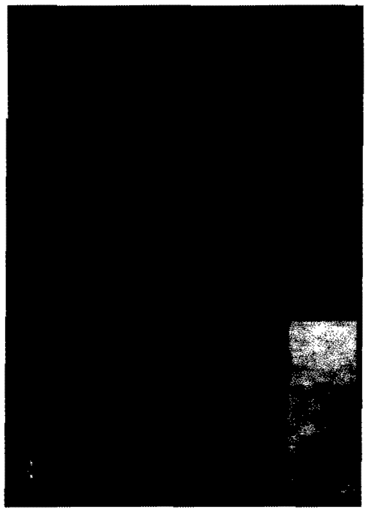

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A review of Kenneth L. Judd's *Numerical Methods in Economics*, Cambridge, MA: MIT Press 1998. Pp. xiii + 633. ISBN 0 262 10071 1.

1 INTRODUCTION

There are still economists, who believe that only results derived analytically with pencil and paper are of any value, and who look down upon results derived numerically. That is ridiculous. Of course, it is usually preferable to obtain the same result analytically rather than numerically, and to obtain a general theorem instead of numerical examples. But often, that is not possible or forbiddingly costly. Numerical tools allow the scientist to explore vast new areas of questions, issues and models, where analytical tools are of little help. Numerical methods are now routinely used by the most advanced researchers to obtain quantitative answers from sufficiently interesting and therefore often demanding theoretical models, and to make meaningful contact with the data: this is a sign of maturity and progress in the field.

But with this advent of the numerical age in economics comes the need to put the numerical tools themselves on a solid, scientific basis. It will no longer be enough to rely only on a list of sensible 'recipes': what is required instead is a systematic and useful theory and treatment of numerical methods in economics.

This is exactly what Kenneth L. Judd's book *Numerical Methods in Economics* provides. It is a landmark achievement, an instant classic and a watershed. I like it a lot, I learned a lot from it, and it now has a firm place among my most favorite books in economics such as Stokey-Lucas, with Prescott, Hamilton and the two (soon three) textbooks by Sargent. There is a lot to be liked about it: fairly short and reasonably self-contained chapters, brief descriptions of the key algorithms, contact with both economic applications on the one hand and numerical mathematics and functional analytic foundations on the other.

2 HOW TO READ THE BOOK

I actually read the book cover to cover, from the beginning to the end. This is one possible way of reading the book, and it is rewarding, building everything from the ground up. It is worthwhile, but also quite an investment. It felt a bit like reading Umberto Ecco's *The Name of the Rose*: there first is an initiation rite of several hundred pages on numerical mathematics before one gets to solve some interesting economic examples on, say page 353. So perhaps, for somebody who wants to learn some new tools quickly, who already knows about some solution method like log-linearization and now wants to see what Judd's book has to offer for solving some particular economic model at hand, it may be best to try to read the book *backwards* instead.

Start with sections 17.5 to 17.7, which solve the standard stochastic dynamic discrete time growth model, and sections 16.4 and 16.5, which solve it without uncertainty. In a future edition of the book, it would be nice if these sections were to offer a complete and shiny exhibition of all that the book has taught with appropriate references to earlier chapters, so that the reader can see what is available, and then go to the relevant details in which they are interested. These sections already go a long way towards that. Additionally, the reader should consult the index and look up 'growth model'. Follow all the references there: because Judd uses the growth model as his prime example throughout the book, one can get a very good idea of what the book has to offer in this way.

3 NEW TOOLS

And indeed, there is a lot the book has to offer. I like to think of the book as viewing the problem of solving an economic, dynamic model as a problem of solving a functional equation. That equation usually arises from calculating the first-order conditions and plugging in the evolution equation for the underlying state. The object of interest is the policy function or decision rule, mapping the state into the current decision. Take for example the deterministic discrete-time growth model

$$\max_{c_t, k_{t+1}} \sum_{t=0}^{\infty} \beta^t u(c_t)$$
$$\text{s.t.} \quad k_{t+1} = F(k_t) - c_t$$

where $F(\cdot)$ is the production function in capital k_t, $u(\cdot)$ is a utility function in consumption c_t and $0 < \beta < 1$ the discount factor. It yields the functional equation

$$0 = u'(C(k)) - \beta u'(C(F(k) - C(k)))F'(F(k) - C(k)) \qquad (1)$$

where the decision rule $C(\cdot)$, is the object of interest. Thus, the aim is to solve this equation (plus a stability condition) for a function $C(k)$, mapping the current capital stock into consumption, see page 476.

