Intergenerational risk-sharing in funded pension schemes under time-varying interest rates

DRAFT

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Abstract

In the economic literature, high potential welfare gains have been reported from intergenerational risk-sharing (IRS) in funded collective pension schemes. In the calculation of these gains a wide range of challenges faced by collective pension schemes however, were ignored. We provide a welfare analysis for a more realistic institutional setup of the collective scheme. Additionally, contrary to earlier papers, we allow for time variation in interest and (expected) inflation. Time-variation provides a challenge for collective pension schemes to optimally allocate risk to individuals, since hedging demands differ across generations. Considering a more realistic institutional setup and more complex financial market, we find that the welfare gains from IRS almost completely disappear. As a matter of fact, we find that an optimally invested individual retirement account, even if subject to several constraints, outperforms the funded collective pension scheme with IRS in welfare terms.

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Introduction

This paper analyses the benefits of intergenerational risk-sharing in funded collective pension schemes. Falling interest rates and increasing longevity have exposed weaknesses in funded pension systems worldwide. In particular systems that feature life-long guaranteed levels of retirement income, such as occupational defined benefit systems (DB), are under great pressure. In response, governments, pension funds and private sponsors of such pension systems are seeking ways to let individuals bear (a bigger part) of the funding risks. A simple way to do so, would be to move to a system based on individual retirement accounts (IRA). Individual accounts or individual property rights over pension wealth help to make transparent who bears what risks and allow for more flexibility compared to traditional DB schemes. The level of investment risk born by the individual can be managed through adjustment of the individual portfolio while, at the same time, longevity risk may still be collectively shared.

An argument heard in policy circles against IRA’s, however, is that IRA’s rule out the benefits of intergenerational risk-sharing (IRS). The idea is that by pooling the assets of different generations together in a single fund, a better risk-return trade-off may be achieved. Plans to create a collective risk-sharing alternative to DB were for example included in the Queen’s speech in 2014 in the United Kingdom. Other countries where these ideas play an important role, for example include The Netherlands and Canada. Both countries saw reforms in recent decades in which collective DB systems were transformed into some form of collective intergenerational risk-sharing arrangements (i.e. see Munell and Sass (2013)).

Policy debates on this topic are inspired by an academic literature on intergenerational risk-sharing. The essence of this literature is that (partially) non-overlapping generations are unable to trade risks with each other through financial markets. Hence, the market is incomplete and therefore inefficient (i.e. see Gordon and Varian (1988), Ball and Mankiw (2007)). This inefficiency could be a justification for policy intervention aimed at establishing the sharing of (investment) risk between these disconnected generations.

Gollier (2008) and Cui et. al. (2010) argue that this missing market problem could be a reason to introduce IRS to funded pension systems. If participation is mandatory, a collective fund could transfer today’s risks (partially) to future generations by (implicitly) reserving surpluses or deficits for them. Both Gollier (2008) and Cui et. al. (2010) estimate welfare gains from IRS within a funded collective pension plan. In Gollier (2008), the optimally designed collective fund achieves a welfare gain equivalent to an increase of 19% in retirement consumption for all future generations. Cui et. al. (2010) report that in their optimal collective pension scheme, the current entering cohort faces a welfare gain between 2 and 4 percent of lifetime consumption. This gain increases to over 30 percent for generations in the far (plus 200 year) future.

These reported gains are significant and hence it is understandable it could make policymakers hesitate to replace existing DB schemes by an institutional setup, such as IRAs, that rules out IRS. An important question however is: are the optimal collective risk-sharing schemes considered in the literature realistic? Economists like to employ a theoretical social planner to solve complex collective problems. Real-life however does not only feature missing markets, but also faces the problem of ‘missing social planners’.

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1 For example, see Donnelly et al. (2014) for a theoretical treatise on how IRAs could be combined with longevity risk sharing. See the PPM unit linked annuity in Sweden for a real-world example (i.e. see Vittas (2011) for a world-wide overview of retirement products in which longevity risk is shared and individuals bear investment risk).

2 In practice misunderstanding may also play a role. People may not always realize that a well diversified portfolio only features systemic market risk and hence further pooling individual retirement savings into a single collective account does not improve the risk-expected return trade-off.
The positive welfare gains of collective schemes with IRS rely on the assumption that wealth is allocated according to an optimal policy function. Will a fund that pools the retirement wealth of multiple generations together be able to consistently follow such a rule over an horizon that stretches multiple generations? Collectivizing such a significant amount of wealth introduces a serious political challenge. How can one commit current and future policy makers to follow the optimal rules? There may be some hope for the collective pension scheme if the collective policy rules could be clearly defined in a long-term contract that - similar to individual property rights in a system of IRAs - could be enforced by a court of law.

A problem with this is that the allocation of wealth in the optimal collective pension schemes depends on variables that are not objectively observable. In Gollier (2008) for example, the amount of wealth distributed to retirees each period relies on the expected return on risky assets and the real interest rate. Both variables are typically not observable. Yet, these variables have a significant impact on the allocation of collective pension wealth (and risk) to different generations. For example, considering the optimal rule in Gollier (2008), an increase in the perceived risk-premium on stocks of one percentage point, will increase current retirement income by roughly 12.5 percent. Similarly, if the expected real interest rate is increased by 1 percentage point, current retirement income would increase by approximately 27 percent. These are extreme levels of redistribution for seemingly fair differences in beliefs about long-run investment opportunities. Hence, it seems likely that interpretation and execution of optimal rules will be subject to significant political controversy. This is exactly what can be observed in practice. In many countries pension discount rates are subject to significant debate. As a result, the intergenerational wealth allocation is more likely to follow the distribution of political bargaining power instead of the theoretically optimal policy functions.

Something that aggravates this problem, is that investment opportunities vary over time. Time-variation introduces discount rate effects to asset prices: i.e. when bond prices rise, interest rates fall and when stock prices rise, expected returns may fall\footnote{See Cochrane (2011) on the empirical relevance of discount rate variation.}. In times of high capital gains, the question for the collective fund will be if these gains reflect higher future cash-flows (a ‘cash-flow effect’) or lower expected returns (a ‘discount rate effect’). Which interpretation is chosen will greatly affect the intergenerational distribution of retirement wealth, as we saw in the example above.

From an alternative perspective, time-variation in expected returns will introduce hedging demands to the optimal portfolio problem\footnote{Boelaars et al. (2014) earlier suggested that IRS in Dutch schemes is rather limited. This paper did not provide a comprehensive welfare analysis however.}. These hedging demands will be horizon dependent and thus introduce heterogeneity in the optimal exposure of individuals to financial market risk. This introduces a challenge for risk-sharing schemes that pool the assets of multiple generations together. Collective schemes could be seen as constraints on the optimal portfolio problem and hence, it is not immediately obvious the collective schemes will outperform an IRA in welfare terms.

Given these problems, it is unclear if collective funded risk-sharing schemes are feasible. One may argue however that such collective risk-sharing schemes can be observed in practice. In particular, the collective schemes in The Netherlands and Canada are often referred to as ‘best practices’ when the reform of traditional DB systems is discussed. The question is though, if these schemes actually feature much IRS to begin with\footnote{Boelaars et al. (2014) earlier suggested that IRS in Dutch schemes is rather limited. This paper did not provide a comprehensive welfare analysis however.} and if collective wealth and risk is allocated in an efficient way.

In this paper we will perform a welfare analysis similar to Gollier (2008) and Cui et al. (2010). We will consider an overlapping generations model with exogenous financial market and compare a collective risk-sharing scheme to an individual retirement account. Our collective setup will be
most comparable to Cui et. al (2010). However, we add some elements of realism. We will add time-variation in interest rates and (expected) inflation. In addition we impose restrictions on the way the collective scheme allocates risk to current and future generations. These restrictions are chosen to roughly match the real-world setups observed in the Netherlands and Canada. \(^5\)

A weakness of our approach will be that we will still ignore the fact that also the real-world schemes that inspire our stylized model, may not be credible in the long-run. Both the Canadian and Dutch schemes only recently transformed from DB into collective risk-sharing schemes. It is not clear if the current setup is credible for several generations to come. The Dutch experience shows that the policies governing these collective schemes have been changed almost annually in the past couple of years. As a matter of fact, Dutch policy makers are currently exploring alternatives because there are concerns the system in its current form is unsustainable. Yet, our welfare calculations will rely on individuals believing that these schemes are credible in the long run.

Even ignoring this credibility problem, we find that the gains from IRS in the more realistic risk-sharing schemes are negligible. As a matter of fact, in our model, a system of IRAs outperforms the collective scheme in welfare terms if we take into account that risk-exposures and contribution levels in the IRA can be optimally managed at the individual level. Our results suggest that a loss of IRS should not be seen as a strong argument against a more transparent and flexible system of IRAs with clearly defined individual property rights over retirement wealth. As we will argue in the final discussion, if one believes in the merits of IRS, there are better ways to organize it than by collectivizing retirement wealth of current generations.

We will now proceed as follows. We will now first introduce and justify the chosen institutional setup of the collective pension scheme with IRS in more detail. Next, we will introduce the financial market model and introduce the individual and collective problem. After that, we will compare how the exposure to investment risk for individuals compares between an IRA and the collective scheme. This will provide an intuitive understanding of how market risk exposure differs in the collective scheme and the IRA. After that, we will move on to discuss the welfare implications of the different setups.

**Modeling real-world funded pension schemes with IRS**

In seek of realism, we will consider a collective pension scheme setup inspired by the funded collective risk-sharing schemes observed in Canada and The Netherlands. \(^6\) Both countries feature large occupational pension funds with some kind of collective risk-sharing. These plans may be referred to in different ways. One may see designations such as ‘Collective Defined Contribution’ (CDC), ‘Defined Ambition’ or ‘Shared Risk Pension Plans’. They all function in essentially the same way though.

