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# THE EIGENFUNCTIONS OF COMPACT WEIGHTED ENDOMORPHISMS OF $C(X)$

HARALD UHLIG

ABSTRACT. In this note we characterize the eigenmanifolds of compact operators $uC_\Phi\colon f \to u \cdot f \circ \Phi$ on $C(X)$ and determine their ascents. As an application we show an easy method for computing the eigenmanifolds of a matrix with at most one nonzero element in each row.

In the sequel $X$ will always denote a compact Hausdorff space, $u$ a function in $C(X)$, and $\Phi$ a continuous function from $X$ to $X$. Let $\Phi_n$ be the $n$th iterate of $\Phi$; i.e., $\Phi_0(x) = x$ and $\Phi_n(x) = \Phi(\Phi_{n-1}(x))$ for $n > 0$ and $x \in X$. $c \in X$ is called a fixed point of $\Phi$ of order $n$ if $n$ is a positive integer, $\Phi_n(c) = c$, and $\Phi_k(c) \neq c$ for $k = 1, \ldots, n-1$.

By $uC_\Phi$ we denote the operator $uC_\Phi\colon f \to u \cdot f \circ \Phi$ on $C(X)$. This is a weighted endomorphism, and every weighted endomorphism may be represented in this way (see Kamowitz [**1**]). Kamowitz [**1**] proved the following result:

THEOREM A. *Suppose $X$ is a compact Hausdorff space, $u$ in $C(X)$, and $\Phi$ a continuous function from $X$ into $X$.*

(1) *The map $uC_\Phi\colon f \to u \cdot f \circ \Phi$ is compact iff for each connected component $C$ of $\{x | u(x) \neq 0\}$ there exists an open set $V \supset C$ such that $\Phi$ is constant on $V$.*

(2) *If $uC_\Phi$ is compact, then $\sigma(uC_\Phi) \setminus \{0\} = \{\lambda | \lambda^n = u(c) \cdots u(\Phi_{n-1}(c))$ for some positive integer $n$ and some fixed point $c$ of $\Phi$ of order $n$, $\lambda \neq 0\}$.*

Our aim here is to characterize the eigenfunctions of a compact $uC_\Phi$. To do that we need some more notation: We always assume that $\Phi$ satisfies the conditions of Theorem A(1) so that $uC_\Phi$ is compact. We call $x, y \in X$ equivalent ($x \sim y$) if there exist $n, m$ in $\{0, 1, 2, \ldots\}$ so that $\Phi_n(x) = y$ and $\Phi_m(y) = x$. The equivalence classes are denoted by $[x]$. For any $\lambda$ in $\mathbf{C} \setminus \{0\}$ let $C_\lambda := \{c$ in $X | c$ is a fixed point of $\Phi$ of order $n$ for some positive integer $n$ and $\lambda^n = u(c) \cdots u(\Phi_{n-1}(c))\}$. Obviously if $x \sim y$ and $x$ in $C_\lambda$, then $y$ is in $C_\lambda$, so let $\tilde{C}_\lambda := \{[x] | x$ is in $C_\lambda\}$ and $m_\lambda$ be the number of equivalent classes in $\tilde{C}_\lambda$. $m_\lambda$ is finite by Theorem B and the compactness of $uC_\Phi$. For every $c \in C_\lambda$ let $h_{c,\lambda}$ denote the following function from $X$ to $\mathbf{C}$ or $\mathbf{R}$ respectively:

$$h_{c,\lambda}(x) := \begin{cases} \lambda^{-r} u(x) \cdots u(\Phi_r(x)) & \text{for every } r \text{ in } \{0,1,2,\ldots\} \text{ and } x \in \Phi_r^{-1}(\{c\}), \\ 0 & \text{otherwise.} \end{cases}$$

It is easy to see that $h_{c,\lambda}$ is well defined (remember that e.g. $c$ is in every $\Phi_{kn}^{-1}(\{c\})$ if $c$ is a fixed point of $\Phi$ of order $n$, but then $\lambda^{kn} = u(c) \cdots u(\Phi_{kn-1}(c))$). Furthermore

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$\{h_{c_1,\lambda},\dots,h_{c_k,\lambda}\}$ is linearly dependent iff, for some $i \neq j$, $c_i \sim c_j$. Finally, let $W_0 := W := \{x|u(x) \neq 0\}$ and $W_k := \Phi(W \cap W_{k-1})$ for $k > 0$. For additional notation see Taylor [**2**].

