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A law of large numbers for large economies*

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Summary. Let $X(i)$, $i \in [0; 1]$ be a collection of identically distributed and pairwise uncorrelated random variables with common finite mean μ and variance σ^2. This paper shows the law of large numbers, i.e. the fact that $\int_0^1 X(i)di = \mu$. It does so by interpreting the integral as a Pettis-integral. Studying Riemann sums, the paper first provides a simple proof involving no more than the calculation of variances, and demonstrates, that the measurability problem pointed out by Judd (1985) is avoided by requiring convergence in mean square rather than convergence almost everywhere. We raise the issue of when a random continuum economy is a good abstraction for a large finite economy and give an example in which it is not.

1. Introduction

In the analysis of economies with a continuum of agents, the following problem often arises. Suppose each agent has to bear a certain risk. The risk of each agent is uncorrelated with the identical risk any other agent faces. Does the risk disappear upon aggregation? Examples in which such a law of large numbers is implicitely or explicitely assumed or used include Bewley (1986), Diamond and Dybvig (1983), Green (1987), Lucas (1980) and Prescott and Townsend (1984).

Formally, the problem can be restated as follows. Let $X(i)$, $i \in [0; 1]$ be a collection of identically distributed and pairwise uncorrelated random variables with common finite mean μ and variance σ^2. One would like to have

$$\int X(i)di = \mu. \tag{1}$$

The contribution of this paper is to show how to make this statement precise and to prove it, avoiding the measurability problem pointed out by Judd (1985).

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These measurability problems are discussed first in section 2. Studying Riemann sums in section 3, a simple proof of equation (1) involving no more than the calculation of variances is given. The measurability problem is avoided because convergence in mean square rather than convergence almost everywhere is used. Section 4 provides a deeper mathematical foundation by bringing the theory of vector valued integration and the Pettis integral to bear on the problem (see Diestel and Uhl, 1977), providing a second proof of (1) and demonstrating the formal equivalence to the Riemann-sum based approach of section 3. Section 5 discusses the issue of when a random continuum economy is a good abstraction for a large finite economy and gives an example in which it is not. Section 6 concludes.

2. The problem with pathwise integration

Judd (1985) pointed out a severe measurability problem which arises when interpreting (1) as pathwise integration, i.e. when interpreting (1) to mean

$$\int X(i)(\omega)di = \mu \text{ with probability one,} \tag{2}$$

where the state ω is drawn from the underlying probability space. Judd (1985) showed that the set of states ω, for which the function $i \mapsto X(i)(\omega)$ is measurable on $[0;1]$, is not even measurable on the probability space usually used for modelling a collection of independent random variables. While it is possible to fix this measurability problem in a somewhat "ad hoc" way and essentially assume (2) to hold, it is not possible to fix the problem in such a way that equation (2) holds on all subintervals of $[0;1]$ simultaneously:

Theorem 1. *Suppose that for some ω, it is true that $i \mapsto X(i)(\omega)$ is Lebesgue measurable in $i \in [0;1]$ and that for all $a, b \in [0;1]$, $a \leq b$*

$$\int_{[a,b]} X(i)(\omega)d\lambda(i) = (b-a)\mu, \tag{3}$$

where λ is the Lebesgue measure. Then $X(i)(\omega) \equiv \mu$ for almost all i, i.e. for all $i \in [0;1]/N_\omega$ for some Lebesgue null set N_ω.

Proof: *Define a measure v on the Borel set of* $[0;1]$ via

$$v(A) = \int_A X(i)(\omega)d\lambda(i).$$

Since v coincides with the Lebesgue measure times μ on all subintervals of the unit interval, the Radon-Nikodym theorem implies that $i \mapsto X(i)(\omega)$, i.e. the Radon-Nikodym derivative or density of v with respect to the Lebesgue measure must equal μ almost everywhere. □

In other words, if the law of large numbers in the sense of (2) is supposed to hold on all subintervals, then $X(i)(\omega)$ must essentially be constant, reducing the validity of the law of large numbers to only trivial cases. This is a fundamental dilemma. Hence, if one does not wish to abandon the law of large numbers as a useful tool for continuum economies, one needs to abandon the interpretation of pathwise integration.