In both countries these pension plans started out as traditional defined benefit schemes. As interest rates fell, longevity increased and the funds went through several financial crises, policymakers realized that funding risk of traditional DB was too big to be born by the plans’ sponsor alone. Consequently, funding risk was shifted towards the plans participants. While the institutional se-

\(^5\) Another paper that considers more realistic collective risk sharing schemes is Westerhout et al. (2014). That paper does not feature time-varying interest rates however, and it does not feature an optimized individual account under borrowing constraints.

\(^6\) For a more extensive introduction to the Canadian and Dutch institutional setup see for example Pensions Policy Institute (2014a) and Pensions Policy Institute (2014b). Legislation in Canada differs by province. For this section we consider in particular the recently introduced legislation on ‘Shared Risk Pension Plans’ in the province of New Brunswick.
tups are still very similar to DB plans, nowadays all funding risk is ultimately born by the plans participants. This happens through two channels. Firstly, benefit levels accrued in the past may be adjustments and, secondly, the fund may charge contributions that differ from the value of new benefit accrual. We will refer to this difference between contributions and the value of new accrual from now on as 'net contributions'. The first measure, adjustment of existing benefit levels, will only affect existing participants in the scheme. The second measure, charging net contributions different from zero, will also affect future participants. This is the channel through which these collective schemes introduce intergenerational risk-sharing. Shocks that are realized today may (partially) be allocated to future participants by making them pay contributions that differ from the economic value of the benefits they will accrue.

How wealth and hence financial market risk is exactly allocated to different generations and individuals is determined to a large extent by financial regulation. The essence of which is very similar in both countries. A central role is played by the so-called 'funding ratio' or 'funded ratio'. This is the ratio of a pension schemes assets over its liabilities, the present value of accrued benefits.

\[
\text{Funding ratio} = \frac{\text{Market value of assets}}{\text{Present value of liabilities}}
\]

The general idea is that, in case of a deficit, net contributions may be increased to levels above zero and/or existing benefits may be cut. The exact details of this process however are complex and depend on subjective assumptions about the distribution of future returns. In addition, legislation in both countries leaves plenty of room for discretionary decision making at the level of the pension fund. For the purpose of this paper we will adopt a stylized setup that captures the main characteristics of the real-world situation, but ignores such subjectivity and discretionary decision making.

Valuation of benefits

A crucial element of financial regulation is the discount rate that is used to determine the present value of the funds liabilities. In DB pension plans, benefits are supposed to be default-free. Hence, the most sensible discount rate would be a default-free term-structure of interest rates. In collective risk-sharing schemes, the choice of a discount rate is not so obvious. The retirement benefits paid by the fund are explicitly contingent on its funding status. The funding status in turn depends on the choice of the discount rate. This introduces a circularity.

In the Netherlands pension benefits are to be valued in a 'market consistent' manner. This is implemented in practice by prescription of a default-free term-structure of nominal market rates. In New Brunswick, Canada, the discount rate "shall be consistent with the purposes of the shared risk plan, the funding policy, the investment policy and the risk management goals and procedures". For example, in the Public Service Shared Risk Plan, a collective plan for public servants, this was translated into a flat discount rate of 4.75 percent in 2014 and 2015.

In this paper we decide to work with a default-free market consistent nominal term-structure. This has three reasons. First of all, it is important that the discount rate is time-varying. If not, the collective fund would not recognize that there is an important distinction for long-term investors between discount rate effects and cash-flow effects. It is important that the fund recognizes that capital gains from discount rate effects do not fundamentally improve the outlook for long-run

See article 126 of the 'Pensioenwet' (English: Pension act)
See New Brunswick Regulation 2012-75 under the Pension Benefits Act, article 6(3)
investors. Secondly, the fact that the discount rate used is default-free allows us to make the scheme actuarially fair. Using a default-free rate will make sure that guarantees, implicitly traded within the fund, will be implicitly priced in a market consistent manner. We will show this more explicitly later on. Finally, nominal term structures are (relatively) objectively observable. As argued in the introduction, this important, since otherwise, the allocation of collective wealth would depend on a very sensitive subjective parameter.

Deficits, surpluses and recovery plans

As said, in both the Dutch and the Canadian setup adjustment of existing benefit levels is somewhat opaque. Both setups imply however that deficits and surpluses are allocated to participants gradually over time. We will refer to this as ‘smoothing’. Dutch legislation prescribes that in case of a deficit, pension funds have to submit a recovery plan to the supervisor. The fund will have to describe in the recovery plan how it will eliminate the deficit within 10 years\(^9\) The recovery measures may include both future net contributions and future reductions of benefits. This 10 year period is a rolling window: each year a new 10-year period starts. Recovery measures taken in the recovery plan have to be spread evenly over this recovery period. This implies that, roughly speaking, net contributions and benefit reductions in any year have to reduce the deficit by 10 percent.

In New Brunswick, Canada, legislation prescribes a recovery window which is shorter: less than 5 years\(^11\) However, the definition of the funding ratio is somewhat different. The present value of projected net contributions in the next 15 years may be added to a funds current assets. Hence, recovery through net contributions may be spread out over a somewhat longer period, but if existing benefits still have to be cut, this will have to be done more abruptly than in the Dutch case.

New accrual and contribution policies

So, in both countries, future net contributions are in theory applied for fund recovery. At the same time however, pension contribution in both countries are subject to political constraints in the collective wage bargaining process. It seems that, in practice, there is not much willingness to let contribution rates fluctuate. In the Netherlands, labor unions, employers and the government jointly agreed to try and keep contribution levels fixed\(^12\) In New Brunswick, Canada, it is formally agreed that contributions may vary, but only within a narrow margin of two percentage points of the pension eligible wage\(^13\) These margins are so tight, that they can not account for typical swings in the value of long-term retirement benefits. In practice we should therefore expect to see frequent renegotiation of contribution and accrual levels.

To keep things simple, we will abstract from such political processes. To make sure the collective contribution policy is sustainable, we will simply assume that the collective scheme sets a fixed contribution level (a percentage of wage income). This also allows us to compare the collective scheme with an IRA with a similar fixed contribution rate. To assure that there remains a link between the contribution level and the value of new benefit accrual, we will assume that the accrual

\(^9\)Of course, variation in the nominal interest rate is just a part of all potential discount rate variation. In practice, risk premia also vary over time. For the other reasons listed, it does not seem desirable though to include such parameters into the discount rate.

\(^10\)See, Pensioenwet article 138

\(^11\)See: New Brunswick Regulation 2012-75 under the Pension Benefits Act, article 11

\(^12\)See the joint statement (in Dutch): Stichting voor de Arbeid (2011)

\(^13\)See New Brunswick Regulation 2012-75 under the Pension Benefits Act, article 9(8)
rate fluctuates with the discount rate. The discounted value of new accrual will always be equal to the level of contribution. Future net contributions will still be different from zero, but this will be a consequence of the fact that newly accrued benefits in an underfunded scheme are (in market terms) worth less than newly accrued benefits in an over-funded scheme. In the next paragraph we will see how this works.

Adjustment of benefits accrued in the past

Recovery plans in practice require some projection of future returns, discount rates, funding ratios and contributions. This gives these plans a somewhat intransparent and subjective nature. The essence however, is that contemporaneous shocks to the funding ratio, do not translate one-to-one into changes in current retirement income. We will therefore think of the adjustment of existing benefits to follow a simple adjustment rule:

\[
\% \Delta \text{retirement benefit} = \frac{1}{\theta} \times (\text{Funding ratio} - 1)
\]

where \( \theta \) captures the level of smoothing that is applied by the fund.

So, if the fund, for example, has a surplus of ten percent and \( \theta \) is 10, the adjustment rule implies that all retirement benefits will be increased by 1 percent in that year.

Economic rational for smoothing

One may argue that these real-world collective schemes are not so much the result of economic optimization, but are the result of political compromise and path dependency. As it became apparent that the DB guarantees were unsustainable, these pension schemes evolved into CDC or 'Defined Ambition' because it was politically impossible to completely redesign the system. The existing guarantees had to be 'softened' in a politically feasible way, and collective risk-sharing schemes were born. One could argue however, that there is at least some economic rational behind the setup. The practice of smoothing seems to serve two purposes. Firstly, it is a way to limit the short-term uncertainty in retirement income. Smoothing implies that retirement benefits that are due in the near future are less exposed to realizations in funding risk than long term benefits. We know from the optimal portfolio choice literature that the optimal exposure to market risk is typically decreasing in the investment horizon of individuals. One reason for this is that financial capital itself is an increasing share of the overall wealth of individuals. Hence, if risk-preferences are constant, to keep the overall exposure to market risk constant, the exposure of financial capital to market risk has to be reduced (Bodie et al., 1992). Smoothing could be an implicit way to introduce such a life-cycle pattern.

Secondly, smoothing is exactly what introduces intergenerational risk-sharing. Smoothing implies that risks that are realized at present, will not fully translate into benefit adjustments right away. Instead, a shock today leads to a gradual adjustment of retirement benefits in the years to come. This means that those who will start accruing benefits in the future, after a shock has been realized, will still be affected by it. Since the price of newly accrued benefits is set independent of the funding ratio, retirement benefits that are accrued within the recovery period, will absorb a part of the risk. The maximum legal length of the recovery period sets a limit on the amount of IRS. An important consequence of this is that participation will have to be mandatory and individuals can not choose their own contribution/accrual rates. Otherwise, they could optimally stop contributing when the fund is in deficit and increase accrual in times of collective surplus.
Welfare effects

Intuitively, we can summarize what the collective risk-sharing scheme does as follows. The fund provides nominal deferred annuities to its participants. If the fund decides to fully hedge these annuities with default-free bonds, that is the end of the story. However, if in practice the fund decides to take mismatch risk between its assets and liabilities, this risk is allocated to current and future fund participants through the adjustment rule. The adjustment rule implies that all participants share proportionally in the return on the funds mismatch, with the exception of smoothing. Smoothing reduces the riskiness of retirement benefits that are due within the recovery period and implies that part of the risk is allocated to participants that will start participating within the recovery period.