The principal result of this note is the following theorem.

THEOREM B. (1) *Let* $\lambda \in \sigma(uC_\Phi) \setminus \{0\}$ *and* $\{c_1,\dots,c_{m_\lambda}\}$ *be representative elements of all equivalence classes in* $\tilde{C}_\lambda$. *Then* $\{h_{c_1,\lambda},\dots,h_{c_{m_\lambda},\lambda}\}$ *is a basis for* $\mathcal{N}(\lambda - uC_\Phi)$ *and* $\alpha(\lambda - uC_\Phi) = 1$, *where* $\alpha(\lambda - uC_\Phi)$ *denotes the ascent of* $\lambda - uC_\Phi$.

(2) *The case* $\lambda = 0$: *If* $n > 0$, *then* $\mathcal{N}((uC_\Phi)^n) = \{f \in C(X)|f(x) = 0$ *for every* $x \in W_n\}$.

Notice that (1) also states that the functions $h_{c,\lambda}$ are continuous.

We will break up the proof by proving several propositions.

PROPOSITION 1. *Let* $\lambda \in \sigma(uC_\Phi) \setminus \{0\}$. *Then* $h_{c,\lambda}$ *is an eigenfunction for* $\lambda$ *for every* $c \in C_\lambda$; *that is,*

(i) $\lambda h_{c,\lambda}(x) = u(x)h_{c,\lambda}(\Phi(x))$ *for all* $x \in X$,

(ii) $h_{c,\lambda}$ *is continuous.*

PROOF. (i) Let $x \in X$. If $x \in \Phi_r^{-r}(\{c\})$ for some $r > 0$, then

$$\lambda h_c(x) = u(x)(\lambda^{-(r-1)}u(\Phi(x))\cdots u(\Phi_r(x))) = u(x)h_{c,\lambda}(\Phi(x)).$$

If $x \notin \Phi_r^{-1}(\{c\})$ for every $r \geq 0$, then the same is true for $\Phi(x)$, so $\lambda h_{c,\lambda}(x) = 0 = u(x)h_{c,\lambda}(\Phi(x))$.

(ii) (1) Since $u$ is continuous, $B = \{x|\ |u(x)| \geq |\lambda|\}$ is compact. As $W$ may be covered with open sets $V_\beta$, so that $\Phi$ is constant on each $V_\beta$, $\Phi(B)$ is finite, of cardinality $N$, say. Let $x \in X$ such that $h_{c,\lambda}(x) \neq 0$, and $r$ the minimal number so that $x \in \Phi_r^{-1}(\{c\})$. Now $x$, $\Phi(x),\dots,\Phi_r(x)$ are distinct, whence

$$\begin{aligned}|h_{c,\lambda}(x)| &= |u(x)/\lambda|\cdot|u(\Phi(x))/\lambda|\cdots|u(\Phi_{r-1}(x))/\lambda|\cdot|u(c)|\\ &\leq \max\{1,(\|u\|_\infty/|\lambda|)^N\}\cdot|u(c)| =: M.\end{aligned}$$

Therefore $h_{c,\lambda}$ is bounded on $X$.

(2) Let $x \in X$. If $u(x) = 0$, then $h_{c,\lambda}(x) = 0$ and for every $\varepsilon > 0$ there is a neighborhood $U$ of $x$ so that $|u(y)| < \varepsilon|\lambda|/M$ for every $y \in U$. Therefore

$$|h_{c,\lambda}(y)| = |\lambda|^{-1}|h_{c,\lambda}(\Phi(x))|\,|u(y)| < \varepsilon$$

for every $y \in U$ and thus $h_{c,\lambda}$ is continuous at $x$. If $u(x) \neq 0$, then $\Phi$ is constant on an open neighborhood $U$ of $x$ and therefore

$$|h_{c,\lambda}(x) - h_{c,\lambda}(y)| = |\lambda|^{-1}|h_{c,\lambda}(\Phi(x))|\,|u(x) - u(y)| < \varepsilon$$

for a suitable neighborhood $U' \subset U$ of $x$ and every $y \in U'$. So $h_{c,\lambda}$ is continuous.

