poster Session

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ABSTRACT
We introduce homogenized unftf varieties and study the degrees of its coordinate projections. These varieties compactify the affine unftf variety differently from the projectivizations studied in [11]. Each are the closures (Zariski) of the set of finite tight unit norm frames. Our motivation comes from studying the frame completion problem.

KEYWORDS
Algebraic frames, unftf, multidegree

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1 ALGEBRAIC FRAME THEORY
Frames are a generalization to a basis of a vector space which have found use in numerous fields of science and engineering. These are typically applications of frames of (infinite dimensional) Hilbert spaces, but in practice only finite frames are used due to the nature of computing. Given a Hilbert space $H$, a frame is a collection of elements $\{f_k\}_{k \in I} \subset H$ such that there exists $A, B$ with $0 < A \leq B < \infty$ and for every $h \in H$

$$\forall \|h\|^2 \leq \sum_{k \in I} |\langle h, f_k \rangle|^2 \leq B\|h\|^2.$$ These frame conditions were originally given by Duffin and Schaeffer in [4]. If $A = B$, then the frame is called tight. If $H$ is $n$-dimensional, then any frame will have $r \geq n$ elements. Additionally, if the frame consists of elements each of norm one, then we say the frame is a unit norm frame.

Frames which are both tight and unit norm are the focus of much research as these frames minimize various measures of error in reconstruction [6, 10]. Algebraic frame theory uses the powerful tools of computational algebraic geometry to solve problems in frame theory. Such approaches have found success in [2, 5, 11, 13]. In algebraic geometry the problem is made easier by working over the algebraically closed field of complex numbers and by considering compact projective varieties.

In this extended abstract we define a projective variety called the homogenized unftf variety. Introducing affine constraints makes this an affine variety and a non-compact space. We calculate examples of multidegrees of this homogenized unftf variety which captures the degree of the fiber of each coordinate projection. The multidegrees that are computed in this paper differ from those appearing in [11] as our variable groups arise from a different homogenization of the affine unftf variety [3].

Our paper is structured as follows. In Section 2, we define the homogenized unftf variety. In Section 3, we define the multidegree of the homogenized unftf variety and what we mean by the support of the multidegree. Throughout the paper we give examples and computational results to motivate our work.

2 HOMOGENIZED UNTF VARIETIES
In this section we define the homogenized unftf variety, and we show how to go from a projective points of this variety to a frame.

2.1 Projective points to frames
Given $n$ and $r$ such that $r \geq n \geq 2$, let $W$ denote an $n \times r$ matrix. The $i$-th row of $W$ is denoted by $W_i$, and the $j$-th column is denoted by $w_j$. Let $Z_{n,r}$ denote the set of matrices $W \in \mathbb{P}^{rn-1}$ such that

\begin{align*}
(1) & \quad W_i^TW_i = W_{i+1}^TW_{i+1}^T, \quad i = 1, \ldots, n-1, \\
(2) & \quad w_j^Tw_j = w_{j+1}^Tw_{j+1}, \quad j = 1, \ldots, r-1, \text{ and } \\
(3) & \quad \text{the off diagonal entries of } WW^T \text{ equal zero.}
\end{align*}

Let $M_i$ denote the hypersurface defined by $W_i^TW_i = 0$, and let $N_j$ denote the hypersurface defined by $w_j^Tw_j = 0$. We say $M_i$ (resp. $N_j$) are the $i$-th (resp. $j$-th) isotropic row (resp. column) hypersurfaces. Let $Y_{n,r}$ denote the Zariski closure of $Z_{n,r}$, saturated by the union of $M_i$ and $N_j$ for $i = 1, \ldots, n$ and $j = 1, \ldots, r$. We say $Y_{n,r}$ is the $(n,r)$ homogenized unftf variety. This is a projective variety.

We let $X_{n,r}$ be the demonogenization of $Y_{n,r}$ by introducing the conditions $W_i^TW_i = \frac{1}{\lambda}$ and $w_j^Tw_j = 1$; this is the affine unftf variety.

Lemma 2.1. Introducing the affine constraint $w_1^Tw_1 = 1$ on the points of $Y_{n,r} \subset \mathbb{P}^{rn-1}$ dehomogenizes $Y_{n,r}$ to $X_{n,r}$. 