It is not immediately obvious whether expected life-time utility under the collective setup will be higher or lower than the expected utility an individual could achieve in an IRA. The collective scheme can create welfare gains by allowing future participants to participate in contemporaneous market risk. Additionally, it can also benefit current participants, for example, if they are borrowing constrained, and the collective scheme allows them to be more exposed to market risk than they could be otherwise. On the other hand the collective fund will not be able to explicitly implement the optimal portfolio allocation for its individual members. The fund does implicitly recognize individuals hedging demands, by assigning (deferred) annuity type retirement benefits, which are valued using a time-varying interest rate. Furthermore, the fund does introduce an implicit lifecycle pattern through the introduction of smoothing. However, it remains to be seen how all this translate into expected life-time utility. In order to illustrate the overall welfare implications within an expected utility framework, we will need a quantitative model, which is what we will introduce next.

Model specification

To perform a welfare analysis, we will need to consider a model of overlapping generations. We will think of two alternative institutional setups. Individuals save and invest for retirement either through a mandatory collective risk-sharing scheme or they save and invest through an individual retirement account. To avoid a potential debate about behavioral aspects and cost-efficiency, we will think of both the collective scheme and the IRA as being organized by the same organization with the same board of trustees that act on behalf of the participants. If organized in IRAs, the board can implement the individually optimal strategy by varying individuals share in some collective stock and bond funds. In the case of the collective scheme, the board can only set the collective asset mix. This creates a conflict of interest between individuals, as different participants will prefer different asset allocations. We will assume that the board of trustees in this case has an objective function defined over the collective funds funding ratio and chooses the asset mix accordingly.

We will think of both individuals and collective pension schemes as modest players relative to the financial market, which is therefore assumed to be exogenous. Individual utility will be of the standard CRRA form and individuals will receive an exogenous level of income. We will start by introducing the financial market model. After that, we will introduce individual preferences, demographic assumptions and the individual and collective optimization problems in detail.
The financial market

We will rely on the financial market model introduced by Brennan and Xia (2002). This model features three sources of risk: equity risk, real interest rate risk and expected inflation risk. The model is formulated in continuous time and these three risk sources are represented by three standard Brownian motions, that may be correlated: \( dz_r, \ dz_\pi \) and \( dz_S \). The model features two state variables: the instantaneous real interest rate, \( r_t \), and the expected instantaneous rate of inflation, \( \pi_t \). All asset prices in the model are captured by a single (real) pricing kernel, \( M_t \).

The different processes describing the financial market are summarized in table (1). The table also contains the parameter values we will use in our base line calculations. Where possible, we have picked values similar to Gollier (2008), for comparability. This means that we set the average real interest rate to 2 percent per year and the risk-premium on equity risk to 4 percent per year.

The processes for the instantaneous inflation rate and real interest rate are mean-reverting Ornstein-Uhlenbeck processes. Mean-reversion is needed to make sure the model is stationary. It is not intended as a realistic description of real-world financial markets. An unintended consequence of mean-reversion is that the effective duration, meaning the sensitivity of bond prices to the interest rate state, tends to be low. This could bury the fundamental uncertainty long term investors face with respect to long term interest rate movements. In order to leave plenty interest rate risk in bond prices, we set the volatility of the interest rate process to one percentage point per year and the half-time for interest rate shocks to 20 years. This results in a maximum effective duration close to 30. For the inflation process we set the annual volatility of 1 percentage point and a mean of 2 percent. We set the half-time of the expected inflation process lower, to 5 years. So, expected inflation shocks are assumed to be less persistent than interest rate shocks.

Bond prices will depend on the prices of interest rate risk and expected inflation risk. A downside of the limited number of state variables in the model is that there is a tight relationship between the instantaneous bond risk-premia and long-run spot and forward rates. We have to accept higher short term bond-premia if we want to keep the yield curve from going negative for long horizons. If the real forward rate for infinitely long horizons is restricted to be positive (and hence bond prices are non-increasing in horizon), the minimum real instantaneous risk premium on long term nominal bonds is around one percent. As a compromise, we will set the prices of risk such that the instantaneous real risk-premium on long term nominal bonds is 1.1 percent while the real forward rate for infinite horizons is 0.7 percent.

Finally, in our baseline calculations, we will assume that the three sources of risk, \( dz_S, \ dz_r \) and \( dz_\pi \), are uncorrelated. This implies that the instantaneous return on stocks will be uncorrelated to the instantaneous return on bonds. In the sensitivity analysis later on, we will explore the effect of correlation between equity and interest rate risk.

Individual utility, life-time and income

Similar to the earlier papers in the literature we will assume individuals have a CRRA utility function with a coefficient of relative risk-aversion of 5. Because we will have to rely on numerical methods when solving the problem, we define utility in discrete time. Expected utility at time \( t \) will be:

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\[^{14}\text{Brennan and Xia (2002)} \text{ also allows for unexpected inflation risk. We will abstract from this feature. As } \text{Brennan and Xia (2002) point out, the welfare effect of the unexpected inflation shocks is relatively low, even for long horizons. For the purpose of our analysis, it would affect both the collective pension plan and the IRA in the same way. Therefore, its effect on the welfare difference would be negligible.}\]

\[^{15}\text{More precisely, the duration of a zero-coupon bond goes to 28.85 years as time to maturity goes to infinity.}\]
Table 1: The financial market

<table>
<thead>
<tr>
<th>Real interest</th>
<th>( dr_t = \kappa(\bar{r} - r_t)dt + \sigma_r , dz_r )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Unconditional mean</td>
<td>( \bar{r} )</td>
</tr>
<tr>
<td>Volatility</td>
<td>( \sigma_r )</td>
</tr>
<tr>
<td>Mean-reversion parameter</td>
<td>( \kappa )</td>
</tr>
<tr>
<td>Half-life (( \bar{r} - r_t ))</td>
<td>( \frac{\ln(0.5)}{-\kappa} )</td>
</tr>
<tr>
<td>Price of risk</td>
<td>( \lambda_r )</td>
</tr>
<tr>
<td>Real ultimate forward rate</td>
<td>0.7%</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Expected inflation</th>
<th>( d\bar{\pi}<em>t = \alpha(\bar{\pi} - \pi_t)dt + \sigma</em>{\pi} , dz_{\pi} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Unconditional mean</td>
<td>( \bar{\pi} )</td>
</tr>
<tr>
<td>Volatility</td>
<td>( \sigma_{\pi} )</td>
</tr>
<tr>
<td>Mean-reversion parameter</td>
<td>( \alpha )</td>
</tr>
<tr>
<td>Half-life (( \bar{\pi} - \pi_t ))</td>
<td>( \frac{\ln(0.5)}{-\alpha} )</td>
</tr>
<tr>
<td>Price of risk</td>
<td>( \lambda_{\pi} )</td>
</tr>
<tr>
<td>real risk premium on LT bond</td>
<td>1.1%</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Stock index</th>
<th>( \frac{dS_t}{S_t} = (R_f + \lambda_S \sigma_s)dt + \sigma_S , dz_s )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Volatility</td>
<td>( \sigma_S )</td>
</tr>
<tr>
<td>Price of risk</td>
<td>( \lambda_S )</td>
</tr>
<tr>
<td>Risk premium</td>
<td>( \lambda_S \sigma_s )</td>
</tr>
</tbody>
</table>

| Price index | \( \frac{d\Pi_t}{\Pi_t} = \pi_t \, dt \) |

<table>
<thead>
<tr>
<th>Real pricing kernel</th>
<th>( \frac{dM_t}{M_t} = -r_t , dt + \phi_S , dz_S + \phi_r , dz_r + \phi_{\pi} , dz_{\pi} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \phi_S )</td>
<td>-0.2</td>
</tr>
<tr>
<td>( \phi_r )</td>
<td>0.1</td>
</tr>
<tr>
<td>( \phi_{\pi} )</td>
<td>-0.2</td>
</tr>
</tbody>
</table>

\[
U_t = E_t \sum_{i=0}^{T-t} \beta^i \frac{c_{t+i}^{1-\gamma}}{1-\gamma}
\]

where \( c_t \) is consumption in real terms at time \( t \), \( \beta \) is the parameter of time-preference, \( \gamma \) is the coefficient of relative risk aversion and \( T \) is the last period the individual is alive. We are abstracting from longevity risk and hence \( T \) will be deterministic. In our baseline setup, we will set the rate of time preference equal to the mean real interest rate (\( \beta = e^{-0.02} \)).

We will assume that the life of an individual can be divided into two parts: the work phase and the retirement phase. The first phase will last \( T_w \) years and the second phase will last \( T_r \) years. In our baseline calculations, we will set these values to 45 and 20 years respectively. During the work phase, the individual will earn a constant, exogenous real wage of \( y_w \) per year. After that the individual will receive a fixed (in real terms) government pension of \( y_r \) per year. In our baseline calculations we will set \( y_r = 0.4y_w \).
The individual problem

In the individual problem the individual has an IRA. Contributions, withdrawals and portfolio allocations can be chosen at the individual level. We assume that the individual faces a no-borrowing constraint and a no-short-selling constraint.

Formally the individual problem at time $t$ can therefore be written as:

$$\max_{\{c_t, x_t\}} \sum_{s=0}^{T-t} \beta^s E_t \frac{c_{t+s}^{1-\gamma}}{1-\gamma} \quad \text{s.t.} \quad \begin{align*}
  w_{t+s+1} &= (w_{t+s} - c_{t+s}) \bar{x}_{t+s} \vec{r}_{t+s+1} + y_{t+s+1} \\
  \bar{x}_{t+s} \vec{1}_{n \times 1} &= 1 \\
  \bar{x}_{t+s} &\geq \vec{0}_{n \times 1} \\
  0 &< c_{t+s} \leq w_{t+s}
\end{align*}$$

where $w_t$ denotes real wealth, $c_t$ real consumption, $\bar{x}_t$ is an $n \times 1$ vector of portfolio weights (with $n$ the number of assets), $\vec{r}_{t+1}$ a $n \times 1$ vector of gross real returns between $t$ and $t+1$ on all $n$ assets available to the investor.

