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The dimensionality of discrete factor analyses

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Abstract This article discusses a solution to Coombs's project of a discrete, ordinal factor analysis for dichotomous data that is structurally homologous to Galois lattice analysis and to other related algebraic approaches. It compares this approach to the better known "biorder" approach to the same problem. In contrast to the biorder approach which is *NP*-hard, here the set of minimal solutions can be determined with a reasonably simple coloration algorithm. The dimensionality of the resulting solution may be larger than that retrieved by the closely related biorder approach, but the underlying space may be more parsimonious in that there are fewer possible regions. In a class of reasonably important cases, the two are equivalent.

Keywords Algebraic · Lattice · Coombs · Discrete · Biorder

1 Introduction

In many cases in the social and behavioral sciences, we confront a rectangular data matrix consisting of the responses of a set of persons to a set of items, and we attempt to find a parsimonious representation of the data by factoring the matrix. While in some cases, such a factoring is used merely for purposes of data reduction and/or prediction, it is often used to generate a plausible model of a response process. The assumption guiding the application of the technique is that the factors retrieved represent psychological traits that vary semi-independently across subjects.

How we approach the factorization problem depends upon whether our data are discrete or continuous, whether the response process is deterministic or stochastic, and whether or not the relation between the factors is seen as compensatory, in that having more of one

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factor makes up for having less of another (for compensatory approaches to dichotomous data, see Takane and de Leeuw 1987; Reise 1999: 224f; also see the interesting approach of Magdison and Vermunt 2001).

Here I wish to re-consider the case of factoring dichotomous data assumed to result from a deterministic, non-compensatory response process. Such a case was originally proposed by Coombs (1964) in his *Theory of Data*, and turns out to have formal homologies to other algebraic techniques currently of interest in the social and behavioral sciences (e.g., Butts and Hilgeman 2003; Degenne and Lebeauz 1996; Duquenne 1995, 1996; Ganter and Wille 1999; Martin and Wiley 2000; Mische and Pattison 2000; Mohr and Duquenne 1997; Pattison 1993, 1995; Van Mechelen et al. 1995; White 1996; Wiley and Martin 1999).

This case was re-explored under the name of "biorders" by Doignon et al. (1984), Doignon and Falmagne (1984), Chubb (1986), and Koppen (1987); Leenen et al. (1999) have recently incorporated this approach in a more general system. In the biorder approach, the emphasis is placed on finding the solution of lowest dimensionality that could re-create the observed patterns. However, this is not the only criterion of parsimony that one might consider when choosing solutions. A second may be minimizing the number of *unobserved* patterns that are considered acceptable combinations by the factorization. A third criterion may be the total number of distinctions made between persons. Here I consider a solution to Coombs's problem recently discussed by Martin (2014), and demonstrate that where it diverges from the biorder approach, it is more parsimonious in these latter two senses, though less parsimonious in terms of the number of dimensions. Most important, I demonstrate under what conditions these two approaches yield the same solution. I begin by replicating Coombs's logic of the response process.

2 Noncompensatory response processes and Galois lattices

Imagine that we give to N subjects a set of J items which draw upon K abilities. More specifically, we follow Coombs (1964) and propose that any dichotomous item requires that the subject be above a particular threshold on each and every trait in order to answer the item in a positive direction (which could be "correct" for a test of ability or "agree" for an attitude). Thus the response process is noncompensatory and deterministic. Consider the ith individual to have a position in a K-dimensional trait space $\mathbf{Y}$ (=$\mathbb{R}^K$) which we can denote as a real vector $\mathbf{y}_\mathrm{i} = [y_{\mathrm{i}1}, y_{\mathrm{i}2},\ldots, y_{\mathrm{iK}}]$. This individual's responses to the J items can be summarized as a Boolean vector $\mathbf{x}_\mathrm{i} = [x_{\mathrm{i}1}, x_{\mathrm{i}2}, \ldots, x_{\mathrm{i}J}]$ with $x_{\mathrm{ij}} = 1$ if the ith respondent answered the jth item in a positive direction, and $= 0$ otherwise. Coombs's logic says that the jth item possesses a set of K thresholds[1] $\{d_{1\mathrm{j}}, d_{2\mathrm{j}}, \ldots\, d_{Kj}\}$ such that

$$\left(x_{ij} = 1\right) \Leftrightarrow y_{ik} \geq d_{kj} \quad \forall k \in \{1, \ldots, K\}. \tag{1}$$

This process then generates a data matrix $\mathbf{X}$; we can also consider $\mathbf{X}$ a set of rows $\mathbf{x}_1, \mathbf{x}_2, \ldots\, \mathbf{x}_N$. We assume that all rows and all columns are distinct: persons (items) with identical distributions across items (persons) may be treated as a single person (item). The set of thresholds for any item can also be interpreted as defining a "discrete item curve

[1] The degenerate case in which a threshold is at $-\infty$ will (following Coombs) here be noted as the threshold being 0, and the trait space being confined to non-negative traits. When we speak of the set of thresholds in a Coombs factorization, we will not include such degenerate cases.

(DIC)", a set of connected points that divides the space **Y** into regions in which the item is answered positively from regions in which it is not.

Figure 1 is an example constructed by Coombs (1964: p. 256). Each item's thresholds appear as a point in this space with lines extending upwards and outwards so that only those persons located "inside" the corner corresponding to any item answer it positively. These L-shapes are the DICs in a two-dimensional space. In a three-dimensional space, each DIC would be a corner composed of three semi-planes, and so on. Table 1 contains the possible response patterns produced organized by the score (the number of positive responses) and indexed according to the numbers in Fig. 1.

Of course, in an actual application, we begin with a data matrix **X**, and not the factorization; our factorization ala Coombs is equivalent to embedding the response vectors in a multidimensional space. Note that this embedding may imply the possibility of unobserved response patterns. If we let $\mathbf{C}_J$ denote the Cartesian product $\{0,1\}^J$ for J items, since $\mathbf{X} \subseteq \mathbf{C}_J$, we can consider the non-observed possibilities $\overline{\mathbf{X}} = \mathbf{C}_J \backslash \mathbf{X}$. Some of these (call them $\overline{\mathbf{X}}_e$), if added to **X**, would imply a different factorization, while the others (call them $\overline{\mathbf{X}}_p = \overline{\mathbf{X}} \backslash \overline{\mathbf{X}}_e$) would not; hence, under the factorization of **X**, the former are *excluded* possibilities and the latter *permitted.* In an analysis of some $\mathbf{X}^* = \mathbf{X} \cup \overline{\mathbf{X}}_p$, we will refer to the rows of $\mathbf{X}^*$ as "states," as they are possible states that a respondent could be in, though they have not all necessarily been observed. All the algebraic approaches cited above permit the Boolean intersections of any observed elements; that is, for any $\mathbf{x}_1$, $\mathbf{x}_2 \subseteq \mathbf{X}$, with $\mathbf{x}_3 = \mathbf{x}_1 \cap \mathbf{x}_2$, if $\mathbf{x}_3 \subseteq \overline{\mathbf{X}}$, then $\mathbf{x}_3 \subseteq \overline{\mathbf{X}}_p$. Because they also all permit the universal upper bound **1** ($x_{\mathrm{ij}} = 1\ \forall$ j) and the universal lower bound $\varnothing$ ($x_{\mathrm{ij}} = 0\ \forall$ j), all assume that the permitted states form a lattice (see "Appendix" for conventional algebraic definitions).