Proof. The affine funtif variety is defined by the matrix equations
\[ W W^T = \frac{r}{n} I_{dn} \]
\[ w_j^T w_j = 1, \quad j = 1, \ldots, r. \]
Since \( w_j^T w_j = w_{j+1}^T w_{j+1} \) for all \( j \), setting \( w_1^T w_1 = 1 \) implies that the second set of equations is true.
It is clear that the equations defined by the off diagonal entries of the matrix equation above are already satisfied by the definition of the homogenized funtif variety. Since \( w_j^T w_j = 1 \) we see that
\[ \sum_{j=1}^{r} w_j^T w_j = r = \text{trace}(W W^T) = \sum_{i=1}^{n} W_i W_i^T. \]
Since \( W_i W_i^T = W_i^T W_i \), \( n W_k W_k^T = \sum_i W_i W_i^T = r \) for any \( k \).
Therefore the diagonal entries of \( W W^T \) are equal to \( \frac{r}{n} \).

This lemma shows why the degree of \( Y_{n,r} \) is half that of \( X_{n,r} \) as seen in the computations next.

**Lemma 2.2.** Consider the dense Zariski open set of \( Y_{n,r} \) defined by be removing the union of row and column isotropic surfaces \( \mathcal{U} := Y_{n,r} \setminus (\cup_{i=1}^{n} M_i) \setminus (\cup_{j=1}^{r} N_j) \). Let \( W \) denote a representative of a point in \( \mathcal{U} \). Let \( c \) denote a nonzero constant such that
\[ \text{trace}(W W^T) = cr. \]
Then, \( W/\sqrt{c} \in Y_{n,r} \) and \( W/\sqrt{c} \in X_{n,r} \).

### 2.2 Computing homogenized funtif varieties

For \( n = 2 \), we compute \( Z_{n,r} \) and \( Y_{n,r} \) for \( r = 3, \ldots, 10 \). We find that each component of these varieties has the expected codimension of \( n(n-1)/2 + (n-1) + (r-1) \). Moreover we see that there are two components of \( Z_{n,r} \) contained in an isotropic hypersurface. These two components each have degree two and are with multiplicity two. We find that the homogenized funtif variety \( Y_{n,r} \) has degree \( 2^r + n - 1 - 8 \). With a straightforward argument using intersection theory, we believe the following theorem will be true.

**Theorem 2.3.** The degree of \( Y_{n,r} \) is \( 2^r + n - 1 - 8 \) for \( r > 2, n = 2 \).

Our computational results agree with those reported in Table 1 of [11] in the following sense. For \( r = 3, \ldots, 7 \) (larger \( r \) are not reported) the degrees reported are of \( X_{n,r} \) which are twice that of \( Y_{n,r} \). We determined the degrees by computing a witness set [12]; this is nothing more than the intersection of a variety with a general linear space. For the largest example we computed (\( r = 10 \) with \( \deg Y_{10,2} = 2040 \)), it took us 213 seconds to compute a witness set using Bertini\(^2\). Our method was to compute a witness set of \( Z_{n,r} \) and then remove the witness points that are in an isotropic hypersurface; this results in a witness set for \( Y_{n,r} \). We used [1], a Macaulay2 [7] package to process the files and remove the witness points. If one tries to perform a standard numerical irreducible decomposition, we found that it took an additional 1313 seconds.

For \( n = r = 3 \), we find that \( Z_{3,3} = Y_{3,3} \) is an irreducible three dimensional projective variety of degree eight. If we change \( r \) to 4, the variety \( Y_{n,r} \) consists of eight components, each of degree eight. The residual components in \( Z_{3,4} \setminus Y_{3,4} \) consists of 12 irreducible components of degree four and one irreducible component of degree 12 with multiplicity 12. On the other hand, changing \( r \) to 5 leads to \( Y_{n,r} \) being an irreducible variety of degree 512.

### 3 Degrees of Projections

#### 3.1 Indexing the degrees

Let \( X \) denote an irreducible affine variety. Then, we are interested in the degree of the fiber of a coordinate projection of \( X \). In particular we are interested in dominant projections where the fiber is finite. To compute the degrees of such a fiber, it’s sufficient to compute the degree of the fiber over a general point.

We consider the case when \( X \) is a homogenized funtif variety \( Y_{n,r} \) restricted to a general affine chart of \( \mathbb{P}^r \). We index the coordinate projections of \( X \) using 0/1 matrices. Let \( \Omega_{n,r,k} \) denote the set of 0/1-matrices of size \( n \times r \) with exactly \( k \) ones. For \( \omega \) in \( \Omega_{n,r,k} \), let \( \omega(a) : X \to \mathbb{C}^k \) denote the projection to the coordinates with a one as an entry in the matrix \( \omega \). We denote the degree of the projection \( \omega(a) : X \to \mathbb{C}^k \) by \( d_{\omega}(X) \). We say \( d_{\omega}(X) \) is a coefficient of the multidegree of \( X \).