Even though the pricing kernel in the financial market model does price real bonds, we will assume real bonds are not available to the investor. This assumption is made to reflect the lack of inflation-linked investments in most financial markets. A worry could be that, since bond prices are determined by just two state variables, investors can easily create a synthetic real bond by appropriately combining two nominal bonds. This is however ruled out by the no-shortsales constraints. Replicating a real bond would involve a short position and is therefore impossible.

The constraints make an analytic solution to this problem unfeasible. We will therefore have to rely on numerical dynamic programming. Appendix A will provide more details on the solution technique. In order to reduce the number of dimensions in the optimization, we will restrict the asset menu. Instead of allowing the individuals (and the collective scheme, later on) to pick bonds of all durations, we will only allow them to invest in a single long-term nominal bond. This will not significantly change the nature of the optimal solution, if we pick the maturity of the bond wisely.

Increasing a bond's maturity will make the bond more effective as a real interest rate hedge while, at the same time, exposing the individual more to expected inflation risk. The 'inflation duration', by which we mean the derivative of the bond price with respect to expected inflation, converges to its maximum more quickly than the interest duration as the bonds time to maturity increases. This is due to the stronger mean reversion in the inflation process in our baseline calibration. As a consequence, longer maturity bonds provide a better trade-off between interest rate risk and expected inflation risk to the investor. In addition, a higher maturity bond allows for higher real interest rate exposures within the portfolio constraints. We will therefore set the maturity of the bond relatively high at 60, which implies a real interest duration of approximately 25 and an inflation duration of 7.2. Summarizing, the individual and the collective pension scheme can allocate their financial wealth to either the stock index, the long-term bond or to cash (a one year nominal bond).

\[\text{See equation A4 and A5 in } \text{Brennan and Xia (2002)}\]
Table 2: Population parameters

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Rate of time preference $\beta$</td>
<td>0.98</td>
</tr>
<tr>
<td>Coefficient of relative risk-aversion $\gamma$</td>
<td>5</td>
</tr>
<tr>
<td>Work phase (years) $T_w$</td>
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</tr>
<tr>
<td>Wage income $y_w$</td>
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</tr>
<tr>
<td>Retirement phase (years) $T_r$</td>
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</tr>
<tr>
<td>Government pension $y_r$</td>
<td>0.4</td>
</tr>
<tr>
<td>Contribution to CDC scheme $s$</td>
<td>0.1</td>
</tr>
</tbody>
</table>

The collective defined contribution plan

We have already informally introduced the collective pension schemes setup. Here we will describe in a more formal matter how the collective scheme is implemented.

Pension benefits

In the collective scheme, individuals do not accumulate assets individually. Instead, they accumulate a 'retirement benefit'. This benefit is expressed as a flat nominal deferred annuity. Once the retirement age is reached, the participant every year receives a payment equal to the contemporaneous face value of her retirement benefit. We will denote the benefit level of individual $i$ at time $t$ by $b^i_t$.

While the CDC pension plans observed in practice record a single benefit level for each participant that is paid until death, we find it useful later on to decompose this benefit into a sequence of single horizon benefits. So, similar to stripping coupon bonds into zero-coupon strips, we will write the individuals holdings of retirement benefits within the CDC scheme as a sequence of single horizon benefits, $b^i_{t,\tau}$, where $\tau$ indicates the time at which the benefit is due. Consequently, the actual payment at time $\tau$ in retirement will be: $b^i_{t,\tau}$. This 'expansion of the state' will help us in a moment to establish the actuarial fairness of the collective scheme.

Pension contributions and benefit accrual

As explained earlier, we will assume that within the collective scheme individuals pay a fixed proportion, $s$, of their income to the fund. In exchange for this contribution, the individuals benefit level is increased. The value of new benefit accrual will be determined based on the nominal default-free interest rate as follows:

$$\Delta b^i_{t,\tau} = \frac{s \cdot y_w}{\sum_{j=T_T+1}^{T} P_{t,j}}$$

where $\Delta b^i_{t,\tau}$ is the benefit increase for individual $i$ and $P_{t,j}$ denotes the price of a nominal default-free zero-coupon bond that pays one nominal dollar at time $j$. Hence, the numerator is the contribution paid and the denominator is the price of a flat, deferred, nominal annuity.
Present value of liabilities

The present value of the funds liabilities at time $t$, will be denoted by $L_t$. It will be equal to the value of all accrued benefits discounted at the default-free nominal discount rate:

$$L_t \equiv \sum_{i} \sum_{\tau=t}^{\infty} b_{i,\tau}^t P_{t,\tau}$$

(4)

Within period time line

As in the individual problem, time in the collective problem will evolve in discrete one-year steps. Within each time period, we will have the following sequence of events:

- Returns realized (1)
- Benefits adjusted (2)
- Retirees paid
- New benefit accrual (3)
- Rebalance portfolio

The period starts with the realization of the market returns. Subsequently the pension scheme determines its funding ratio and pension benefits are updated according to the adjustment rule. Next, retirement benefits due this period will be paid, contributions are collected and new benefit accrual will take place. Finally the fund sets its portfolio allocation for the next year.

Due to benefit adjustment, pay-out of benefits and new benefit accrual, the level of accrued benefits, assets and liabilities will change within the period. It will therefore be useful to introduce notation that allows us to distinguish between the levels of variables before and after these events. When we want to denote a variable after benefit adjustment, we will indicate this by adding (2) in the superscript. Likewise, (3) in the superscript will indicate the value of a variable at the very end of the period.

Benefit adjustment and actuarial fairness

The adjustment of benefits will be a slightly modified version of the simple rule we introduced earlier (Eq. 1). We will choose a specification such that the market value of pension benefits in the CDC scheme can be determined analytically. The market value of benefits will not be of immediate relevance to individuals, since they can not trade them. Knowing the market value will be useful to us however, for two reasons. First of all, it allows us to perform checks on our numerical work. It enables us to see if simulated values closely match their analytical counterparts.

Secondly, knowing the market value will help us to assure that the collective pension scheme is actuarially fair or 'arbitrage-free'. By this, we mean that that ex ante, the market value of future contributions is equal to the market value of future benefits. If so, the collective pension plan does not, from an ex-ante perspective, redistribute market value between participants or provide free lunches. This implies that the welfare effects we find for different individuals will not be driven by redistribution of wealth between individuals or generations, but purely reflect efficiency gains or losses.
A first challenge when determining the market value of a retirement benefit is that its value depends on the current funding ratio. Collective pension benefits are worth more if the funding ratio is higher. Earlier we introduced that we will keep track of benefit levels for each horizon separately. We will employ this here to make the market value of benefits independent of the funding ratio, while at the same time maintaining the smoothing of funding shocks.

Instead of smoothing shocks by carrying funding deficits and surpluses over to future periods, we will allocate these surpluses and deficits explicitly, right-away. Benefits for all horizons will be adjusted such that the funding ratio is back to 1 right away. So, we will impose that \( A_t^{(3)} = L_t^3 \) for all \( t \). To make sure the properties of smoothing are maintained, benefit adjustment will be made horizon dependent. Furthermore, to preserve the fact that new benefits, accrued within the recovery period, are exposed to contemporaneous shocks, we will already add these as a liability to the balance sheet. The balance sheet is balanced by also adding the value of the corresponding future contributions as an asset. By this modification of the adjustment process, we eliminated the funding ratio as a state variable in the market value of benefits.

In order to determine the market value of a retirement benefit, there is still one more hurdle to take: the funding ratio is a non-linear function of stochastic variables (assets and liabilities). As a consequence, pension benefits are exposed to market risk in a non-linear way. Since benefit adjustment will take place in discrete time, this makes it difficult to find an analytical expression for the market value of benefits. We will therefore get rid of this non-linearity by linearizing the funding ratio. Specifically, we will specify the adjustment rule as follows:

\[
b_{t+1,\tau}^{(2)} = b_{t,\tau}^{(3)} \times \left( 1 + w(\tau - t, \theta) \frac{A_{t+1}^{(1)} - L_{t+1}^{(1)}}{L_t^{(3)}} \frac{P_{t,\tau}}{P_{t+1,\tau}} \right)
\]

where \( w(\tau - t, \theta) \) is a horizon-dependent weight. This weight will capture the effect of smoothing. The exact value of the smoothing weight will depend on the length of the recovery period and the due date of the benefit, \( \tau - t \). The term \( \frac{A_{t+1}^{(1)} - L_{t+1}^{(1)}}{L_t^{(3)}} \) is the linearized funding ratio. It can also be interpreted as the excess return on the funds mismatch between assets and liabilities. The term \( \frac{P_{t,\tau}}{P_{t+1,\tau}} \) is probably better understood if we rewrite (5), as follows:

\[
b_{t+1,\tau}^{(2)} P_{t+1,\tau} = b_{t,\tau}^{(3)} P_{t,\tau} \times \left( \frac{P_{t+1,\tau}}{P_{t,\tau}} + w(\tau - t, \theta) \frac{A_{t+1}^{(1)} - L_{t+1}^{(1)}}{L_t^{(3)}} \right)
\]

This expression says that the value of the retirement benefit at \( t + 1 \) is equal to the value of that benefit at \( t \), times the return on a zero-coupon bond (with similar horizon) plus a share of the excess return on the funds mismatch. This equation also makes it straightforward to see that participants in the collective fund will always earn a rate of return that is market consistent. The rate of return is simply the sum of a (market consistent) bond-return and some weight times the (market consistent) excess return on the funds mismatch risk.