For example, let us imagine that we have three persons asked a short test of three items, with one person answering only the first item (*a*) correctly ($\mathbf{x}_1 = [1,0,0]$), the second answering only the second item (*b*) correctly ($\mathbf{x}_2 = [0,1,0]$), and the third answering only the third item (*c*) correctly ($\mathbf{x}_3 = [0,0,1]$; hence $\mathbf{X} = \{[0,0,1], [0,1,0], [1,0,0]\}$. By the closure of intersection as well as our inability to rule out someone having *no* degree of *any* trait, our $\overline{\mathbf{X}}_p$ would also include [0,0,0]; by our inability to rule out someone having *complete* possession of *all* traits, our $\overline{\mathbf{X}}_p$ would also include [1,1,1].

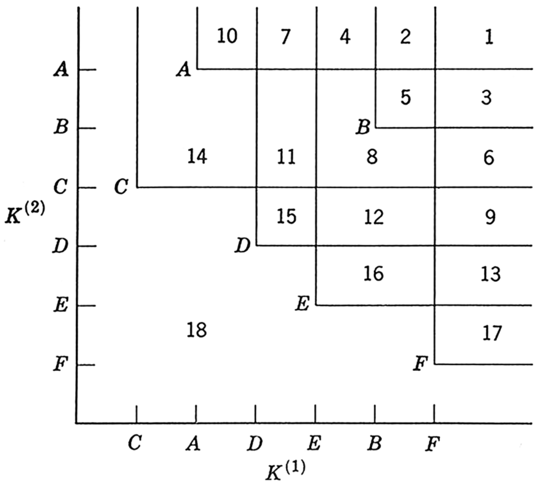

Fig. 1 Coombs's example of noncompensatory, discrete, ordinal space

Table 1 Response patterns associated with Fig. 1

Pattern number	Item A	Item B	Item C	Item D	Item E	Item F	Score
1	1	1	1	1	1	1	6
2	1	1	1	1	1	0	5
3	0	1	1	1	1	1	5
4	1	0	1	1	1	0	4
5	0	1	1	1	1	0	4
6	0	0	1	1	1	1	4
7	1	0	1	1	0	0	3
8	0	0	1	1	1	0	3
9	0	0	0	1	1	1	3
10	1	0	1	0	0	0	2
11	0	0	1	1	0	0	2
12	0	0	0	1	1	0	2
13	0	0	0	0	1	1	2
14	0	0	1	0	0	0	1
15	0	0	0	1	0	0	1
16	0	0	0	0	1	0	1
17	0	0	0	0	0	1	1
18	0	0	0	0	0	0	0

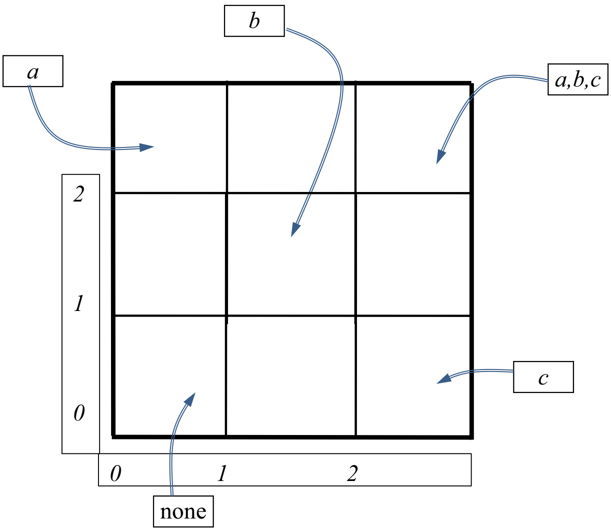

Fig. 2 A two-dimensional factorization

Now let us go on to attempt to place this resulting **X*** = {[0,0,0], [0,0,1], [0,1,0], [1,0,0], [1,1,1]} in a space (we will go over this more technically below; here we use an impressionistic discussion to bring a crucial contrast to light). First, consider one factorization solution (which will turn out to be that reached by the biorder approach) in which we propose two traits, each with three levels, producing nine possible regions (see Fig. 2).

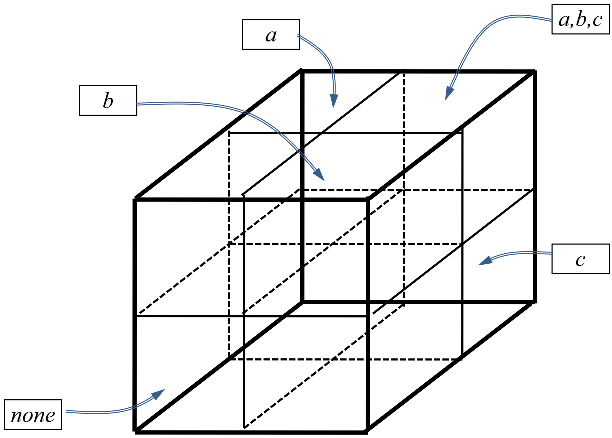

Fig. 3 A three-dimensional factorization

Item *a* requires that a subject exceed the second threshold on the vertical dimension (which is why neither the second nor the third subject answers this positively); item *c* requires that a subject exceed the second threshold on the horizontal dimension (which is why neither the first nor the second subject answers this positively), while item *b* requires that a subject exceed the first threshold on both the vertical and the horizontal dimension (which is why neither the first nor the third subject answers this positively).

A different factorization (which will turn out to be that termed a "Coombs factorization" below) is graphed in Fig. 3. Here we propose three dimensions, each with a single threshold dividing it into high and low ability. Every item requires that a subject exceed this threshold on *two* of the three dimensions. Which solution is to be preferred? Here we may be interested both in issues of parsimony, and in issues of plausibility. The first solution has only two dimensions, as opposed to three. But it allows for 9 (=3 $\times$ 3) possible underlying ability states as opposed to 8 (=2 $\times$ 2 $\times$ 2). Most troublingly, the first solution implies that we well could have observed [1,1,0] and [0,1,1] but did not.[2]

If our data are sparse, then the omission of such observations is quite plausible, but if we had a great number of observations, we might wonder why we find persons with values on our two dimensions of [2,0], and others with values of [1,1] and [2,2], but not, it seems, anyone with values of [2,1]. Such empty regions troubled Coombs (1964: p. 278). (While the factorization of Fig. 3 allows for 8 possible ability states, four of these ([0,0,0], [1,0,0], [0,1,0], [0,0,1]) all map onto the same observed response pattern [0,0,0] and thus there are no missing observations, or $\overline{\mathbf{X}}_p = \varnothing$.) Thus it is not always obvious that we will prefer a factorization of minimal dimensionality as a model of the response process.