3. A simple solution

Throughout, let (Ω, Σ, P) denote the underlying probability space. It is useful to think about (1) as a problem of integrating a function X, which maps $i \in [0;1]$ or, more generally, $i \in [a,b]$ for $a \leq b \in \mathbb{R}$ into the space $L_2(\Omega, \Sigma, P)$ of random variables with finite variance on the probability space (Ω, Σ, P). Consider integrating such a function using a Riemann-type approach. A partition $p = (n, i_0, \ldots, i_n, \psi_1, \ldots, \psi_n)$ is a grid $a = i_0 < i_1 < \cdots < i_n = b$ on the interval $[a,b]$ and midpoints $\psi_j \in [i_{j-1}, i_j]$, $j = 1, \ldots, n$ for the grid intervals. The mesh $\zeta(p) = \max(i_j - i_{j-1})$ of a partition is the maximal length of a grid interval. Define

$$S(p) = \sum_{j=1}^{n} X(\psi_j)(i_j - i_{j-1})$$

to be the Riemann sum for the partition p. Note, that the Riemann sum is itself a random variable. Given a convergence criterion for random variables, the Riemann integral $Y = \int_a^b X(i)di$ can be defined to be a random variable Y such that

$$\lim_{\zeta(p) \to 0} S(p) = Y, \tag{4}$$

provided the limit exists. There are several convergence criteria to choose from. Judds approach outlined in the previous section amounts to using pointwise convergence almost everywhere. As has been discussed above, this results in a fundamental dilemma. The solution to this dilemma is to use a weaker convergence criterion, namely norm-convergence in $L_2(\Omega, \Sigma, P)$ or, equivalently, convergence in mean square, interpreting (4) as

$$\lim_{\zeta(p) \to 0} E[(S(p) - Y)^2] = 0. \tag{5}$$

If a random variable Y satisfying (5) exists, call the function X L_2-Riemann integrable and Y the L_2-Riemann integral of X.

One immediately obtains.

Theorem 2 (the law of large numbers) *Let $X: [a,b] \to L_2(\Omega, \Sigma, P)$ be a function, mapping the interval $[a,b]$, $a \leq b$ into the space of random variables with finite variances. Suppose that the random variables $X(i), i \in [a,b]$ are pairwise uncorrelated, have the same common mean μ and that their variances are bounded above by $\sigma^2 < \infty$. Then X is L_2-Riemann integrable and the L_2-Riemann integral Y of X is almost everywhere constant,*

$$\int_a^b X(i)di \equiv (b-a)\mu \text{ with probability one}$$

Proof: *Calculate*

$$\begin{aligned} E[(S(p)-(b-a)\mu)^2] &= \sum_{j=1}^{n} E[((X(\psi_j)-\mu)(i_j-i_{j-1}))^2] \\ &\leq \sum_{j=1}^{n} (i_j-i_{j-1})^2\sigma^2 \\ &\leq \zeta(p)\sum_{j=1}^{n} (i_j-i_{j-1})\sigma^2 \\ &= \zeta(p)(b-a)\sigma^2, \end{aligned}$$

converging to zero as $\zeta(p)\to 0$. □

In analogy to the law of large numbers for sequences, the approach here yields a version of Khinchines law of large numbers rather than a strong law of large numbers as aimed for in Judd (1985). The measurability problem is avoided and the proof of the theorem becomes remarkably simple. The definition of the L_2-Riemann integral does not require uncorrelatedness of the $X(i)$, of course. For example, if the function $X:[0;1]\to L_2$ is norm-continuous or indeed pathwise integrable and satisfies some weak boundedness condition, the L_2-Riemann integral exists and coincides with the pathwaise integral, if it exists. An example for this is the Brownian motion: the integral is a normally distributed random variable equal to the pathwise integral. However, the key advantage of the L_2-Riemann integral is its applicability to the common situation of mutually uncorrelated random variables as stated in the theorem above.

Interesting variations and applications can be shown in the same manner or building on this theorem. For example, suppose that the $X(i)$, $i\in[0;1]$ are independent and identically distributed. Denote their common distribution function by F. Does F also represent the population distribution? To analyze this question, choose some $x\in\mathbb{R}$ and consider the indicator functions $1_x(i)$ with $1_x(i)(\omega)=1$, if $X(i)(\omega)\leq x$ and $1_x(i)(\omega)=0$ otherwise. The theorem above then yields directly

$$\int 1_x(i)di = \text{Prob}(X(i)\leq x) = F(x),$$

answering the question affirmatively.

4. Vector-valued integration

This section embeds the law of large numbers as stated above deeper in the mathematical literature. Viewing the law of large numbers for a large economy as a problem of integrating a function X from the unit interval into the vector space $L_2(\Omega, \Sigma, P)$ naturally leads to the theory of vector-valued integration as reviewed in Diestel and Uhl (1977). There are two general integration concepts for vector-valued functions: the Pettis integral and the Bochner integral. After defining the Pettis integral for the sake of completeness, it is shown that the Pettis integral again delivers the law of large numbers and is essentially equivalent to the L_2-Riemann integral introduced above, whereas the Bochner integral does not exist for the continuum of uncorrelated random variables.