**The smoothing weight**

For benefits that mature within the recovery period, we will set the smoothing weight as follows:

\[
w(\tau - t, \theta) = \frac{\tau - t}{\theta} \quad \text{for } \tau - t \leq \theta
\]
This choice implies that benefits paid next year \((\tau - t = 1)\) only bear a \(\frac{1}{\theta}\) share of the funding risk. As the horizon increases, the weight increases linearly, which is consistent with a gradual adjustment of benefits as time progresses. The weight on benefits outside the recovery period will be set to:

\[
w(\tau - t, \theta) = 1 + \theta \sum_{i=1}^{\theta} (1 - w(i, \theta))P_{t, t+i}b_{t, t+i} \sum_{i=\theta+1}^{\infty} P_{t, t+i}b_{t, t+i} \quad \text{for } \tau - t > \theta
\]

This choice implies that any surpluses or deficits that remain after benefits within the recovery period are adjusted, are absorbed by benefits that are due after the recovery period. This assures that \(A(3)^t = L^3_t\) at the end of each period. The fraction on the right-hand side is strictly positive. This highlights the fact that the long-run benefits are leveraged in their exposure to the funds mismatch risk. How strong this leverage is, will depend on the share of benefits with short horizons relative to long-run benefits in the scheme. Technically the value of long-term benefits could turn negative, due to this leverage. Given our choice of parameters, the probability this happens will be approximately zero though. In our simulations later on, this will not happen in any of the scenarios.

**Pension scheme demographics**

For simplicity, we will assume that there will be a constant inflow of new generations in the fund. Since individuals die deterministically at the end of the retirement phase, there will be an equal number of participants of each age group. Hence, in our calculations the fund will at any point in time have 65 active representative participants. Albeit convenient from a modeling perspective, we should keep in mind of course that these assumptions do create an overly optimistic view of the collective plan. In practice the collective scheme will have to worry about uncertainty in the funds population. In particular, there may be behavioral effects that make the inflow of new participants endogenous. IRS implies that the collective pension scheme imposes implicit taxes and subsidies on its participants. This could induce participants to avoid participation in the collective scheme. When interpreting our results later on, we have to bear in mind that such effects are ignored.

**Collective portfolio choice**

A last aspect of the collective scheme we need to address is how it sets its asset allocation. With multiple generations in one fund and future generations participating in current risks, this, in practice, is a complex governance challenge. We have made sure that individuals always receive a market consistent return and hence the asset mix will not have redistributional effects in terms of market value. In terms of expected utility there will still be conflicts of interest, however. This will be simply the case due to differences in age and wealth. In practice this would be even more the case, if preferences and investment believes of individuals are heterogeneous. For simplicity we will ignore this kind of heterogeneity, and assume all individuals are equal except for their age and level of their accumulated benefits.

In theory, we could choose a set of welfare weights for all individuals and we let a social planner determine an optimal solution to the collective portfolio choice problem. This however would be a problem in over 65 state variables with the funds adjustment mechanism as a complicated restriction on the allocation of risk. Solving this would not only be unfeasible, it would probably not be a realistic description of actual behavior of pension fund boards. It is more likely that boards of
trustees mainly focus on projections of funding deficits and surpluses. We will assume that the board of trustees of the collective scheme has a utility function defined over next periods funding ratio. Specifically, we will assume the board solves:

$$\max_{x_t} \mathbb{E}_t \frac{1}{1 - \gamma} \left( \frac{A^{(1)}_{t+1} + H_{t+1}}{L^{(1)}_{t+1} + H_{t+1}} \right)^{1-\gamma} \text{ s.t.}$$

$$A^{(1)}_{t+1} = A^{(3)}_t x'_{t+1} \bar{R}_{t+1}$$

$$x'_{t+1} 1_{n \times 1} = 1$$

$$x_t \geq 0_{n \times 1}$$

where $H_{t+1}$ the present value of future contributions and outside retirement income of all fund participants that share in the risk realized at $t+1$. By including this term, we make sure that the board recognizes the fact the wealth in the collective fund is only a part of participants total retirement wealth. By including the future contributions of future participants that share in todays risk, the fund will, consistent with Gollier (2008), tend to invest in a more risky asset mix than the individuals would by themselves.

One may worry that, by assuming that the board of trustees solves a one-period problem, the fund becomes myopic. This is not the case though. Since the objective function includes the funds liabilities, the trustees will care about time-variation in interest rates.

We will assume that the collective fund can choose from the same asset menu as we introduced in the individual problem: cash, stocks and the nominal long-term bond. Notice that the above problem is still a problem in many state variables. The fund has liabilities for many different horizons. The relative share of each horizon within the total liability will determine the distribution of the present value of liabilities next period. To reduce the dimensionality of the problem, we will summarize the state of the liabilities using a single state variable: its modified duration. By doing so, we have a maximization problem in three state variables, namely: the real interest rate, the relative size of the pension scheme compared to $H_t$ and the duration of the liabilities.

**Initial conditions**

In the numerical calculations that follow, we will assume that the initial wealth of individuals is equal to the level of wealth she would have accumulated, if she had saved the fixed collective contribution rate in an individual account and returns would have been equal to their median. At initiation of the collective scheme, this wealth is treated as a one-time initial contribution. This assumption implies that in the initial situation, the fund has assets and liabilities worth $210y_w$, which is approximately four times aggregate annual income. Furthermore, we will assume that the state variables start at their unconditional means.

**Risk exposure over the life-cycle**

Before we turn to the welfare analysis, let us first examine the portfolio allocation in the IRA setup and the (implied) exposure to financial market risk in the collective setup. For this purpose, it will be useful to define different measures of wealth we will be referring to. First of all there is 'financial wealth'. This refers to the value of the assets an individual accumulated so far in her IRA or the
value of her accrued benefits in the collective scheme. Secondly, we will refer to 'human capital'. This will be the present value of all future income. The sum of human capital and financial wealth will be referred to as 'total wealth'. We may also refer to 'retirement human capital'. In the context of the collective scheme or an IRA with a fixed contribution similar to the collective scheme, this refers to the present value of future pension contributions plus the present value of the future income from the government pension. In other words, this is the value of all future income that will be used for consumption during retirement. Together, financial wealth plus retirement human capital, will be called 'total retirement wealth'.

**Optimal exposure in the IRA**

A property of the Brennan and Xia (2002) financial market is that the optimal asset allocation does not depend on the expected level of inflation. Expected inflation is already priced into all asset prices and does not affect the risk-premia and expected real returns. The optimal portfolio allocation does depend on the real interest rate, the level of financial wealth and the investment horizon.

Figure 1 shows the optimal portfolio allocation in the IRA over the life-cycle. We assume that working life starts at the age of 20 and hence the retirement phase starts at the age of 65. The graphs is drawn for an interest rate equal to its unconditional mean and wealth levels from a median scenario (cf. the initial wealth distribution described before).

If we first look at the period in which the individual is retired, we notice that the equity exposure is constant. This matches the well known result from Merton (1969) and Samuelson (1969). This may be somewhat surprising, since the Merton/Samuelson result applies to a setup without outside income. Here, the individual does receive an outside income during retirement in the form of the government pension. In a median scenario however, financial wealth is an approximately constant part of the individuals total wealth during retirement.

The constant exposure does not apply to the fixed income part of the portfolio (long-term bond and cash). We see that the fraction invested in the long-term bond increases with the remaining investment horizon. As we would expect, with the investment horizon increasing, the individual has an increasing hedging demand for the long-term bond. Notice that demand for the long-term bond does not go all the way to zero as the investment horizon goes to zero. This is due to the fact that the long-term bond has a positive instantaneous risk-premium and hence, the myopic demand for the bond is positive.

If we move further back in the life-cycle, we see that the exposures kink around the retirement date. From this point, moving backwards in the life-cycle, financial wealth is not a constant fraction of the individuals total wealth anymore. The younger the individual, the smaller financial wealth is, as a fraction of total wealth. As a result, we see that the individual compensates for this fact by increasing its exposure to the stock and long-term bond. Around the age of 60, we can see that the no-borrowing constraint starts to bind. As a result, below this age, the individual faces a trade-off. She will have to choose between allocating more wealth towards stocks or to bonds.

Considering different levels of interest would not significantly change the general shape of these graphs. Since risk-premia are constant, the real interest rate only affects the optimal portfolio allocation through its effect on the duration of future consumption. This influence tends to be modest.

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17 This is compatible with the findings of Bodie et al. (1992).

18 See Brennan and Xia (2002). Intuitively: the CRRA utility function implies that as interest rates rise, consumption growth rises and hence the duration of consumption increases. As a result, the hedging demand for the long-term bond tends to go up.
The level of wealth will have a significant impact, in particular on the point at which constraints are binding. Those interested to see the relation between wealth and the optimal portfolio allocation, are referred to Appendix B. There we provide some three dimensional graphs of the optimal asset allocation.

**IRA with fixed contribution rate**

Now let us consider what happens to the optimal portfolio allocation if the individual is restricted to save a fixed share of her income during her working life, as would be the case in the collective setup. A fixed contribution, implies that the individual can not control her level of consumption during the work phase. As a consequence, the individual will become effectively more risk averse because all risk in her investment portfolio will now have to be absorbed during retirement only. In addition, the duration of consumption financed by her financial wealth increases. The life-cycle portfolio choice under this regime is shown in figure 2. The main difference is that individuals now balance their portfolio much more towards the bond when young.