I go on to briefly describe the logic of the second solution; without denying that the biorder approach is also related to Coombs' problem, I term this second solution a "Coombs factorization" out of eponymous motives. This approach may be of interest for five reasons. The first is that, as shown in Martin (2014), the permitted states are homologous to those explored using seemingly different methods, namely the "concept" or "Galois" lattices that have attracted attention as a theoretically generative descriptive

[2] It is of course true that in this case, the biorder approach is indifferent to which of the three items we treat as requiring two as opposed to one trait. Similarly indeterminacies can occur in all Boolean factorizations.

technique in the social sciences (see Birkhoff 1967: p. 124; Ganter and Wille 1989, 1999; also Duquenne 1995, 1996; Freeman and White 1993). The second reason is that as we will see below, the solution for the Coombs factorization is simpler than that of the biorder approach. The third reason is that the spatial basis may prove important for generalizing to other response processes. For example, an "unfolding scale" type response process, where a respondent chooses an item if it is "close enough" to his own ideal point in a latent space, generalizes to the rectangle graphs and their higher-dimensional analogues which may also have an algebraic analogue (Trotter 1983: 256f; see Doignon and Falmagne 1994 for a solution to the one-dimensional case). The fourth reason is that there is a second criterion of a solution's parsimony (in addition to low dimensionality), and the biorder approach may lead to solutions that are less parsimonious according to this criterion. But the most important reason is that in some cases, we may find that the Coombs solution provides greater insight as to the nature of the underlying response process.

We begin by considering the set of responses generated by Eq. (1). The remainder of this section briefly outlines the "Coombs factorization" that, for any set of Boolean vectors, re-creates the space in which a Coombs-style response process could have produced the observed data along with other possible unobserved but permitted states.

Definitions Definitions of poset and lattice are found in the "Appendix". We denote a graph G as a set of vertices V and edges E. Given a poset $(A, \leq_A)$, we can define an "incomparability graph" $S(A) = (A, E)$ in which the elements are the elements of A and for some $a, b \in A$, $(a, b) \in E$ iff (not $a \leq_A b$) and (not $b \leq_A a$). Given a graph $G = (V, E)$, a subset C of V is said to form a clique if $(a, b) \in E$ for all $a, b \in C$. Define the "clique number" $Q(G)$ of some graph G as the size of the largest clique present in it. Define a "coloration" of some graph $G = (V, E)$ as a partitioning of vertices Ψ such that for any vertex a in V, $\Psi(a)$ can be labeled as a color or consecutive positive integer, and that if $(a, b) \in E$, $\Psi(a) \neq \Psi(b)$. Let the chromatic number for G be the smallest number of colors in an adequate colorization of the graph. Note that the chromatic number of any incomparability graph such as $S(A)$ is its clique number (see Lovásv 1983: 57f, Theorem 2.6; also Dilworth 1950).

Given a lattice L, an element $m \in L$ is said to be a meet irreducible element (or MIRE) of L if $m = a \wedge b$ implies that $a = m$ or $b = m$. Martin (2014) shows that Coombs's process (given observations in all regions of the trait-space) produces a set of response patterns that form a lattice. In this lattice, meet is equivalent to intersection or element-wise Boolean multiplication. Given such a set of response patterns we can determine the minimum dimensionality as follows. First, we construct the lattice of states and select the MIREs. Second, we construct a matrix $\mathbf{D}_0$ by stacking the MIREs of $\mathbf{X}$ and taking their complement; thus the (i, j)th element of $\mathbf{D}_0$ is '1' iff for the ith MIRE, $x_{ij} = 0$. Third, we construct the matrix $\mathbf{P} = \left(\mathbf{D}_0^c \mathbf{D}_0^T\right)^c$, where $\mathbf{D}_0^c$ indicates the complement of $\mathbf{D}_0$ (i.e. $d_{0ij}^c = 1 - d_{0ij}$) and $\mathbf{D}_0^T$ indicates the transpose of $\mathbf{D}_0$ (i.e. $d_{0ij}^T = d_{0ji}$) and addition is Boolean $(1 + 1 = 1)$. By construction, $p_{ij} = 1$ iff $\mathbf{x}_i \leq \mathbf{x}_j$. Then, interpreting $\mathbf{P}$ as a graph P^M (in which the vertices are the MIREs and an edge connects vertices i and j iff $p_{ij} = 1$), we construct the incomparability graph $S(P^M)$. As shown by Martin (2014: p. 969, theorem 2), the chromatic number of $S(P^M)$ is then the minimum possible dimensionality of a Coombs factorization of the data $\mathbf{X}$. This number (denoted K) is not, however, the same as that considered the minimum dimensionality according to the biorder approach, to which we now turn.

2.1 Comparison to biorders

2.1.1 Biorder approach reviewed

We begin with a general overview of the differences between the biorder approach and the Coombs factorization. The biorder approach was discussed by Doignon et al. (1984), Doignon and Falmagne (1984), Chubb (1986), and Koppen (1987); we use the last of these to frame the problem and the solution. Here the task was to determine for any $N \times W$ set of Boolean data **X** the smallest number K^B such that we may express **X** as the intersection of K^B $N \times W$ biorders. A biorder, in turn, may be seen as a matrix **B** in which there is no h, i, j, and k such that $b_{ij} = b_{hk} = 1$; $b_{ik} = b_{hj} = 0$. This is as much as to say that the rows and columns of **B** may be permuted so that for any i, there is some j such that $b_{i1}, b_{i2}, \ldots, b_{ij} = 0$; $b_{i,j+1}, b_{i,j+2}, b_{iW} = 1$. This clearly can be understood involving the same basic equation as (1) above: in the terms of Koppen (1987: 158; notation adapted for consistency with this exposition), if we let $A = \{1, 2,\ldots, N\}$ and $D = \{1, 2,\ldots, W\}$ then the biorder dimension (or bidimension) of some **X** is the smallest number K^B for which there are two mappings $f = (f_1, \ldots, f_{K^B}) : \mathrm{i} \rightarrow \mathbb{R}^{K^B}$ and $g = (g_1, \ldots, g_{K^B}) : \mathrm{j} \rightarrow \mathbb{R}^{K^B}$ such that for all $\mathrm{i} \in A$ and $\mathrm{j} \in D$, $x_{ij} = 1$ iff $f_k(\mathrm{i}) \leq g_k(\mathrm{j})$ for k = 1, 2,…, K^B.

Finding this number K^B has proven to be *NP*-hard for more than two dimensions (Doignon et al. 1984), although there are some possible simplifications. In particular, Koppen (1987) shows that the problem can be seen as reducible to finding the chromatic number of a hypergraph $H(X^c)$, where the vertices of this graph are the failures in **X** (that is, all pairs (i,j) for which $x_{ij} = 0$), and the edges are alternating cycles in the complement of **X**, interpreted as sets. Thus let us say that we can permute four rows and four columns of **X** so that the following pattern emerges

0xx1
10xx
x10x
xx10

where the x's can be either 1 s or 0 s. Then this represents an edge connecting the four zeros found on the diagonal. We go on to review Koppen's approach only insofar it is necessary for the subsequent results.