To solve this functional equation, the book offers three approaches: perturbation methods, projection methods and iterative methods. The last of these three encompasses such techniques as the den-Haan-Marcet method of parameterized expectations, value function iteration or time domain simulation. Judd describes them in section 12.8, section 16.4, starting on page 551, and section 17.7. Also, for deterministic continuous time models, equation (1) becomes a differential equation, which can be solved with standard tools, see chapter 10. To assure stability of the solution, reverse shooting is the method of choice. Section 10.7 is an extremely useful description, and is a must-read.

However, for this review, I will concentrate on the first two methods; the perturbation methods and the projection methods. I shall describe them in some detail, since they most clearly demonstrate the book's research agenda and philosophy. I imagine that you already know how to solve, say, the standard stochastic neoclassical growth model or some other stochastic dynamic model with a single endogenous state variable, using linearization techniques. These techniques are described in many places: obviously, my favorite one is Uhlig (1999). I shall briefly recall this benchmark solution method first, which provides me with a good starting point to describe what Judd's book has to offer.

4 A BENCHMARK LINEARIZATION METHOD

The benchmark linearization method of undetermined coefficients can be stated here as follows.[1]

Algorithm 1 Goal: calculate an approximate linear law of motion.

1. Calculate the non-stochastic steady state. E.g. for (1) set $k = k^*$, $C(k^*) = c^*$, observe $k^* = F(k^*) - c^*$, and solve: $\beta F'(k^*) = 1$, etc.
2. Linearize all equations around the steady state. 'Guess' that the solution can be written as a recursive linear law of motion, in which the variables are linear functions of the state variable(s) with yet undetermined coefficients. Repeatedly substitute in the recursive linear law of motion, until the equations only involve the state variables of one date only. One typically gets a quadratic equation. In the example here, the repeated substitution is already done. Thus, differentiate (1) with respect to k, and set $k = k^*$. This gives a quadratic equation in the 'undetermined coefficient' $\eta_{ck} = C'(k^*)$
$$0 = u''\eta_{ck} - \beta u''\eta_{ck}[F' - \eta_{ck}] - \beta u' F''[F' - \eta_{ck}]$$
which has two solutions.
3. To pick the right solution of the quadratic equation, impose stability. In the example here, $c_t - c^* = \eta_{ck}(k_t - k^*)$ and linearization of $k_{t+1} = F(k_t) - c_t$ implies $k_{t+1} - k^* = \eta_{kk}(k_t - k^*)$ with $\eta_{kk} = F' - \eta_{ck} = 1/\beta - \eta_{ck}$. Pick the solution η_{ck}, which yields $|\eta_{kk}| < 1$, i.e. pick the positive among the two solutions for η_{ck}.

The key step here is the selection of the stable root; it is there, where unstable and economically non-sensical solutions to the linear system are ruled out. General solutions to the linearized system involve both the stable and the unstable root: restricting the part stemming from the unstable root to be identical to zero is restricting the set of solutions to a lower-dimensional subspace. A (numerical) mathematician is unlikely to impose that restriction, when just blindly attacking the linear system directly: he needs to be told that there is an additional restriction. The same holds true, when attempting to directly solve the original non-linear system numerically: a sensible method has to make sure that the solution is on the stable arm, but it will not, unless one is careful about it. Put differently, the first-order necessary conditions are necessary, but not sufficient: one also needs to satisfy the transversality condition.

Judd is always careful about all this in his book, of course, and indeed often even states the additional transversality condition. But Judd also rightly assumes that a reader of the book ought to already be familiar with this, and does not need much of a reminder or explanation, although a nice discussion is provided on page 358. It is important to always keep this additional stability condition in mind when reading the book on pages 339, 358, 459, 464, 466, 476, 478, 538, 550–62, 589–602, for example, as it explains some short, but crucial remarks and initialization choices.

4.1 Perturbation methods

Many researchers stop after calculating the linearized law of motion around the steady state, as described above; this is where Judd keeps going. Judd first points out, that solving the model by linearization is really a perturbation method. Perturbation methods work by solving an equation first at a point, where it is easy to solve (the steady state, in this case), and then use Taylor expansions to solve for points nearby. Indeed, the linearization method above is nothing but a first-order Taylor expansion of the functional equation around the steady state.