**Risk exposure in the collective scheme**

The asset allocation chosen by the collective fund depends on three state variables: The ratio of assets over aggregate retirement human capital, the duration of the funds liabilities and the real interest rate. Figure 3 and figure 4 show the allocation by the fund to the stock index and the long-term bond as a function of the funds wealth. These figures correspond to the situation in which the interest rate is at its unconditional mean and the distribution of wealth (and hence, duration of liabilities) is equal to the initial wealth distribution. In the initial situation, aggregate financial wealth divided by aggregate retirement human capital is approximately 0.5. The variation of wealth on the horizontal axis is created by multiplying the initial wealth of all participants by a
common factor. For comparison, the graphs also include the aggregate exposure in case wealth was held in IRAs instead (with the same fixed contribution rate as the collective scheme).

For sufficiently high levels of financial wealth, we see that the stock exposure in both the IRAs and in the collective scheme converges to the same level. At the same time, the exposure to long-term bonds is higher in the collective scheme. At high levels of financial wealth, constraints are no longer binding and human capital becomes an ever smaller component of total wealth. Hence, both the collective scheme and the IRA converge to the 'Merton/Samuelson level' of stock exposure. The higher bond exposure is due to the fact that the collective scheme also, to some extent, takes into account the hedging demands of future participants.

As financial wealth decreases, we see that the aggregate stock exposure in the collective scheme and the IRAs diverge. Individual no-borrowing constraints start to bind earlier than the no-borrowing constraint at the collective level. At some point the collective will also hit its borrowing constraint, however. In these graphs this happens, roughly speaking, when financial wealth falls below 30 percent of aggregate human capital. Once this happens, the fund is fully invested in stocks and its long-term bond and cash position is zero. As a consequence, we can see in figure [4] that the funds equity position starts to cannibalize its long-term bond portfolio.

**Internal risk allocation: Equity risk**

To get a sense how the collective fund allocates the overall risk to individuals, now let us see how the risk is distributed across generations in the collective fund.

First, we will look at equity risk. Figure [3] shows the implied exposure of individuals, by age, to equity risk. This exposure is expressed as the portfolio share the individual would have to choose
Figure 3: Aggregate stock exposure in CDC versus IRA

Figure 4: Aggregate LT bond exposure in CDC versus IRA
in an individual account in order to replicate the implied exposure in the collective scheme. Note that the vertical axis is now defined as the share of total retirement wealth invested in equity. The collective scheme allocates some risk to future participants. These individuals did not accumulate any financial wealth yet and hence the replicating portfolio share of financial wealth would be undefined.

In the figure we see that indeed, the collective scheme implies that future participants are exposed to equity risk. This exposure starts at the age of ten, exactly ten years before they start to contribute. This is a consequence of the funds ten year recovery period. Allowing these young individuals to be exposed beyond their borrowing constraint will create efficiency gains. However, the equity exposure seems to turn sub-optimally high for most working generations. For older participants the exposure gradually falls again as an increasing part of their benefits start to fall within the funds recovery period. For the very old, the exposure seems to turn sub-optimally low.

All this illustrates the weakness of the collective pension scheme: exposures can not be optimally controlled at the individual level. The overexposure for the middle aged participants, for example, could only be reduced by reducing the share of equity for the fund as a whole or by reducing the recovery period. The first would reduce the exposure for all participants, including those who would actually prefer a higher exposure, while the latter would reduce risk sharing with future participants. In addition such changes would also affect the allocation of interest rate risk. The collective setup acts as a restriction on optimal risk management at the individual level.

**Internal risk allocation: Real interest rate risk**

Next, let us look at real interest rate risk. In order to present this in a meaningful way, we show the difference between the duration of a (deferred) retirement annuity that pays a constant real cashflow during retirement to the individual and the implied duration of the individuals total retirement
wealth. If this measure is zero, duration of retirement wealth is equal to the duration of a flat real annuity. If this measure is one, an increase in the real interest rate by one percentage point, decreases the price of the annuity by one percent more than it decreases the value of total retirement wealth. Hence, roughly speaking, this implies a gain in retirement consumption of one percent.

In the graph we see that up to the age of 10, the IRA and the collective scheme are equivalent, since in neither systems the individual is exposed to financial market risk and only has its own human capital. Since the duration of human capital is lower than the duration of a deferred retirement annuity, an increase in the real-interest rate would be beneficial to the individual in either system.

Once the individual starts to share in the collective schemes risk, it starts to receive a partial hedge. As a result the duration mismatch in the collective scheme starts to fall. In the IRA, the individual can only start to hedge herself at the age of 20, but due to the no borrowing constraint, the real interest exposure does not start to decrease until a couple of years later. The individual with an IRA keeps decreasing her real interest rate exposure, until it even turns negative. The reason for this pattern is that overall demand for the long-term bond consists of a hedging demand and a myopic demand. The hedging demand disappears as the investment horizon goes down, while the myopic demand (except for the constraints) is constant. So, while the duration of consumption falls to zero, the duration of the long term bond position remains positive.

In the collective scheme, the interest rate exposure remains approximately flat, as long as the individuals retirement date lies outside of the smoothing window of the collective scheme. Once the retirement date is within the the smoothing window, the individuals exposure to interest rate risk starts to fall more or less linearly to zero.

Thinking about the welfare implications, the high interest exposure in the IRA at young levels of age is most likely welfare decreasing compared to the collective scheme, since the high exposure is a consequence of the individuals no-borrowing constraint. At higher levels of age, it is the other way around. The individual is no longer borrowing constrained and optimally chooses a lower exposure than would be implied by participation in the collective scheme.

**Internal risk allocation: Expected inflation risk**

The final source of financial market risk in the model is expected inflation risk. Notice that both the stock index and human capital (future income) are not exposed to shocks in expected inflation. All (expected) inflation risk therefore comes from the (nominal) retirement benefits in the collective scheme or nominal bond holdings in the IRAs.

Like we did in the case of real interest rate risk, we will think of the exposure to expected inflation risk again in terms of the difference in inflation duration between a real (deferred) annuity and the inflation duration of total retirement wealth. Since the inflation duration of a real annuity is zero, figure 7 simply shows the inflation duration of total retirement wealth. If the duration of total retirement wealth in this graph is one, for example, an increase in the expected rate of inflation by one percentage point reduces retirement consumption by approximately one percent in real terms.

Figure 7 shows that the inflation exposure follows a similar pattern in the IRA and the collective scheme. Initially, the individual is not exposed to expected inflation risk, since her wealth consists of human capital only. Once the individual is exposed to financial market risk, the exposure starts to rise. As more and more human capital is converted into long term bonds or benefits in the collective

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19The reason the duration gap is initially increasing, is that duration, as a function of horizon, asymptotically approaches a maximum. As a consequence, the duration of the deferred annuity, which only makes cash-flows in the very long run is initially near the maximum. It does not fall much initially, as age increases.
scheme, this exposure keeps rising. At some point this exposure starts to fall again. In the IRA this is a consequence of the fact that the portfolio share of long-term bonds, as a percentage of total retirement wealth, starts to fall. In the collective scheme this happens because the (inflation) duration of an individual’s benefits keep falling as the individual turns older.

As before, from these graphs it is hard to tell what the welfare implications will be of the differences in exposure we see. We will therefore now turn to the welfare analysis.

**Welfare effects**

In the previous section we saw that the exposure in the collective pension scheme differs in several ways from the exposure individuals would prefer in an IRA setting. These differences were both desirable in some cases and undesirable. In particular during the later stages of the life-cycle the IRA will most likely perform better than the collective scheme. Since restrictions are no longer binding at this point, exposures can be chosen optimally at the individual level. In earlier parts of the life-cycle, the collective scheme may perform better, since it allows for exposures that go beyond the constraints individuals face.

We will now compare the different institutional setups determining expected life-time utility through the simulation of 50,000 economic scenarios. We start each scenario at the unconditional mean for the real interest rate and expected inflation. We will calculate the expected life-time utility both for the IRA and the CDC scheme. To represent the difference in expected utility in a meaningful way, we present in terms of an equivalent variation measure. We will denote this measure by \( \delta \) and define it implicitly as follows:
where $c_t$ is consumption in the benchmark setup and $\tilde{c}_t$ is consumption in the alternative scheme. This definition implies that $\delta$ is the percentage of lifetime consumption that the individual is willing to give up in the alternative scheme in order to be indifferent between the benchmark and the alternative.

**Welfare effects by generation**

Figure (8) shows $\delta$ (multiplied by 100) for individuals at different levels of initial age. The benchmark scheme here is the IRA with a fixed contribution level. The graph shows two alternative schemes: the collective scheme and an IRA with flexible contribution rates.

Let us start by considering the retired individuals first. Since there are no contributions anymore during retirement both IRAs are exactly the same here. The collective scheme however shows a welfare loss. In the IRA, constraints are not binding anymore and hence the collective scheme can only do worse. The welfare loss increases with the remaining investment horizon.

When we look at individuals at a working age, we see that the IRA with flexible contribution clearly outperforms the IRA with fixed contribution. The longer the remaining horizon, the more valuable it is to have an unconstrained saving rate, even in relative terms. For the collective scheme we see that the relative welfare loss peaks around the age of 55. At this point, an individual would be willing to give up more than 2.5 percent of lifetime consumption in order to get an IRA instead of the collective scheme. Below this point the relative welfare loss starts to fall (although the absolute loss peaks around the age of 50). The younger the individual, the more likely it is that the no-borrowing constraint binds in the IRA setup. This is where the collective scheme may
add value. Only around the age of 15 this benefit of the collective scheme completely compensates for its downside later in life. Hence, all current participants of the collective scheme would rather leave and get their own IRA. If we look at generations with an initial age below 10, we see that the relative welfare loss remains more or less constant. As the individual gets younger, uncertainty about the initial conditions increase, but this has limited impact on the relative welfare from the different institutional setups.

**Aggregate welfare effect**

In addition to the equivalent variations per generation, we can also determine an aggregate welfare effect. By multiplying $\delta$ for each individual with her total wealth, we obtain an equivalent variation in present real terms. We can sum this value for all current and future generations to get an aggregate welfare measure. Table (3) shows this aggregate equivalent variation, expressed as a percentage of aggregate total wealth of all current and future generations (which equals $3851y_w$).