PROPOSITION 2. *Let* $\lambda \in \sigma(uC_\Phi)$, $\lambda \neq 0$, *and* $f$ *an eigenfunction for* $\lambda$. *Then*

(i) *For every* $c \in C_\lambda$ *there exists* $\alpha(c)$ *such that* $f(x) = \alpha(c)h_{c,\lambda}(x)$ *for every* $r \geq 0$ *and* $x \in \Phi_r^{-1}(\{c\})$.

(ii) *If* $x \notin \Phi_r^{-1}(\{c\})$ *for every* $c \in C_\lambda$ *and* $r \geq 0$, *then* $f(x) = 0$.

PROOF. (i) Let $c \in C_\lambda$ and $\alpha(c) := f(c)/u(c)$ (remember $\lambda \neq 0$!). Then for $r \geq 0$ and $x \in \Phi_r^{-1}(\{c\})$ we have by iteration

$$f(x) = \lambda^{-r}u(x)u(\Phi(x))\cdots u(\Phi_{r-1}(x))f(\Phi_r(x)) = \alpha(c)h_{c,\lambda}(x).$$

(ii) This part of the proof is actually the same as for Proposition 4 in [**1**] and is repeated here for the sake of completeness:

Let $x \notin \Phi_r^{-1}(\{c\})$ for every $c \in C_\lambda$, $r \geq 0$. If $x$ is a fixed point of $\Phi$, of order $n$, say, then by iteration $f(x) = \lambda^{-n} u(x) \cdots u(\Phi_{n-1}(x)) f(x)$ and, since $x \notin C_\lambda$, we conclude that $f(x) = 0$.

If $x \in \Phi_r^{-1}(\{c\})$ for some fixed point $c \notin C_\lambda$ and $r \geq 1$, then, since $f(c) = 0$, we have $f(x) = \lambda^{-r} u(x) \cdots u(\Phi_{r-1}(x)) f(c) = 0$.

Finally, we may suppose that all $\Phi_r(x)$ are distinct. Let $\delta := |\lambda|/2$. Since $B := \{x | \, |u(x)| \geq \delta\}$ is compact and by Theorem A $W$ may be covered by open sets on which $\Phi$ is constant, $\Phi(B)$ is finite, of cardinality $N$, say. Therefore for every $n > N$

$$\begin{aligned} |f(x)| &= |u(x)/\lambda| \, |u(\Phi(x))/\lambda| \cdots |u(\Phi_{n-1}(x))/\lambda| \, |f(\Phi_n(x))| \\ &\leq (\|u\|_\infty / |\lambda|)^N 2^{N-n} \|f\|_\infty \to 0 \qquad (n \to \infty). \end{aligned}$$

Thus $f(x) = 0$. Q.E.D.

Let $\{c_1, \ldots, c_{m_\lambda}\}$ be representative elements of all equivalence classes in $\tilde{C}_\lambda$. Then $\{h_{c_1,\lambda}, \ldots, h_{c_{m_\lambda},\lambda}\}$ is a basis for $\mathcal{N}(\lambda - A)$ if $0 \neq \lambda \in \sigma(uC_\Phi)$. So what remains to be done for part (1) of Theorem B is

PROPOSITION 3. *Let $0 \neq \lambda \in \sigma(uC_\Phi)$ and $f \in \mathcal{N}((\lambda - uC_\Phi)^2)$. Then $f \in \mathcal{N}(\lambda - uC_\Phi)$.*