Let \( \Omega(X) \) denote the following set
\[ \Omega(X) := \{ \omega \in \Omega_{n,r,k} : k = \dim X, \quad \pi_{\omega}(X \to \mathbb{C}^k) \text{ is dominant} \}, \]
and we say that \( \Omega(X) \) is the support of \( X \). The support of \( X \) has the following relevance for algebraic frame completion. If \( \omega \in \Omega(X) \), then a matrix \( W \) with general entries where \( \omega \) has a one can be completed to a point \( p \in Y_{n,r} \). If \( p \) has all real entries, then it is frame up to a constant factor.

**Example 3.1.** Consider \( Y_{3,5} \) for \( n = 3, r = 5 \), which has codimension nine and projective dimension five. Let \( X \) denote the restriction of \( Y_{3,5} \) to a general affine chart. Then, \( \Omega(X) \) is a subset of \( \Omega_{3,5,5} \) and contains \( \{ \omega_1, \omega_2, \omega_3, \omega_4 \} \), where \( \omega_i \) are as follows,

\[
\begin{align*}
\omega_1 &= \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix} \\
\omega_2 &= \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \\
\omega_3 &= \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix} \\
\omega_4 &= \begin{bmatrix} 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}
\end{align*}
\]

We find that
\[
\begin{align*}
d_{\omega_1}(X) &= d_{\omega_2}(X) = 48, \\
d_{\omega_3}(X) &= d_{\omega_4}(X) = 96.
\end{align*}
\]

In the example, we see many of the coefficients are the same. One reason for this is because there is a natural group action of permutation matrices on the left by \( \mathcal{S}_n \) and the right by \( \mathcal{S}_r \). Such an action preserves the multidegree. Thus, we have an equivalence relation on the 0/1 matrices \( \Omega_{n,r,k} \). Each equivalence class is determined by the row sums and column sums reordered to be non-increasing. Thus, we are able to organize the \( d_{\omega} \) in the Cartesian product of dominance ordering of partitions of size \( k \).

#### 3.2 Computing degrees of projections

We used Bertini to compute the degrees of these fibers using witness sets of projections [9]. In the case of \( n = 3, r = 3 \), there are only
two possible outcomes for these multidegree computations. If \( \omega \) has ordered row and column sums of \((111, 111)\) or \((21, 21)\) then the degree of the fiber of the projection is eight. In all other cases, the degree of the fiber is four.

In Figure 1 we give the multidegrees for the case where \( n = 3, r = 5 \). The diagram is organized by recording the ordered row sums and ordered column sums of the 0/1 matrices \( \omega \in \Omega(X) \). These ordered row (column) sums form partitions which may be partially ordered by the dominance ordering on partitions. We then use this partial ordering to give a partial ordering of the ordered row sum, column sum pairs.

\[
\begin{bmatrix}
1 & 1 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0
\end{bmatrix}
\]

\[
\begin{bmatrix}
221, 311, \{192\} \\
311, 311, \{144\} \\
32, 221, \{304\} \\
32, 2111, \{416\} \\
41, 21111, \{256\}
\end{bmatrix}
\]

\[
\begin{bmatrix}
41, 11111, \{206\}
\end{bmatrix}
\]

\[
\begin{bmatrix}
221, 2211, \{288\} \\
311, 21111, \{416\} \\
32, 211111, \{448\}
\end{bmatrix}
\]

\[
\begin{bmatrix}
221, 11111, \{448\}
\end{bmatrix}
\]

Figure 1: Multidegrees for frames with \( n = 3 \) and \( r = 5 \) ordered by product ordering on pairs of partitions representing row sums and column sums

### 3.3 Computing monodromy groups

We can associate a monodromy group to each of the coordinate projections we considered in the previous subsection. This monodromy group can capture interesting structure of the algebraic variety. Indeed, recent work in [8], has shown that this occurs in algebraic varieties coming from kinematics, algebraic statistics, and formation shape control. Part of our future work is to investigate whether the monodromy group captures interesting structure of homogenized funtf varieties.

### 4 OUTLOOK

In this extended abstract we presented preliminary computations regarding homogenized funtf varieties and their coordinate projections. Our results are based on a new homogenization of the affine funtf variety, which led to equations defining a complete intersection consisting of a union of components \( Y_{n, r} \) of interest and residual components. By understanding the residual components, we are able to make conclusions about components of interest. In Theorem 2.3, we believe that the intersection theory is straightforward and that we can compute the degree of \( Y_{n, r} \). We are hoping for similar results for \( n > 2 \). Since the degree of \( Y_{n, r} \) appears to grow exponentially, we wish for a simpler way to describe the homogenized funtf variety that is related to algebraic frame completion. Our approach was to consider coordinate projections.

### REFERENCES