So, what about a second-order or even higher-order Taylor expansion? If a first-order expansion resulted in a quadratic equation, will one get a cubic equation for a second-order expansion, etc.? Here, Judd clearly teaches us in chapter 13, that this is not so. Instead, the higher order expansion terms result in *linear* equations, regardless of the order of the expansion! More precisely, given the linearized solution, i.e. the solution to the first order expansion, calculate the second-order expansion and evaluate at the steady state: one obtains a linear equation in the to-be-calculated second derivative $C''(k^*)$ at the steady state as well as expressions involving the already calculated first derivative. The same will be true for the third-order expansion, etc.: one can keep on going as long as one pleases. It might have been nice if the book gave the explicit recursion formula for calculating the higher-order derivatives for a somewhat general problem – this would be high on my wish list for a future revision of the book – but it is straightforward (though tedious) to do it yourself, and the book points at publications and programs which can do it for you.

In any case, the linearity of the higher-order expansion equations is really a powerful and useful insight, and which immediately provides users of the benchmark (log-)linearization method with an expansion of their toolkit.

Chapter 13 deals with both discrete-time and continuous-time control problems, does both in the deterministic and stochastic version, and offers some additional and general material on top. I found sections 13.5 through 13.7 particularly useful and enlightening, and encourage every reader to study them carefully. A bit more in section 13.6 on stochastic continuous-time optimal control would have been even better. Chapter 14 of the book takes the reader through the multidimensional version: for higher-order derivatives, that means lots of indices and summation signs. Judd introduces the language of tensors as well as Einstein summation notation, which, roughly stated, encourages you to be lazy and forget writing summation signs in all the formulae, assuming instead that you always mean there to be a summation sign, if a potential summation index occurs in several places. No magic here: this is just an ink-saving device. Chapter 15 provides some advanced, intriguing material. It shows how perturbation methods can still be applied, even if the derivative with respect to the choice variable is zero at the steady state. It applies this extension to solve portfolio choice problems, thereby dealing with the 'problem' that portfolios of risky assets are indetermined at the deterministic steady state. Chapter 15 also introduces generalizations of Taylor expansions, called gauge functions.

4.2 Projection methods

Projection methods push the functional analytic perspective all the way. Think of the right hand side of (1) as a non-linear operator $N(C)$, mapping functions $C = C(k)$ into another function $(N(C))(k)$ of k

$$(N(C))(k) = u'(C(k)) - \beta u'(C(F(k) - C(k)))F'(F(k) - C(k)) \quad (2)$$

Solving the problem means solving the equation

$$N(C) = 0 \quad (3)$$

(plus the stability condition, i.e. a condition regarding the derivative at the steady state). Since this is an equation in infinite-dimensional space, one needs to 'project' it down into a finite dimensional subspace to make this all feasible.

First, 'project' the function C into an n-dimensional subspace, replacing it with

$$\hat{C}(k; a) = \sum_{i=1}^{n} a_i \Psi_i(k)$$

where $\Psi_i, i = 1, \ldots, n$ are functions, forming the basis of this subspace. How should one choose Ψ_i? That's a good question, to which we return below in section 5. Let's ignore it for the moment.

Second, 'project' the result delivered by the operator N into an n-dimensional subspace. To avoid clutter, write

$$R(k;a) = N(\hat{C}(\cdot;a))(k)$$

or, less abstractly

$$R(k;a) = u'(\hat{C}(k;a)) - \beta u'(\hat{C}(F(k) - \hat{C}(k;a);a))F'(F(k) - \hat{C}(k;a)) \quad (4)$$

see equation (16.5.2) in Judd. Now, rather than demanding $R(k;a) = 0$ to be true for all k, demand instead either that

$$R(k_j;a) = 0, \quad j = 1, \ldots, n$$

for some suitably chosen values $k_j, j = 1, \ldots, n$ or demand that

$$P_i(a) = 0, \quad i = 1, \ldots, n$$

where

$$P_i(a) = \int R(k;a)\psi_i(k)w(k)dk$$

for some suitably chosen weighting function $w(k)$. The first is called the collocation method, whereas the second is called the Galerkin method. You may wonder, how to pick the k_j in the collocation method, or how to pick $w(k)$ and how to compute the integral in Galerkin's method. Good question: again, we shall return to that in section 5 below, and ignore it for now.