PROOF. Since $g := (\lambda - uC_\Phi) f$ is an eigenfunction for $\lambda$, we know by Proposition 2 that if $x$ is not in $\Phi_r^{-1}(\{c\})$ for some $c \in C_\lambda$ and $r \geq 0$, then $g(x) = 0$. If $c \in C_\lambda$ there exists $\alpha(c)$ so that $g(x) = \alpha(c) h_{c,\lambda}(x)$ for every $r \geq 0$ and $x \in \Phi_r^{-1}(\{c\})$ by Proposition 2, so we have to show that $\alpha(c) = 0$. Let $c$ be of order $n$. Since by iteration

$$f = n \cdot \frac{g}{\lambda} + (uC_\Phi)^n \frac{f}{\lambda^n},$$

evaluation at $c$ yields

$$f(c) = n\alpha(c) h_{c,\lambda}(c)/\lambda + f(c),$$

for $g$ is an eigenfunction and $\lambda^n = u(c) \cdots u(\Phi_{n_1}(c))$. Therefore $\alpha(c) = 0$.

So far we have proved Theorem B(1). Part (2) follows from

PROPOSITION 4. $(uC_\Phi)^k f = 0 \Leftrightarrow f(x) = 0$ *for every* $x \in W_k$.

PROOF. By induction:

($\Rightarrow$) Let $k = 1$ and $uC_\Phi f = 0$. Then for any $x \in W$ we have $0 = u(x) f(\Phi(x))$, whence $f(\Phi(x)) = 0$. If $k > 1$ and $(uC_\Phi)^k f = 0$, we know by induction that $u(x) f(\Phi(x)) = 0$ for every $x \in W_{k-1}$. Furthermore, if $x \in W$, then $u(x) \neq 0$, so that $f(\Phi(x)) = 0$. Thus $f$ vanishes on $W_k$.

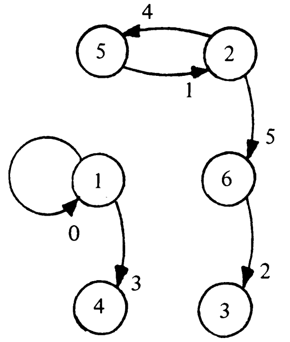
($\Leftarrow$) Let $k = 1$ and $f(x) = 0$ for every $x \in W_1$. For $x \in X$ either $x \in W$ and therefore $f(\Phi(x)) = 0$ or $x \notin W$ and $u(x) = 0$. Thus $uC_\Phi f = 0$. Now let $k > 1$ and $f(x) = 0$ for every $x \in W_k$. We have to show that $u(x) f(\Phi(x)) = 0$ for every $x \in W_{k-1}$, because then the assertion follows by induction hypothesis. But this is trivial since either $x \notin W$ and $u(x) = 0$, or $\Phi(x) \in W_k$ and $f(\Phi(x)) = 0$, if $x \in W_{k-1}$.

EXAMPLE 1. We want to give an example for Theorem B(2) that the case $\mathcal{N}((uC_\Phi)^n) \neq \mathcal{N}((uC_\Phi)^{n+1})$ for ever $n$ may occur. Let $X := \{0\} \cup \{1/n | n \in \mathbf{N}\}$ with the topology induced by the usual topology on $\mathbf{R}$ so that $X$ is compact. Let

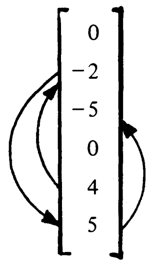
$u(x) = x$ and $\Phi(1/n) = 1/(n+1)$, $\Phi(0) = 0$. These are continuous functions satisfying the conditions of Theorem A. Therefore $uC_\Phi$ is a compact operator on $C(X)$, where $C(X)$ may obviously be identified with $c(\mathbf{N}) := \{(a_n)_{n\in\mathbf{N}} \mid \lim_{n\to\infty} a_n$ exists$\}$. Since there are no fixed points $c \neq 0$ of $\Phi$ of any order, $\sigma(uC_\Phi) = \{0\}$ by Theorem A. Now $W_k = \{x \in X \mid 0 < x < 1/k\}$, so $\mathcal{N}((uC_\Phi)^k) = \{(a_n) \mid a_n = 0$ for every $n > k\}$ and the union of all $\mathcal{N}((uC_\Phi)^k)$ is exactly the set of all $(a_n)$ satisfying $a_n = 0$ for all but finitely many $n$.