The problem of coloring this hypergraph, though quite complex, can be simplified somewhat. Most importantly for our purposes, we do not need to examine all the vertices. One vertex may be dominated by another so that if we color the graph excluding the dominated vertex, we can extend this coloring to include the graph with the dominated vertex.[3] Determining the relations of domination in the hypergraph is itself rather complex. But this task too can be simplified. Koppen defines an "enemies" graph $G(X^c)$ in which the elements are again the "zeros" in **X** and a tie is present between two elements (i, j) and (i*, j*) iff $x_{i*j} = 1$ and $x_{ij*} = 1$. The enemies graph $G(X^c) = (V^G, E^G)$ is therefore a subhypergraph of $H(X^c) = (V^H, E^H)$ where $V^G = V^H$ and $E^G = \{e \in E^H$ such that $|e| = 2\}$. Koppen shows that we can actually extend the coloration of the hypergraph minus a vertex that is dominated in $G(X^c)$ to the hypergraph that adds this vertex; hence, we need only determine the relations of dominance in the "enemies graph."

[3] More technically, in this hypergraph $H = (V^H, E^H)$ for any two vertices, v dominates w if for any edge e in E^H that contains w, the set $B_{vw} = (e \backslash w) \cup v$ is "non-stable," meaning that there is no edge $e^* \in E^H | e^* \subseteq B_{vw}$.

In particular, we can ignore any element (a zero) that is "implied" by another. For example, consider some $\mathbf{x}_1$, $\mathbf{x}_2$, $\mathbf{x}_3 \in \mathbf{X}$, where $\mathbf{x}_3 = \mathbf{x}_1 \cap \mathbf{x}_2$. If for some j, $x_{3j} = 0$, this is failure and hence (3,j) is a vertex in the enemies graph. However, by the definition of intersection, either $x_{1j} = 0$ or $x_{2j} = 0$, and hence either (1, j) or (2, j) is a vertex in the graph which implies that (3, j) would also be present; hence (3, j) is an implied zero. In particular, a zero x_{ik} is said to be "row implied" if there is some x_j such that $\mathbf{x}_i \leq \mathbf{x}_j$ and $x_{jk} = 0$. Despite this simplification, the problem remains *NP*-complete. Indeed, the biorder dimensionality turns out to be equivalent to Ferrers dimensionality as explored in graph theory (Cogis 1982).

2.1.2 Comparison of the coombs factorization to the bidimension problem

One difference between the Coombs factorization and the biorder approach is that the problem of computing the bidimension is *NP*-complete, while the Coombs factorization, as it relies on finding the clique number of an incomparability graph, is much easier and can be completed in polynomial time (Jungnickle 2005: p. 256). Thus even though the dimensionality retrieved is not necessarily the lowest possible, there may be advantages for tasks that involve determining dimensionality for a large number of vertices, such as lattice drawing (see Müller-Hannemann 2001). A lattice that may be embedded in a two-dimensional space ($K \leq 2$) as above can be represented using a Hasse diagram on a two-dimensional plane without crossed lines.

More important, the two have different principles of parsimony. Following the notation of Martin (2014), we can say that a Coombs Factorization is "*M*-minimal" if the total number of thresholds in all dimensions is the same as the number of MIREs in *X*. We call a Coombs Factorization is "*K*-minimal" if number of dimensions is the chromatic number of $S(P^M)$ as defined above. The biorder approach is always at least *K*-minimal; the Coombs factorization will never retrieve fewer dimensions than the biorder approach, while the biorder approach may retrieve fewer dimensions than the Coombs factorization. (This necessarily follows from the fact that the biorder dimensionality is equivalent to the Ferrers dimensionality, and the Coombs dimensionality is not). However, the Coombs factorization but not necessarily the biorder approach is always *M*-minimal; thus the Coombs factorization may retrieve fewer total thresholds than the biorder approach.

Consider data arising through a response process outlined in Eq. (1) where the density of subjects is sufficient to ensure that all possible response patterns are observed and hence our resulting **X** is a lattice closed under intersection. In this case, the biorder approach may be *overly* parsimonious, in that it allows us to construct a space of dimensionality *lower* than that which generated the data. (Of course, it is also possible for a Coombs factorization to retrieve a lower *K* than that responsible for generating the data; consider, for example, a perfect chain that happens to arise in a three-dimensional space.) Thus although the biorder approach is always at least as parsimonious in terms of dimensions, the Coombs factorization may (or may not be) more parsimonious in terms of thresholds and the number of possible observations. More importantly, there may be cases in which we find no unobserved permitted regions ($\overline{\mathbf{X}}_p = \emptyset$) for the Coombs representation but not for the biorder, in which case we may suspect that the Coombs factorization is a better representation of the dimensionality of the trait space that *generated* the observations, even if the biorder approach is a better representation of the minimum dimensionality necessary to *factor* the observations.

Table 2 Example Data and Biorder Solution

	a	b	c
(a) Observed data			
1	1	0	0
2	0	0	1
3	0	1	0
(b) Biorder 1			
1	1	1	1
2	0	0	1
3	0	1	1
(c) Biorder 2			
1	1	0	0
2	1	1	1
3	1	1	0

Take, for example, the simple data portrayed in Table 2a (which we have already analyzed in Fig. 2 above). It can easily be seen that such data can be decomposed into two biorders (presented as Table 2b, c respectively above). However, if we were to follow Coombs's logic and turn this into a geometric model with ordinal thresholds (see Fig. 4, a recasting of Fig. 2), we find that two regions are empty (these are shaded). (This lattical structure of three incomparable elements with a common meet and join is known as an M_3 sub-lattice; this fact will be used below.) Such empty regions might be understood as a lack of "convexity" in the geometric representation; we show below (corollary 2.2) that such gaps are sufficient but not necessary to destroy convexity in a technical sense. Coombs (1964: p. 278) had actually suggested from his experiments that one criterion for a proper model was the absence of such empty regions. If we refuse to entertain these particular empty regions, we would find that three, not two dimensions are necessary, since not one of the three rows in the data matrix in Table 2a can be reduced to the intersection of other rows; each row therefore corresponds to a unique meet-irreducible element. The Coombs factorization then would consist of a space of three dichotomous dimensions (as in Fig. 3). There is an additional dimension of the Coombs factorization, but fewer possible

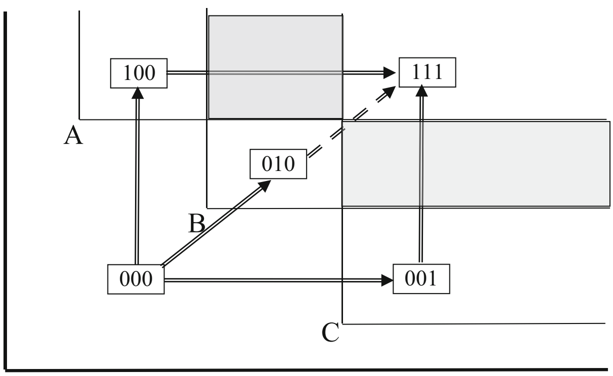

Fig. 4 Absences in biorder representation

observations than the two trichotomous dimensions of the biorder approach (as noted above; $2 \times 2 \times 2 = 8$ versus $3 \times 3 = 9$).

Further, we can determine when the Coombs factorization reaches the same results as a biorder approach. It turns out that in a large set of (perhaps) substantively important cases, the two are equivalent, and hence the biorder dimensionality can be determined in polynomial time. Even more important, consideration of the conditions of this equivalence sheds light on the relation between our data and our models. We close with this.