All this generalizes to the stochastic case, see section 17.6 in Judd: aside from adding one more variable, it also adds an expectations operator in e.g. (4): one therefore needs to find a way to calculate these expectations, see section 5.

In any case, what one now has at the end is a non-linear system of n equations in n unknowns $a_1, \ldots, a_n$, which one can now attempt to solve. How? Go to section 5. The key point is that this is now in principle a familiar, well-understood problem.

The projection methods are explained mainly in section 16.4 and 16.5 for the non-stochastic case, in particular page 550, and sections 17.5 and 17.6 for the stochastic case. Section 11.3 explains the general principle, and chapter 11, together with section 10.8 perhaps, provides the background. Section 15.6 offers a hybrid perturbation-projection method. Table 17.4 on page 606 provides an overview.

5 'BASIC' NUMERICAL MATHEMATICS

In several places above, additional 'mundane' numerical issues have arisen. How should base functions be picked for the projection methods? How can integrals and expectations be calculated? How can systems of non-linear equations be solved?

Most of these issues belong in the realm of 'basic' numerical mathematics: my favorite reference here is the classic text by Stoer and Burlisch (1980). Judd provides an excellent self-contained treatment of these issues in the first nine chapters of his book: these chapters offer the additional advantage that a number of issues particularly relevant to economics but perhaps less relevant in numerical mathematics are treated here in detail, drawing on the relevant specialized literature. For example, when doing value function iteration, one needs some way of interpolating a value function given at a few points only in such a way, that the interpolation function is concave or monotone. This is dealt with in section 6.11 in Judd and on page 440 (a reference on page 440 to section 6.11 would be useful).

There is a lot of material in these nine chapters on 'basic' numerical analysis, and it is probably best to pick and choose when trying to solve a particular economic problem. For example, if one uses software that already has a fast, canned routine for solving nonlinear equations, and if it is working well, then there probably is no need to study chapter 5 on solving systems of non-linear equations in detail. On the other hand, it is handy to have that chapter at hand, if something goes wrong, if one has to write some equation-solving code oneself or if one simply wants to understand what the canned routines are doing anyways.

There are a number of topics one should study closely, though. One is chapter 6, in particular, sections 6.3 through 6.7, because they are crucial to the philosophy of this book, and explain how functions can be approximated well. The short summary is: Chebyshev polynomials are great. Actually, one cannot quite see why they are great until section 16.4, in particular table 16.1. There, it turns out, that using Chebyshev polynomials in the collocation method delivers coefficients that do not change much as the dimensionality n of the approximation is increased. This is nice: it makes it easier in practice to choose a suitably accurate dimensionality for the approximation. What could have been given perhaps a bit more space is the question whether and how Chebyshev polynomials are good approximations to, say, policy functions which one knows to be increasing: there is a little bit in section 17.8 towards the end, there are comparisons like those in figure 15.2 and practical discussions like those in section 15.6, and there are, of course, sections 6.10 and 6.11. Still, this is an issue which is probably worth investigating further and more prominently. Of course, Judd did not hide anything here: there simply is not much yet that could have been reported, i.e. this is a still underresearched area.

The second part that one should study closely is section 7.2, which links Chebyshev polynomials to numerical integration and Gaussian quadrature, thereby addressing some issues that have arisen in the description of the Galerkin method above.

Finally, chapter 8 strikes me as very promising and useful material. Ultimately, one wishes to deal with stochastic dynamic models of medium dimensionality: in those cases, calculating e.g. expectations can be practically impossible, when using standard integration procedures. Here, new ideas are needed, and the Monte Carlo Methods in this chapter together with e.g. recent advances in Neuro-dynamic programming probably offer the solution, see Bertsekas and Tsitsiklis (1996). The attention that leading researchers in the area like Ken Judd, John Rust or Ariel Pakes are giving it, suggests that this will be an active and fruitful area of future research.

6 THE FUTURE

Ken Judd has set himself a lofty goal. In his introduction he writes: 'This is not a backward-looking textbook describing the numerical methods that have been used in economics, nor an intellectual history of computation in economics, but instead the book attempts to focus on methods that will be of value . . . My priority is to give a unified treatment of important topics, and base it on standard mathematical practice' (p. 19). The book admirably achieves this lofty goal, and more. It contains a lot of excellent new and exciting tools which may not have been used much in the past, simply because economists did not have a good place to study and learn them; they do now.