This section requires tools from functional analysis and measure theory. The following definitions are taken from Diestel and Uhl (1977). Let V be a Banach space, V' its dual and (L, Λ, λ) a finite measure space without atoms. A function $f: L \to V$ is called weakly λ-measurable, if for each $v' \in V'$ the function $v'f$ is λ-measurable. It is called Pettis integrable, if in addition there is a vector $y \in V$, called the Pettis integral and denoted by $y = (P) - \int f d\lambda$, such that

$$v'y = \int v' f d\lambda \text{ for all } v' \in V'$$

The Pettis integral was introduced by Pettis (1938).

Theorem 3 (the law of large numbers: The Pettis integral version) *Let $X: L \to L_2(\Omega, \Sigma, P)$ be a function, so that the random variables $X(i)$ are pairwise uncorrelated, have a common finite mean μ and so that their variances are bounded above by $\sigma^2 < \infty$. Then X is Pettis-integrable with*

$$(P) - \int X d\lambda \equiv \mu\lambda(L) \text{ with probability one.}$$

Proof: *The dual space of L_2 is (naturally isomorphic to) L_2. Let $Z \in L_2$ be an element of the dual space. Z operates on $X \in L_2$ via $ZX = E[ZX]$. It needs to be shown that the function $g(i) = E[X(i)Z]$ is measurable with respect to λ and that*

$$0 = \int E[(X(i) - \mu)Z] d\lambda.$$

But this is trivial: observe that $E[(X(i) - \mu)Z] = 0$ for almost every $i \in L$ since

$$\sum_{j=0}^{\infty} (E[(X(i_j) - \mu)Z])^2 \leq E[Z^2]\sigma^2$$

for any countable selection of different i_j's by Bessel's inequality. □

It is possible to generalize this result to the case where the random variables $X(i)$ themselves take values in some Banach space rather than the real line by reducing this more general situation to the simpler situation above again by means of functionals in the dual space. Details can be obtained from the author.

The link between the result above and the result in the previous section is established by the following theorem.

Theorem 4. *If $X: [a, b] \to L_2(\Omega, \Sigma, P)$ is L_2-Riemann integrable, then X is Pettis integrable and the L_2-Riemann integral equals the Pettis integral.*

Proof: *Let $Y = \int_a^b X(i) di$ be the L_2-Riemann integral. Take $Z \in L_2$ and define f_Z by $f_Z(i) = E[ZX(i)]$. For any partition p, the Cauchy-Schwarz inequality implies that*

$$\left| E[ZY] - \sum_{j=1}^{n} f_Z(\psi_j)(i_j - i_{j-1}) \right| \leq E[Z^2]^{1/2} (E[(Y - S(p))^2])^{1/2}.$$

Taking $\zeta(p) \to 0$ shows that the real-valued function f_Z is Riemann integrable and therefore Lebesgue integrable with $\int f_Z(i) di = E[ZY]$. Since real-valued Riemann integrable functions are measurable, this also implies that X is weakly λ-measurable.

An application of Lemma II.3.1. *in Diestel and Uhl* (1977) *together with the reflexivity of* L_2 *proves the claim about Pettis integrability. The equality of the two integrals is now clear.* □

The Bochner integral is the other general integration concept for vector valued functions. It is defined via norm-approximations by simple functions. If such a norm-approximation exists, the function is called λ-measurable. This approach is similar to the usual way of defining measurability or the Lebesgue integral for real-valued functions by in essence replacing absolute values of real numbers with norms of vectors. The reader is referred to Diestel and Uhl (1977) for the details. The Bochner integral has many appealing properties, and it would therefore be nice to state a law of large numbers using the Bochner integral rather than "just" the Pettis integral. Unfortunately, this is impossible as the next theorem tells us. That theorem also sheds light on a fundamental difficulty of analyzing large random economies: the underlying probability space needs to be "very large".

Theorem 5. *Let* $X:[0;1] \rightarrow L_2(\Omega, \Sigma, P)$ *be function, mapping the unit interval into the space of random variables with finite variance. Assume that the* $X(i)$ *are pairwise uncorrelated with common finite mean and variance* $\sigma^2, 0 < \sigma^2 < \infty$. *Then the function* X *is not* λ*-measurable, where* λ *is the Lebesgue measure, and is not Bochner-integrable.* $L_2(\Omega, \Sigma, P)$ *is not separable. The set* Ω *is not a separable metric space.*

Proof: *The norm distance between* $X(i)$ and $X(j)$ *for* $i \neq j$ *is* $\sqrt{2}\sigma$. *Thus, for any uncountable subset* A *of* $[0;1]$, *the set* $\{X(i)|i \in A\}$ *cannot be separable in* $L_2(\Omega, \Sigma, P)$. *Pettis' measurability theorem (Theorem II.1.2 in Diestel and Uhl, 1977) thus implies that* X *cannot be* λ*-measurable and hence not Bochner-integrable. Furthermore,* Ω *cannot be a separable metric space.* □

The Bochner integral exists for continuous functions $X:[0;1] \rightarrow L_2$ for example. In these cases, it equals the Pettis integral and the L_2-Riemann integral. An Example is the Brownian motion on $[0;1]$.