2.2 The simplicity condition

2.2.1 Simplicity defined

The requirements of the jth item in terms of the K dimensions may be represented as a (column) vector $\mathbf{d}_j$ with values $d_{1j}, d_{2j}, \ldots d_{Kj} \in \mathbb{R}$; the set of all W column vectors is then a $K \times W$ matrix $\mathbf{D}$. We can use Eq. (1) to define a mapping between any position $\mathbf{y} \in \mathbb{R}^K$ to $\mathbf{x} \in \{0,1\}^W$, denoted $\mathbf{x} = D(\mathbf{y})$ and hence the matrix $\mathbf{X} = \{\mathbf{x}\}$ as the set of all possible observed responses stacked to form a matrix; hence we also can say $\mathbf{X} = D(\mathbf{Y})$ where $\mathbf{Y}$ is the (infinite) set of all positions similarly stacked. Here and elsewhere we shall assume that the density of $\mathbf{Y}$ is such that every possible response vector $\mathbf{x}$ is observed. Martin (2014) demonstrated that a set of points $z = \{\mathbf{y} \mid D(\mathbf{y}) = \mathbf{x}\}$, where $\mathbf{x}$ is some particular observed response state, may be termed a "region" and denoted by its unique minimum point; hence each region can be denoted as a vector with one value for each of K dimensions.

Call any $\mathbf{D}$ "simple" if there is no i, j and k such that $d_{ki} = d_{kj}$. This is to say that no two items have precisely the same threshold on any dimension.[4] Given a poset P and $a, b \in P$, a is said to "cover" b if $b \leq a$ and $b \leq c \leq a$ implies $a = c$ or $b = c$. Let the "score" of any Boolean vector $\mathbf{x}$ be defined $h(\mathbf{x}) = \Sigma_i x_i$. We say that a lattice is "graded by score" or "score-graded" when, if whenever element $\mathbf{x}_1$ covers $\mathbf{x}_2$ then $h(\mathbf{x}_1) = h(\mathbf{x}_2) + 1$. (We note that a lattice of Boolean vectors can be graded without being score-graded.) A lattice L is said to be "lower semi-modular" if for distinct elements $a, b, c \in L$, [c covers a, c covers b] implies that both a and b cover $a \wedge b$ (Birkhoff 1967: 14f). A lattice is said to be "lower locally distributive" if it is lower semimodular and does not contain a sublattice isomorphic to M_3 (which is homomorphic to Fig. 4) (Monjardet 2003: theorem 16, p. 132). We may say that two regions z_1 and z_2 are "contiguous" if there are points $y_1 \in z_1$ and $y_2 \in z_2$ such that there is a curve connecting y_1 and y_2 that intersects only one DIC.

Note that not all Coombs factorizations have simple $\mathbf{D}$ matrices; the importance of the simplicity criterion has to do with the relation between Coombs factorizations and biorders. The first theorem simply helps illustrate the effect of the simplicity criterion on the mapping between Z and $\mathbf{X}$.

Theorem 1 *If* $\mathbf{D}$ *is simple, given any two regions* z_1 *and* z_2 *as defined above, if* z_1 *and* z_2 *are contiguous with* $\mathbf{x}_1 = D(z_1)$ *and* $\mathbf{x}_2 = D(z_2)$, *then either* $\mathbf{x}_1$ *covers* $\mathbf{x}_2$ *or* $\mathbf{x}_2$ *covers* $\mathbf{x}_1$ *in the lattice X.*

Lemma 1.1 *If any two regions* z_1 *and* z_2 *as defined above are contiguous, then either* $z_1 \leq z_2$ *or* $z_2 \leq z_1$.

[4] We note that this implies that each dimension can be considered a permutation of the items; this has implications for the relation to incomparability graphs, though we do not make use of this here (though see Golumbic et al. 1983). Also note that we here exclude degenerate thresholds as discussed in note 1.

Proof If not, then there is at least some k such that $z_{1k} > z_{2k}$ and some k* such that $z_{1k*} < z_{2k*}$; let j and j* be the items whose DICs contribute the kth dimension boundary for z_1 and the k*th dimension boundary for z_2 respectively. Then the DIC of j separates z_1 and z_2 and the DIC of j* separates z_1 and z_2 and hence by definition z_1 and z_2 are not contiguous.

We now restate two findings from the previous paper as Lemmas; for proofs we direct the reader to the earlier treatment.

Lemma 1.2 *For any two regions z_1 and z_2, if $\boldsymbol{x}_1 = D(z_1)$, $\boldsymbol{x}_2 = D(z_2)$, and $z_1 \leq z_2$, then $\boldsymbol{x}_1 \leq \boldsymbol{x}_2$.*

Proof See Martin (2014), Corollary 2.

Lemma 1.3 *If $\boldsymbol{x}_1 = D(z_1)$, $\boldsymbol{x}_2 = D(z_2)$, and $\boldsymbol{x}_1 \leq \boldsymbol{x}_2$, then $z_1 \leq z_2$.*

Proof See Martin (2014), Theorem 6.

Proof of Theorem 1 Let z_1 and z_2 as defined in Theorem 1 be contiguous regions with $\mathbf{x}_1 = D(z_1)$ and $\mathbf{x}_2 = D(z_2)$; since we are indifferent to the labeling, by Lemma 1.1 we may assume that $z_1 \leq z_2$ and hence by Lemma 1.2, $\mathbf{x}_1 \leq \mathbf{x}_2$. Now imagine that there is some distinct $\mathbf{x}_3 = D(z_3)$ such that $\mathbf{x}_1 \leq \mathbf{x}_3 \leq \mathbf{x}_2$; this means that (1a) $\exists$ i | $x_{2i} = 1$; $x_{3i} = 0$; (1b) $\exists$ i*|$x_{3i*} = 1$; $x_{1i*} = 0$ and (2) $\exists$ k | $z_{3k} < d_{ki} \leq z_{2k}$ and k* | $z_{1k*} < d_{k*i*} \leq z_{3k*}$. Either (a) $z_{2k*} < z_{3k*}$ or (b) $z_{3k*} < z_{2k*}$; by the simplicity condition we know that $z_{2k} \neq z_{3k}$. If (a) then z_3 not $\leq z_2$ and by Lemma 1.3 $\mathbf{x}_3$ not $\leq \mathbf{x}_2$, a contradiction. If (b) then in order for z_1 and z_2 to be contiguous there must be some dimension j in which either (b1) $z_{3j} < z_{1j} < z_{2j}$ or (b2) $z_{1j} < z_{2j} < z_{3j}$ both of which are impossible given Lemma 1.3 since $\mathbf{x}_1 \leq \mathbf{x}_3 \leq \mathbf{x}_2$.