The book occasionally beats up too much on the benchmark loglinearization approach and linear-quadratic approach or other numerical approaches which may be familiar to an interested reader, as in Danthine and Donaldson (1995), Hansen and Prescott (1995) or Uhlig (1999). Those approaches often work quite well and, in any case, are usually the methods of choice for anybody starting to use numerical tools: there is nothing wrong with that. The book could have used these methods as a launching pad to show readers how to go from there. This review has been written with these researchers in mind. I see Judd's book as offering a variety of options for such researchers on how to take the next step of employing more sophisticated and more powerful tools. It is my hope that this review provides a guide to accessing this book, and thereby aids in making the use of these new tools more widespread.

Judd's book offers a powerful perspective that can even be pushed much further, I believe. The perspective taken is one of solving functional equations. Judd mostly focuses on a functional equation in the decision rule, mapping the state into the current decision. One can imagine other functions to be the object to be solved for. The value function is an obvious one, and so is its first derivative. Also, the expectation of the value function or the expectation of its derivative are often really the key to solving dynamic economic models: this immediately suggests to focus on 'parameterized expectations', as is done by Albert Marcet and his collaborators, see e.g. Marcet and Lorenzoni (1999). This fits in well with the theme of this book. Marcet uses simulation methods to calculate these expectations: given Judd's book, one can now perhaps imagine many alternative approaches to calculating these integrals, and they should all be tried out eventually. Judd discusses these issues somewhat, but more would be nice: what are the merits and disadvantages of focusing on the decision as opposed to focusing on the expectation as a function of the state?

There is still too little that we know about the numerical quality of approximations. Judd's book offers a number of evaluations, which is very nice, mostly focusing on some measure of distance between the decision rules obtained. An alternative is the simulation-based approach, using the den-Haan-Marcet statistic, see den Haan and Marcet (1994), which was used by Taylor and Uhlig (1990) in a comparison of a number of solution methods. Judd discusses this on pages 574 and 600 (although the den-Haan-Marcet paper is missing in the list of references). In any case, here too more research should be done. As economists, we need to make sure that the numerical methods used are accurate in the dimensions which are relevant for the economic interpretation of the results. A systematic 'theory' of evaluation methods is called for.

Finally, there is now a new class of models, which researchers wish to solve, and these involve dynamic, stochastic economies with many heterogeneous agents. Recent breakthroughs here include Krusell and Smith (1998) and Rios-Rull (1999). Certainly, much more needs to be done. These models are too recent to have made it into Judd's book yet. It will be exciting to see new methods and insights emerging from applying the systematic perspective offered in Judd's book to these new classes of models.

7 CONCLUSION

My conclusion can be brief. This is a wonderful book. Go buy it, read it and apply it.

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NOTES

1 I am using linearization for expositional reasons rather than log-linearization, which I find preferable in applications. Alternately, rewrite everything in terms of log-deviations, and follow the same steps: nothing of substance changes.

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WHEN DO EMPIRICAL DATA PROVIDE RELIABLE EVIDENCE FOR A HYPOTHESIS (THEORY)?

A review of Deborah G. Mayo's *Error and the Growth of Experimental Knowledge*, Chicago: The University of Chicago Press, 1996. Pp. 493, $29.95.

Economists (and other scientists) interested in the philosophy and methodology of science are likely to find Mayo's book entitled *Error and the Growth of Experimental Knowledge*, a recipient of the Lakatos Prize in 1998, highly pertinent to their discussions on methodology. This book sheds ample light, as well as offers effective ways to tackle, several methodological issues of paramount interest to the methodology of economics literature (see Backhouse 1994; Blaug 1980/1992; Caldwell 1982/1993), because it offers:

1 a coherent and constructive account of what can be 'salvaged' from the Popperian, Lakatosian and Kuhnian traditions;
2 a detailed discussion of 'what's wrong with falsificationism' and 'what it should be replaced with';
3 a wealth of compelling arguments against the Bayesian way as a vehicle of inductive inference;
4 effective ways to deal with or sidestep several methodological problems that plagued the Popperian and the growth of knowledge traditions: demarcation of scientific inquiry criteria, non-accumulation of knowledge, theory-ladenness, incommensurability, underdetermination and the Duhem-Quine problems; and