EXAMPLE 2. We give an application of our results to the finite-dimensional case. Let $X = \{1, \ldots, n\}$ with the discrete topology. Then $C(X)$ will be identified with $\mathbf{K}^n$, where $\mathbf{K} = \mathbf{C}$ or $\mathbf{K} = \mathbf{R}$ is the underlying scalar field. Every linear operator may (and will) be identified with the matrix $(a_{ij})_{1\leq i,j\leq n}$ with $a_{ij} = (A\delta_j)(i)$, where $\delta_j(j) = 1$, $\delta_j(i) = 0$ if $i \neq j$.

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 \\ 3 & 0 & 0 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 & 0 & 0 \\ 0 & 5 & 0 & 0 & 0 & 0 \end{bmatrix}$$




$\sigma(uC_\Phi) = \{-2, 0, 2\}$




$= h_{5,-2}$.

$$\mathcal{N}(A) = \{(x_n) \mid x_1 = x_2 = x_5 = x_6 = 0\}, \quad \mathcal{N}(A^2) = \{(x_n) \mid x_5 = x_2 = 0\} = \mathcal{N}(A^3).$$

If $A = uC_\Phi$, then $a_{ij} = u(i)$ if $j = \Phi(i)$ and $a_{ij} = 0$ otherwise, so there is at most one nonzero element in each row. Conversely let $A$ have this property. Then for $i = 1, \ldots, n$ let $j = \Phi(i)$ and $u(i) = a_{ij}$, if $a_{ij}$ is the unique nonzero element in row $i$. If $a_{ij} = 0$ for all $j = 1, \ldots, n$ we let $i = \Phi(i)$ and $u(i) = 0$. Then obviously $A = uC_\Phi$.

Now the eigenvalues and eigenvectors are easily determined: first find out all cycles of $\Phi$. e.g. by drawing $n$ dots with numbers $1, \ldots, n$ and an arrow from dot $j$ to dot $i$ if $\Phi(i) = j$, adding $u(i)$ to that arrow for later purposes. For each cycle multiply all the $u(i)$ of this cycle and calculate the $k$th roots, where $k$ denotes the number of elements of this cycle: these are all eigenvalues possibly except 0.

Take one eigenvalue $\lambda \neq 0$ and a cycle corresponding to that $\lambda$. Choose an arbitrary dot $j$, say, of that cycle and set $x_j := u(j)$. Now follow the arrows. If you reach dot $i$ from dot $k$ let $x_i$ be the product of $\lambda^{-1}u_i$ and $x_k$. When you are done with all the dots which belong to the "connected component" containing the cycle set all other $x_i = 0$. This is an eigenvector for $\lambda$.

If you do this for every cycle corresponding to $\lambda$ you get a basis for the eigenspace $\mathcal{N}(\lambda - A)$.

In order to determine $\mathcal{N}(A^r)$ remove all arrows where $u_i = 0$. Now $\mathcal{N}(A)$ consists of all $(x_k)$, where $x_k = 0$ if there is a directed path of length one starting in dot $k$ (to dot $k$ itself or any other dot), and $x_k$ is arbitrary otherwise. Similarly for $\mathcal{N}(A^r)$, $r > 1$: "one" has to be replaced by "$r$" and it is allowed to "use" the same arrow more than one time.

There is a diagonalization for $A$ iff $\mathcal{N}(A) = \mathcal{N}(A^2)$. Of course all these results are easily obtained by direct verification as well.

## REFERENCES

1. H. Kamowitz, *Compact weighted emdomorphisms of $C(X)$*, Proc. Amer. Math. Soc. **83** (1981).
2. A. E. Taylor, *Introduction to functional analysis*, Wiley, New York, 1958.

FACHBEREICH 3 (MATHEMATIK), SEKR. 6-4, TECHNISCHE UNIVERSITÄT BERLIN, STRASSE DES 17 JUNI 136, D-1000 BERLIN 12, WEST GERMANY

*Current address*: School of Economics, University of Minnesota, Minneapolis, Minnesota 55455