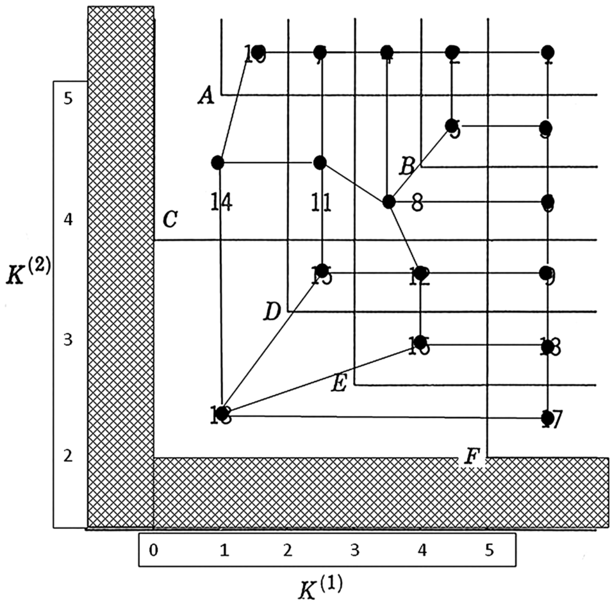

Fig. 5 Contiguous regions connected

Comment We have now established that when no DICs overlap, the relation of proximity between regions corresponds to the relation of "covering" in the lattice **X** of observed states. The lattice for Coombs's example is drawn in Fig. 5 which connects adjacent regions; Coombs's example is thereby a simple one.

Theorem 2 *If* $\mathbf{D}$ *is simple,* $\mathbf{X} = D(\mathbf{Y})$ *is lower semi-modular.*

Lemma 2.1 *If* $\mathbf{D}$ *is simple,* $\mathbf{X} = D(\mathbf{Y})$ *is a score-graded lattice; that is, if element* $\mathbf{x}_1$ *covers* $\mathbf{x}_2$ *then* $h(\mathbf{x}_1) = h(\mathbf{x}_2) + 1$.

Proof By definition of "cover" we know that $h(\mathbf{x}_1) > h(\mathbf{x}_2)$ so we need only demonstrate that there is no i, j| $x_{1i} = x_{1j} = 1$; $x_{2i} = x_{2j} = 0$. Assume there is; consider the points corresponding to columns of $\mathbf{D}$, $\mathbf{d}_i$ and $\mathbf{d}_j$; both cannot be less than the other; since we are indifferent to labeling we can assume that $\mathbf{d}_i$ not $\leq \mathbf{d}_j$, i.e. $\exists$ a|$d_{ai} > d_{aj}$.

Let $\mathbf{Y}^*$ be the set of all points $\mathbf{y}$ such that $\mathbf{x}_2 = D(\mathbf{y})$ and let $\mathbf{y}_2$ be some minimal member of $\mathbf{Y}^*$; construct the point $\mathbf{y}_3$ such that $y_{3k} = \max(y_{2k}, d_{kj})$ for all k and consider $\mathbf{x}_3 = D(\mathbf{y}_3)$. By Eq. (1) $\mathbf{x}_3$ is such that $\mathbf{x}_2 \leq \mathbf{x}_3 \leq \mathbf{x}_1$; thus unless either $\mathbf{x}_2 = \mathbf{x}_3$ or $\mathbf{x}_3 = \mathbf{x}_1$, $\mathbf{x}_1$ will not cover $\mathbf{x}_2$, a contradiction. We know that $\mathbf{x}_2 \neq \mathbf{x}_3$ because $x_{2j} = 0$ while $x_{3j} = 1$ by construction. If $\mathbf{x}_3 = \mathbf{x}_1$ then $x_{3i} = 1$. Let $A = \{g|x_{2g} = 1\}$; because $x_{2i} = 0$ we know $\exists$ b | $d_{bi} > d_{bg}\ \forall\ g \in A$), but because (we now assume) $x_{3i} = 1$, $y_{3b} \geq d_{bi}$ and hence $d_{bj} \geq d_{bi}$. In words, there is some dimension on which item j's requirements are high enough to meet item i's requirements, and there is no other item present in $\mathbf{x}_2$ about which we can say this. Let B be the set of all such b's and recall $d_{bj} \geq d_{bi}$ for b $\in B$. The question then is whether this inequality is a strict one or not. Imagine that $d_{bj} > d_{bi}\ \forall$ b $\in B$ and construct the point $\mathbf{y}^*$ as follows: $y^*_k = \max(y_{2k}, d_{ki})$ and consider $\mathbf{x}_4 = D(\mathbf{y}^*)$. Note that $\mathbf{x}_2 \leq \mathbf{x}_4$ and $\mathbf{x}_4 \leq \mathbf{x}_1$; further, since $x_{4i} = 1$ and $x_{2i} = 0$ by construction, $\mathbf{x}_2 \neq \mathbf{x}_4$, and since we have assumed $d_{bj} > d_{bi}$, $x_{4j} = 0$ and so $\mathbf{x}_4 \neq \mathbf{x}_1$. Thus $\mathbf{x}_2 < \mathbf{x}_4 < \mathbf{x}_1$ and so $\mathbf{x}_1$ does not cover $\mathbf{x}_2$, a contradiction. This means that there is at least one b such that d_{bj} not $> d_{bi}$, that is, $d_{bj} = d_{bi}$ and so $\mathbf{D}$ is not simple.

Corollary 2.1 *If* $\mathbf{D}$ *is simple, and hence* $\mathbf{X} = D(\mathbf{Y})$ *is a score-graded lattice, X contains no sublattice isomorphic to* M_3. *Because this result is not used in the results below but rather to indicate the relation to other work, the simple proof of this is left to the reader as an enjoyable exercise.*

Proof of Theorem 2 In some $\mathbf{X} = D(\mathbf{Y})$ with $\mathbf{D}$ simple let $\mathbf{x}_1$ cover $\mathbf{x}_2$ and $\mathbf{x}_3$; let $\mathbf{x}_4 = \mathbf{x}_2 \wedge \mathbf{x}_3$. If $\mathbf{x}_2$ and $\mathbf{x}_3$ do not cover $\mathbf{x}_4$, $\mathbf{X}$ is not lower semi-modular. Assume (since we are indifferent to labeling) $\mathbf{x}_2$ does not cover $\mathbf{x}_4$. By lemma 2.1, $\mathbf{X}$ is score-graded which means that there is a distinct i (j) such that $x_{1i} = 1$, $x_{2i} = 0$ ($x_{1j} = 1$; $x_{3j} = 0$); for k $\neq$ i, j, $x_{3k} = x_{2k} = x_{1k}$. Because meet is equivalent to intersection we know that $x_{4i} = x_{4j} = 0$; $x_{4k} = x_{1k}$, k $\neq$ i, j. Hence $h(\mathbf{x}_4) = h(\mathbf{x}_2) - 1 = h(\mathbf{x}_3) - 1$. If $\mathbf{x}_2$ does not cover $\mathbf{x}_4$, $\exists\ \mathbf{x}_5$ such that $\mathbf{x}_4 \leq \mathbf{x}_5 \leq \mathbf{x}_2$; ditto $\mathbf{x}_3$, but this is a contradiction as it requires that $h(\mathbf{x}_4) = h(\mathbf{x}_5) - 1 = h(\mathbf{x}_2) - 2$ and hence $h(\mathbf{x}_2) - 2 = h(\mathbf{x}_2) - 1$ or $2 = 1$, a contradiction.

Corollary 2.2 *If* $\mathbf{D}$ *is simple,* $\mathbf{X} = D(\mathbf{Y})$ *is lower locally distributive. This follows from Theorem 2 and Corollary2.1.*

Comment Note that every lower locally distributive lattice is isomorphic to the lattice of closed sets of a convex geometry (Monjardet 2003: Theorem 17, p. 132; Stern 1999: p. 279; Theorem 7.2.27). Hence corollary 2.2 implies that "unoccupied spaces" (as in Fig. 4) which cause the lattice not to be score graded destroy convexity.