In addition to the two IRAs and the collective scheme we saw before, we added a collective scheme in which no risk is allocated through net contributions. This is sometimes referred to as a 'closed' CDC scheme. In a closed CDC scheme, no risk is allocated to future participants and hence there is no intergenerational risk-sharing. This allows us to roughly decompose the overall welfare effect of the CDC scheme into three causes: the contribution restriction, the portfolio restriction, and the effect from risk-taking on behalf of future generations. The impact of the contribution restriction is measured by the difference between the IRA with and without fixed contribution. The cost of the portfolio restriction is measured by the difference between the IRA with fixed contribution and the closed CDC scheme. The benefit of intergenerational risk-sharing is measured by the difference between the open and closed CDC scheme.

The table shows that abolishing inter-generational risk sharing is equivalent to a 0.2 percent
loss in total wealth of all current and future generations. The benefit of the IRA that exposures can be set at the individual level more than compensates for this loss however. The IRA with a fixed contribution rate outperforms the collective scheme with IRS by 0.6 percent of life-time consumption for all current and future generations. To put this in perspective: The generations currently alive would be willing to pay $19.5y_w$ to be allowed to have an IRA with a fixed contribution rate, instead of the collective scheme. This is approximately 9 percent of the collective funds assets. If in addition, we would allow the contribution level to be set at the individual level, an overall welfare gain equivalent to a 3.0% increase in total wealth for all current and future generations results. This is equal to $115.5y_w$ in real terms and more than 50 percent of the funds assets.

### Sensitivity Analysis

The results in the previous section depend on a long list of assumptions. In order to see how robust our results are, we will look into the sensitivity of the results here with respect to several variations in the input parameters. Table 4 provides a list of all the alternative specifications we will consider. Table 5 provides the aggregate welfare effects under these alternative specifications.

<table>
<thead>
<tr>
<th>Scheme</th>
<th>Aggregate EV (% of life-time consumption)</th>
</tr>
</thead>
<tbody>
<tr>
<td>CDC</td>
<td>-0.6%</td>
</tr>
<tr>
<td>closed CDC (no IGR)</td>
<td>-0.8%</td>
</tr>
<tr>
<td>IRA fixed contribution</td>
<td>-</td>
</tr>
<tr>
<td>IRA flexible contribution</td>
<td>3.0%</td>
</tr>
</tbody>
</table>

Table 3: Aggregate welfare gain relative to IRA with fixed contribution

<table>
<thead>
<tr>
<th>Sensitivity Parameters</th>
<th>(1)</th>
<th>(2)</th>
<th>(3)</th>
<th>(4)</th>
<th>(5)</th>
<th>(6)</th>
<th>(7)</th>
<th>(8)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1 Baseline with $\lambda_S = 0.3$</td>
<td>-0.2%</td>
<td>-0.2%</td>
<td>-0.1%</td>
<td>-0.7%</td>
<td>-0.4%</td>
<td>-0.9%</td>
<td>-0.2%</td>
<td>-0.3%</td>
</tr>
<tr>
<td>2 Baseline with $\lambda_r = -0.2$</td>
<td>-0.5%</td>
<td>-0.5%</td>
<td>-0.5%</td>
<td>-1.0%</td>
<td>-0.6%</td>
<td>-0.9%</td>
<td>-0.7%</td>
<td>-0.9%</td>
</tr>
<tr>
<td>3 Baseline with $\lambda_r = 0.1$</td>
<td>-0.2%</td>
<td>-0.2%</td>
<td>-0.2%</td>
<td>-0.7%</td>
<td>-0.4%</td>
<td>-0.9%</td>
<td>-0.2%</td>
<td>-0.3%</td>
</tr>
<tr>
<td>4 Baseline with $\rho_{S,r} = -0.5$</td>
<td>-0.5%</td>
<td>-0.5%</td>
<td>-0.5%</td>
<td>-1.0%</td>
<td>-0.6%</td>
<td>-0.9%</td>
<td>-0.7%</td>
<td>-0.9%</td>
</tr>
<tr>
<td>5 Baseline with halftime $\tau_t$ 10 years</td>
<td>-0.2%</td>
<td>-0.2%</td>
<td>-0.2%</td>
<td>-0.7%</td>
<td>-0.4%</td>
<td>-0.9%</td>
<td>-0.2%</td>
<td>-0.3%</td>
</tr>
<tr>
<td>6 Baseline with halftime $\pi_t$ 10 years</td>
<td>-0.2%</td>
<td>-0.2%</td>
<td>-0.2%</td>
<td>-0.7%</td>
<td>-0.4%</td>
<td>-0.9%</td>
<td>-0.2%</td>
<td>-0.3%</td>
</tr>
<tr>
<td>7 Baseline with recovery period ($\theta$) 15 years</td>
<td>-0.2%</td>
<td>-0.2%</td>
<td>-0.2%</td>
<td>-0.7%</td>
<td>-0.4%</td>
<td>-0.9%</td>
<td>-0.2%</td>
<td>-0.3%</td>
</tr>
<tr>
<td>8 Baseline with recovery period ($\theta$) 20 years</td>
<td>-0.2%</td>
<td>-0.2%</td>
<td>-0.2%</td>
<td>-0.7%</td>
<td>-0.4%</td>
<td>-0.9%</td>
<td>-0.2%</td>
<td>-0.3%</td>
</tr>
</tbody>
</table>

Table 4: Sensitivity specification
First of all, we look at the prices of risk. For future generations being exposed to current market risk, is more valuable if risk-premia are higher. Therefore, one may worry that changing these parameters could significantly shift the relative welfare effects.

The first sensitivity shows the impact of a 50 percent increase in the equity risk-premium. As expected, the relative welfare from IRS increases. The open CDC scheme performs 0.3 percentage point better than the closed CDC scheme. The open CDC scheme still performs worse than the IRA with fixed contribution, though. Additionally, the higher equity risk-premium actually increases the value of a flexible contribution rate. Individuals can profit more from the higher equity premium if they can absorb the increased risk by adapting their contribution rate throughout life.

The second sensitivity changes the price of interest rate risk from -0.1 to -0.2. This change implies that the nominal long-term bond provides a significantly higher instantaneous risk-premium of approximately four percent. This does improve the relative performance of the collective scheme, but qualitatively the situation is not very different, with the IRA still dominating the collective scheme, even if contribution rates are fixed. We see the same picture in the third sensitivity, where the price of inflation risk is changed. Here too, the instantaneous risk premium of the long-term nominal bond goes up and again the collective scheme does improve somewhat in relative welfare terms.

The fourth sensitivity considers a negative correlation between stock returns and interest rate changes. This corresponds to the idea that falling interest rates are correlated with increasing asset prices in general. As we can see this slightly worsens the performance of the collective scheme. An intuition for this result could be that stocks now provide a hedge against adverse shocks in interest rates. As a result, the individual no-borrowing constraint is less often binding, because less long-term bonds are needed for a given level of interest rate hedging.

The sixth sensitivity reduces the presence of real interest rate risk. We see that this tends to move all schemes somewhat closer together. Surprisingly, we see in the seventh sensitivity that increasing (expected) inflation risk moves the open and closed version so close together that the gains from IRS fall below 0.1 percentage points.

Sensitivity seven and eight show what happens if the recovery period is increased. On the one hand this increases the amount of risk that can be taken on behalf of future generations. At the same time it changes the distribution of risk between generations and provides an incentive to take more risk to the board of trustees. We see that the gains from IRS do indeed increase somewhat, but still the IRA outperforms the collective schemes.

Discussion

Earlier papers have reported significant gains from intergenerational risk-sharing in collective funded pension schemes. These papers argued that collective, funded pension schemes could be a substitute for the missing market that allows current and future generations to trade financial market risk. Real world economies do not only feature missing markets however, they also feature missing social planners. Collectivizing retirement wealth introduces a complex governance problem. Every period, political decisions have to be made how this vast amount of wealth is to be allocated. Ideally, this allocation process follows optimal rules, but these rules tend to be very sensitive to subjective beliefs about the investment opportunity set. It is therefore not realistic to assume that collective pension schemes with intergenerational risk-share will follow such optimal rules in practice.

In this paper we therefore determined the welfare gains from collective pension schemes with intergenerational risk-sharing that are more similar to actual schemes observed in practice. In
addition, we added some realism by allowing interest rates and expected inflation to vary over time. Our calculations show that, the large gains from IRS reported earlier almost completely disappear in our somewhat more realistic setup. The overall welfare difference between the collective fund with IRS and the simple IRA even turns negative, due to the fact that the collective scheme struggles to provide optimal risk exposures to individual participants. This welfare loss turns even more negative if we consider the fact that the collective schemes do not grant the individual any flexibility in contribution and/or withdrawal from the retirement plan.

These results seem even more gloomy for collective schemes with IRS, if we bring to mind that our modeling setup is still relatively favorable towards the collective scheme. The model still ignores heterogeneity in individual preferences, heterogeneity in wealth accumulation within generations, heterogeneity in investment beliefs, uncertainty in the future demography of the collective pension schemes, welfare losses due to implicit taxes and subsidies imposed by the collective pension schemes, welfare losses from increased policy uncertainty, and so on and so forth.

Of course there are many caveats to our analysis. The financial market we considered does not seem very realistic. One may want to consider alternative utility functions. One may even have fundamental philosophical objections against applying an expected utility model to unborn individuals. Furthermore, we ignored potential correlation between outside income and financial market risk. Such criticisms would all also apply to the earlier papers on IRS in funded pension schemes. The main point we are making, is that policy makers should not be taken away by the significant gains from IRS in these papers.