Simplicity is sufficient but not necessary for the lattice **X** to be score-graded; the important results regarding the relation of the Coombs factorization to the biorder approach require the score-graded condition as opposed to the simplicity itself. Hence there may be other classes of substantively interesting cases other than "simple" ones in which the two approaches give the same results.

2.2.2 Mappings of score graded lattices to the enemy graph

In the following, when we speak of MIREs, we will imply only the set of non-trivial meet irreducible elements; that is, we exclude the topmost vector **1**. We will speak of the elements of X^c (the failures in X) as "zeroes" and denote them x_{ij}. We begin by stating the central theorem which this section sets out to prove:

Theorem 3 *Given some* $\mathbf{X} = D(\mathbf{Y})$ *with* $\mathbf{X}$ *score graded, if* K^B *is the biorder dimensionality of* $\mathbf{X}$ *and K is the Coombs dimensionality, then* $K^B = K$*. Note that D being simple is sufficient though not necessary for this equivalence.*

Because of the relation between such a lower locally distributive lattice and convex geometries, this theorem is related to work on the relation between convex geometries and the width of their meet irreducible elements (Edelman and Saks 1988: p. 30), but here this result is established in ways that are particularly relevant to the biorder approach for binary vectors, and makes use of the notation that has been employed in algorithms to solve the biorder problem (Koppen 1987). Key is the notion of the domination of one vertex by another (which allows for a simplification of the coloration problem); here, recall, the vertices are zeros in the **X** matrix.

Lemma 3.1 *Given a lattice L of Boolean vectors closed under intersection, (a) if* x_{ik} *is a non-dominated zero,* $\mathbf{x}_i$ *is a MIRE; (b) if* $\mathbf{x}_i$ *is a MIRE, it has some value* x_{ij} *that is a non-row-implied zero.*

Proof Koppen (1987: 164) shows the following (in our notation): given $\mathbf{x}_i$ and $\mathbf{x}_j$ such that $\mathbf{x}_j \leq \mathbf{x}_i$, if $x_{ik} = 0$, then x_{jk} (which also $= 0$) is dominated by x_{ik}. (That is, a "row implied" zero is dominated; not every dominated zero, however, is row implied.) (a) Assume $\mathbf{x}_i$ is not a MIRE, and let J be the set of all j such that $\mathbf{x}_j$ covers $\mathbf{x}_i$ ($|J| > 1$); let x_{ik} be the non-dominated zero. By definition of L, $\mathbf{x}_i = \wedge\{\mathbf{x}_j, j \in J\} = \cap\{\mathbf{x}_j, j \in J\}$, which means $\exists\, j^* \in J | x_{j^*k} = 0$, which means that x_{j^*k} dominates x_{ik}, a contradiction. (b) is reasonably obvious.

Corollary 3.1 *In a graded lattice, every row has at most 1 non-dominated zero.*

Definitions For any (hyper)graph $H = (V, E)$, where $V = \{v\}$, define an "induced sub(hyper)graph" as some $H^* = (V^*, E^*)$ where $V^* \subseteq V$ and $E^* = \{e \in E |$ there is no $v \in e, v \notin V^*\}$ (Harary 1969: 11). That is to say, it is a maximally connected sub(hyper)graph on the vertices V^*, including all edges in H between vertices in V^*.

Lemma 3.2 *Given a lattice of Boolean vectors* $L = (X, \leq)$*, where* $X = \{\mathbf{x}_i\}$*, let* $G(L) = G(X^c)$ *be the enemies graph* $= (V^G, E^G)$*, of elements where every* $v \in V^G$ *is a zero that can be notated* x_{ij}*. Let* P^M *be the poset of the MIREs of L, as defined above. Construct* $S(P^M) = (V^{SPM}, E^{SPM})$ *as the incomparability graph of* P^M*. Consider the mappings* Θ^V*:* $V^G \rightarrow X$ *defined such that* $\Theta^V(x_{ij}) = \mathbf{x}_i$*, and* Θ^E*:* $E^G \rightarrow E^*$ *defined such that* $\Theta^E(x_{ij}, x_{i^*j^*}) = (\mathbf{x}_i, \mathbf{x}_{i^*})$*. That is, we collapse all enemies associated with a single row to a vertex, and establish a relation between vertices that have any enmity between them. Now* *let V* be only those vertices in G(L) that are not row implied, and construct the induced subgraph G*(L) = (V*, E*). Then consider the graph S = (*V^S*, *E^S*) such that* $V^S = \Theta^V(V^*)$ *and* $E^S = \Theta^E(E^*)$*. S = S(*P^M*).*

Proof (a) $V^S = V^{SPM}$. By lemma 3.1a, only MIREs are in $V^S = \Theta^V(V^*)$; further, by lemma 3.1b $\Theta^V(V^*)$ "covers" V^{SPM} (the mapping is surjective). (b) $E^S = E^{SPM}$. By the definition of enemies, it is clear that if two MIREs contain enemy zeros, they are incomparable (and hence $\{\mathbf{x}_i, \mathbf{x}_j\} \in E^S \rightarrow \{\mathbf{x}_i, \mathbf{x}_j\} \in E^{SPM}$). For the reverse ($\{\mathbf{x}_i, \mathbf{x}_j\} \in E^{SPM} \rightarrow \{\mathbf{x}_i, \mathbf{x}_j\} \in E^S$), note that $\{\mathbf{x}_i, \mathbf{x}_j\} \in E^{SPM} \rightarrow \mathbf{S}_{ij} = 1 \rightarrow [\mathbf{P} \cup \mathbf{P}^T]^c_{ij} = 1 \rightarrow [\mathbf{P} \cup \mathbf{P}^T]_{ij} = 0 \rightarrow \mathbf{P}_{ij} = \mathbf{P}^T{}_{ij} = 0 \rightarrow \mathbf{P}_{ij} = \mathbf{P}_{ji} = 0$. Take the first of these and note that since $\mathbf{P} = \left(\mathbf{D}_0^c \mathbf{D}_0^T\right)^c$, $\mathbf{P}_{ij} = 0 \rightarrow \left(\mathbf{D}_0^c \mathbf{D}_0^T\right)^c_{ij} = 0 \rightarrow \left(\mathbf{D}_0^c \mathbf{D}_0^T\right)_{ij} = 1 \rightarrow \Sigma_k \mathbf{D}^c_{0ik} \mathbf{D}^T_{0ij} = 1 \rightarrow \Sigma_k (1 - \mathbf{D}_{0ik})\mathbf{D}_{0jk} = 1$ which means there is at least one k such that $d_{0ik} = 0$ and $d_{0jk} = 1$, which, by the definition of $\mathbf{D}_0$, means that there is at least one k such that $x_{ik} = 1$ and $x_{jk} = 0$. Doing the same for $\mathbf{P}_{ji} = 0$ leads to demonstrating that there is at least one k* such that $x_{ik*} = 0$ and $x_{jk*} = 1$. This is the definition of an enemy relation between zeros x_{jk} and x_{ik*} and so by the definition of the mapping Θ^E, $\{\mathbf{x}_i, \mathbf{x}_j\} \in E^S$.

Lemma 3.3 *If L is graded by score,* Θ^V *is bijective. (That is,* $\Theta^V(a) = \Theta^V(b)$ *iff a = b.)*

Proof This follows from Corollary 3.1 and Lemma 3.2 above. (Every MIRE has at least one non-row-implied zero; in a graded lattice, each has one and only one.)

Corollary 3.2 *If L is graded by score*, *G**(*L*) *as defined in Lemma 3.2 is isomorphic to* *S*(*P^M*).

Lemma 3.4 (*The induced subhypergraph coloration lemma*) *Consider a hypergraph* $H = (V^H, E^H)$ *with chromatic number* $c_H = \psi(H)$*. Let* $H^* = (V^{H^*}, E^{H^*})$ *be an induced subhypergraph* ($V^{H^*} \subseteq V^H$, $E^{H^*} \subseteq E^H$) *with chromatic number* $c_{H^*} = \psi(H^*)$*. Then* $c_{H^*} \le c_H$.

Proof Obvious; any coloration of V^{H^*}in *H* colors *H**.

Proof of Theorem 3 For a score graded lattice *L* let *H*(*L*), *G*(*L*), *G**(*L*), and *S*(P^M) be defined as above, with respective chromatic numbers $K^B = \psi(H(R))$, $K^{B*} = \psi(G(L))$, $K^* = \psi(G^*(L))$ and $K = \psi(S(P^M))$. By corollary 3.2, $K^* = K$. Note that *G*(*L*) is an induced subgraph of *H*(*L*), and that *G**(*L*) is an induced subgraph of *G*(*L*). Then by the induced subhypergraph coloration lemma, $K = K^* \le K^{B*} \le K^B$. Since by definition of Ferrers relation $K^B \le K$, $K^B = K$, and the Coombs reduction reaches the same dimensionality as the biorder reduction.

3 Conclusion

We have investigated the relations between two different, closely related approaches to the discrete factorization of dichotomous items. Both involve assuming a response process that may be seen as deterministic and non-compensatory. Such a discrete factorization may be a useful technique for cases in which we want to reduce the dimensionality of response without losing a sense of qualitative distinctions between sets of persons.

Both the Coombs factorization and the biorder approach may be interpreted as embedding a data structure in a space of unobserved traits. Although the biorder approach is superior when our object is to reduce dimensionality, we generally are less interested in reduced dimensionality for its own sake than in the development of a plausible model in which the dimensions are interpretable characteristics. If we heed Johnson's (1935) call for a more realistic testing of task aptitudes, we may move towards theories of data that attempt to reconstruct *processes* as opposed to building on mathematical *properties*. When we have bad data, neither approach is likely to provide enlightening results. But when we have good data, we may consider it problematic for a factorization if there are unobserved but permitted states, and hence may use this as a criterion to choose between factorizations.

"Good data" may seem too high a bar, for this means not only very low (or no) measurement error, but also a non-stochastic process and a sufficient distribution of persons in the trait space so that all possible response vectors are observed. Sets of data that good may be few and far between, but it is noteworthy that Coombs, setting out his monumental *Theory of Data*, did not think this too fanciful. He suggested that if we found "blank spaces," we perhaps should reject the model; in our terms, this means that a Coombs factorization for data that is *not* closed under intersection is potentially problematic. While methodologists may be happy to construct good models for bad data, Coombs thought that if we had good models we should be able to collect very good data.

Further, it is interesting that as we go from "good data" to "truly excellent data," the two approaches discussed here will converge. That is, the criterion of simplicity is that no two items have exactly the same position on any dimension. We can see that in practice, simplicity will be limited by the smallness of our sample. Thus as $N \to \infty$, it becomes decreasingly likely that two items could be seen as having the same threshold, in that we are more likely to observe people who are "in between" two closely placed thresholds. Thus we might say that when the biorder approach is more parsimonious than the Coombs factorization, we may suspect that our data is not quite as good as it should be.

Alternately, we may suspect that the "leaps" that lead to the non-graded character come from qualitative logical reasoning, and not the nature of the distribution of persons in the latent trait. In such cases, we may not want to pursue a factorial reduction at all; rather, we might want to try to decompose the items into components, using the Haertel and Wiley (1993) inversion. Presumably, the mathematical properties of the matrix can only tell us so much; beyond that, close reading of items and repeated testing is required.

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Appendix: conventional algebraic definitions

The classic treatment of lattice algebra is Birkhoff (1967 [1940]); the exposition here includes only terms needed. The reader familiar with partial orders and lattices may skip this section. A "partially ordered set" (or "poset," for short) is a set of elements $\{\delta_1, \delta_2, \delta_3, \ldots\}$ and a binary relation denoted $\leq$, which satisfies the following three conditions:

(i) transitivity: $\delta_1 \leq \delta_2$, $\delta_2 \leq \delta_3$ implies $\delta_1 \leq \delta_3$;
(ii) reflexivity: $\delta_1 \leq \delta_1$;
(iii) antisymmetry: $\delta_1 \leq \delta_2$ and $\delta_2 \leq \delta_1$ implies $\delta_1 = \delta_2$.

Given a set of elements A and a relation $\leq$, $\delta_1 \in A$ is said to be the "minimum" of A if for any $\delta_2 \in A$, $\delta_2 \leq \delta_1$ implies $\delta_2 = \delta_1$. Note that the set of all points is a poset.

Consider a poset A, consisting of elements δ_1, δ_2, δ_3 etc. together with a binary relation $\leq$ as defined in the text. The "lower bound" of a pair of elements, δ_1 and δ_2, in A is an element δ_3, such that $\delta_3 \leq \delta_1$ and $\delta_3 \leq \delta_2$. Similarly, the "upper bound" of a pair of elements, δ_1 and δ_2, in A is an element δ_3, such that $\delta_1 \leq \delta_3$ and $\delta_2 \leq \delta_3$. The "greatest lower bound" or "meet" of any two elements δ_1 and δ_2 in A, denoted $\delta_1 \wedge \delta_2$, is a unique element δ_3 in A such that $\delta_3 \leq \delta_1$ and $\delta_3 \leq \delta_2$, and there is no δ_4 in A such that $\delta_3 \leq \delta_4 \leq \delta_1$ and $\delta_3 \leq \delta_4 \leq \delta_2$. Similarly, the "least upper bound" or "join" of any two elements, δ_1 and δ_2 in A, denoted $\delta_1 \vee \delta_2$, is a unique element δ_3 in A such that $\delta_1 \leq \delta_3$ and $\delta_2 \leq \delta_3$, and there is no δ_4 in A such that $\delta_1 \leq \delta_4 \leq \delta_3$ and $\delta_2 \leq \delta_4 \leq \delta_3$. A "lattice" is then a poset that is closed under the binary operations of meet and join; that is, for any two elements δ_1 and δ_2 in A, $\delta_1 \vee \delta_2 \in A$, $\delta_1 \wedge \delta_2 \in A$. Given a set of elements $\{\delta_1, \delta_2, \delta_3, \ldots\}$ we may write $\delta_1 \wedge \delta_2 \wedge \delta_3 \wedge \ldots$ as $\wedge\{\delta_1, \delta_2, \delta_3, \ldots\}$.

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