The good news is, that there actually is no necessity to collectivize the retirement wealth of current generations in order to implement IRS. The high welfare gains reported in earlier literature come from the fact that the collective pension funds implicitly borrow on behalf of future generations and use the proceeds to make risky investments. This borrowing is made possible by the fact that future generations mandatorily have to service these implicit debts. If one believes in the benefits of this type of intergenerational risk-sharing, this could very well be organized in a more explicit way, without collectivizing all retirement wealth. A separate entity could be created, which borrows on behalf of future generations and invests the borrowed money on behalf of future generations. This entity could be organized into generational accounts to depoliticize the intergenerational allocation of accumulated assets and/or debt. Just like participation in collective risk-sharing schemes is mandatory, the government would force each generation to repay any potential deficits in their generational account\(^{20}\).

Intergenerational risk-sharing does not seem to be a valid excuse for policymakers to collectivize retirement wealth of current generations. The earlier literature on intergenerational risk-sharing in collective funded pension schemes should not deter policymakers from improving transparency and risk-management over the life-cycle in funded pension systems by introducing more clearly defined individual property rights over retirement wealth.

\(^{20}\)This was suggested earlier by Teulings and De Vries (2006)
References


Prime Minister’s Office (2014). The queen’s speech.


Appendix A Numerical solution method

The individual problem (eq. 2) is solved through numerical dynamic program. In order to explain how this is done, let us first define the value function, $V_t(w_t, r_t)$. In the final period, $T$, it is optimal to consume all remaining wealth, hence the value functions is:

$$V_T(w_T, r_T) = \frac{w_T^{1-\gamma}}{1-\gamma}$$

For all earlier periods, the value function can be represented by the following Bellman equation:

$$V_t(w_t, r_t) = \max_{c_t, x_t} \left\{ c_t^{1-\gamma} + \beta E_t[V_{t+1}(w_{t+1}, r_{t+1})] \right\}$$

subject to the constraints and transition equation for (real) financial wealth.

Following Carroll (2006) we solve the optimization for consumption and portfolio choice sequentially in each period. First we find the optimal portfolio allocation for given levels of wealth after consumption, next we solve for the optimal level of consumption.

Optimizing the portfolio allocation

Let us denote end-of-period real financial wealth as $a_t (= w_t - c_t)$ and define the end-of-period value function, $\tilde{V}_t$, as:

$$\tilde{V}_t(a_t, r_t) = \max_{\tilde{x}_t} \beta E_t[V_{t+1}(w_{t+1}, r_{t+1})]$$

subject to

$$w_{t+1} = a_t \tilde{x}_t' \tilde{r}_{t+1} + y_{t+1}$$

$$\tilde{x}_t' \tilde{1}_{n \times 1} = 1$$

$$\tilde{x}_t \geq 0_{n \times 1}$$

In each period, we determine the optimal values of $x_t$ for a grid in the ratio of $a_t$ over human capital ($HC_t$) and $r_t$. We perform this optimization using standard numerical constrained optimization in Matlab. The expectation in the objective function requires us to integrate over the distribution of next periods shocks. We do so by means of Gauss-Hermite quadrature.

Numerical optimization combined with numerical integration is computationally slow. Therefore, we keep the number of grid points limited at this point. We choose to have 50 grid points in each dimension. Whenever we need the optimal asset allocation for intermediate values of $a_t$ and $r_t$ these will be determined by linear interpolation.

The range of values for $r_t$ is chosen such that we will never need to extrapolate during any of our calculations. So, even if we are on the boundary of the grid at time $t$, mean reversion at the boundary is strong enough to make sure we remain inside the range of the grid at time $t + 1$. The minimum grid value of $a_t$ over human capital will be set to zero for each period. This implies that we will never need to extrapolate on the lower boundary of the wealth grid. The maximum grid value in for each period will be chosen such that we do not have to extrapolate on the upper end of
the grid either. The interior grid points will be set using a triple exponential function to make sure the grid points are concentrated in the lower wealth region.

Some numerical tests show that significantly increasing the number of grid points does not have a measurable impact. This seems to make sense, since small deviations from the optimal asset allocation have only second order effects on the value function.

Optimizing consumption

Once we have established the optimal portfolio allocation for a given period, we solve the optimal consumption for a grid of \( w_t \) and \( r_t \). Given the optimal portfolio allocation, the optimal consumption decision can be determined analytically. Therefore, we can solve optimal consumption on a much finer grid, without increasing the computational costs by too much. Following Carroll (2006) we solve optimal consumption for an endogenous start-of-period wealth grid.

The consumption problem, given optimal portfolio allocation, is:

\[
\max_{c_t} \frac{c_t^{1-\gamma}}{1-\gamma} + \beta E_t[V_{t+1}(w_{t+1}, r_{t+1})] \quad \text{s.t.} \quad w_{t+1} = (w_t - c_t)x_t^*(w_t - c_t, r_t)'R_{t+1} + y_{t+1}
\]

where \( x_t^*(a_t, r_t) \) is the (potentially interpolated) optimal portfolio allocation. The first-order condition to this problem is:

\[
c_t^{\gamma} = \beta E_t[(c_{t+1}^*(w_{t+1}, r_{t+1}))^{-\gamma} x_t^*(a_t, r_t)'R_{t+1}] - \gamma x_t^*(a_t, r_t)'R_{t+1}
\]

Substituting the wealth transition equation in, we get:

\[
c_t^{\gamma} = \beta E_t[(c_{t+1}^*(w_{t+1}, r_{t+1}))^{-\gamma} x_t^*(a_t, r_t)'R_{t+1} + y_{t+1}] - \gamma x_t^*(a_t, r_t)'R_{t+1}
\]

Since we have already determined next periods optimal consumption \( c_{t+1}^*(..) \) and this periods optimal portfolio allocation \( x_t^*(..) \), we can determine \( c_t \) analytically for any value of \( a_t \) and \( r_t \). We do so for a grid of \( a_t \) and \( r_t \) with 500 points in each dimension. We find find the expectation, again, by numerical integration using Gauss-Hermite quadrature.

Once \( c_t \) is determined, the endogenous beginning-of-period wealth grid is found by using the fact that \( w_t = c_t + a_t \). The lower bound of the endogenous grid will correspond to wealth level at which liquidity constraint \( (c_t < w_t) \) stops to bind. Below this level, the individual will simply consume all start-of-period financial wealth available.

Value function interpolation

Once we have established the optimal consumption policy, we determine the value function for all points on the endogenous, beginning-of-period grid. To improve the interpolation of the value function, we transform the function. Instead of interpolating the value itself, we record the certainty equivalent level of life-time consumption as a function of total wealth:

\[
CE_t(TW_t, r_t) = \left( \frac{V_t(w_t, r_t)(1-\gamma)}{\sum_{i=0}^{T-t} \beta^i} \right)^\frac{1}{1-\gamma}
\]
where $TW_t$ is the sum of real (financial) wealth and the present value of the individuals future income. Whenever we need a value of the value function, we linearly interpolate between the values of $CE_t$ on the grid and transform it back by reversing the above transformation. We do so, because we know that, in the absence of constraints, and if human capital was tradable, $CE_t$ would be a linear function of $TW_t$ (the value function would be homogeneous of degree $(1 - \gamma)$ in total wealth).

Solving the individual problem with fixed contribution

We also solve the individual problem for the case in which the contribution to the IRA during working life is exogonously fixed. During retirement we will have exactly the same problem as before. Before retirement the problem changes a bit. Since consumption is fixed, we only need to optimize the portfolio allocation during working life. The appropriate value function during the work phase becomes:

$$V_t(w_t, r_t) = \max_{x_t} \beta E_t [V_{t+1}(w_{t+1}, r_{t+1})] \quad s.t. \quad \begin{align*}
w_{t+1} &= w_t x_t' R_{t+1} + s y_{t+1} \\
x_t' 1_{n \times 1} &= 1 \\
x_t &\geq 0_{n \times 1}
\end{align*}$$

We solve for the optimal portfolio allocation in this case as we did before.

Error propagation

The numerical techniques used, numerical optimization and numerical integration tend to be very accurate if judged by a single iteration. The life-cycle problem requires us to perform each step over
and over again. This means that small numerical errors will propagate with each additional period in the models horizon. Over long horizons the errors may become substantial. Since an analytical solution is not available, we can not be completely sure how big this problem is. To get a sense if error propagation is significant, we compare two measures: The certainty equivalent consumption obtained in the optimization process through sequential Gauss-Hermite integration and the CE obtained in the Monte-Carlo simulation using 50,000 economic scenarios (as used to create figure 8). In figure 9 the relative difference between these two CE measures is given. It shows that the two measures do diverge somewhat, but the difference is at most 1 percent.
Appendix B Optimal exposures in the individual problem

This appendix shows the optimal asset allocation in the IRA. The optimal asset allocation depends on three variables: age (investment horizon), the ratio of wealth to human capital and the real interest rate. As explained earlier, the real interest rate does not have a very significant impact. It only affects the optimal portfolio choice by affecting the duration of future consumption. The graphs below therefore show the optimal asset allocation as a function of wealth and age only for the interest rate being at its unconditional mean. Wealth is measured as the ratio of total retirement wealth divided by the value of human capital. In the case of a fixed contribution rate, human capital is equal to the present value of all future contributions and the present value of outside retirement income (the state pension). Total retirement wealth is the sum of financial wealth in the individual retirement account and human capital. Hence, if the account is empty, the ratio of total retirement wealth over human capital is one. Due to the no-borrowing and no-short sales constraints, all exposures are zero when total retirement wealth over human capital is one. The portfolio allocation is represented as a percentage of total retirement wealth (including human capital).

IRA with fixed contribution rate

Figure 10: Stock exposure in IRA with fixed contribution
Figure 11: LT Bond exposure in IRA with fixed contribution

Figure 12: Cash exposure in IRA with fixed contribution
IRA with flexible contribution rate

Figure 13: Stock exposure in IRA with flexible contribution

Figure 14: LT Bond exposure in IRA with flexible contribution
Figure 15: Cash exposure in IRA with flexible contribution