5. Finite and continuum random economies

The purpose of this section is to raise a question rather than to answer one: what is the nature of the relationship between finite random economies and continuum random economies?

Nonrandom continuum economies are traditionally analyzed as limits of sequences of finite economies, see e.g. Hildenbrand (1974) or Mas-Colell (1986). For example, $x(i) \in \mathbb{R}^k$ might represent the endowment of agent i in some finite-dimensional commodity space $\mathbb{R}^k$. The finite economies converge to the continuum, if (among other things) the endowment distribution of the finite economy converges to that of the continuum. In particular, the closure of the set of endowments of all agents of the finite economies contains the support of the endowments of the continuum economy, i.e. contains the endowment of any agent drawn randomly from the continuum economy with probability one. This necessary condition is a useful check when proposing some notion of convergence to the continuum.

Consider now random continuum economies. As Al-Najjar (1994) has shown, these economies can be "decomposed" into an aggregate risk part, which can be treated in a manner similar to nonrandom economies, and a part with idiosynchratic risk $X(i)$, where all $X(i)$, $i \in [0;1]$ are uncorrelated. Assume additionally that their variance is bounded below by $\sigma^2 > 0$. Generalizing the usual approach runs into a fundamental difficulty. The appropriate commodity space is now the space $L_2(\Omega, \Sigma, P)$ of random variables with finite variance. This space is not only infinite-dimensional, but it needs to be nonseparable as well, since the distance between any two $X(i)$ in L_2-space is at least $\sqrt{2}\sigma$, see theorem 5 above. Since any given sequence of finite economies delivers no more than countably many endowment points in L_2, the norm-closure of the set of endowments of all agents of the finite economies cannot contain the set of endowments of a nonzero set of agents in a continuum economy. In a deep sense, a sequence of finite economies typically cannot approximate a given random continuum economy. One would need to consider generalized notions of convergence, using nets rather than sequences, but such an approach risks becoming quite arcane quickly.

One may consider working around this problem by concentrating on finitely many characteristics of the economies, mapping the infinite-dimensional richness back into finitely many dimensions. Better yet, one might consider the closure of the set of endowments in the weak topology rather than the strong topology and use some convergence concept based on this idea, see Al-Najjar (1994). This seems promising at first: the set of endowments in L_2 of any finite economy where one agent has a nonrandom endowment equal to μ contains almost all endowments of the continuum economy with idiosynchratic risk in a "weak topology" sense, because for any $Z \in L_2$, $E[ZX(i)] = E[Z\mu]$ for all but countably many i, see the proof for theorem 3. But while the requirement of norm-closure above failed because in essence the finite economies were required to deliver too much detail, the requirement here seems too coarse. Using the weak topology view, there is no distinction between a continuum economy with idiosynchratic risk and a continuum economy without any risk: if a given sequence of finite economies converges to one, it will also converge to the other. Perhaps, it is possible to deliver more detail by including descriptions of the random distributions as well in a manner similar to the method described in the second paragraph following theorem 2. Alternatively, one might restrict attention to only the distributional aspects rather than the full description of the randomness: this is done for example in Mas-Colell and Vives (1993). In the special case of continuum economies, in which the risk is independent and identically distributed across agents, a replication argument might yield some insights, where the endowments of the replicated agents need to be drawn independently from the same distribution rather than being copies of the endowments of the "original" agent. Whatever approach is used: for the general case, it should be clear that any sequence of finite economies will fail to capture some features of some given continuum economy. Whether these features are relevant in any particular economic application depends on the particular question being asked, is difficult to pin down by a general approach and seems highly judgemental.

Instead, it seems more fruitful to consider continuum economies as approximations to finite economies rather than the other way around. The quality of the approximation will be the better, the larger the finite economy is. Bewley (1986), who draws finite economies from the given continuum, can be interpreted in this way. Pursuing this line further might be fruitful but is beyond the scope of this paper.

Given the rather murky relationship between large finite economies and continuum economies, how can one judge whether theorem 2 delivers an economically meaningful law of large numbers or not? We claim that one can and that the theorem indeed does accomplish this objective in the following sense. Assume in some finite or continuum economies, that the ex-ante utility functions $U(C)$ are norm-continuous in random consumption $C \in L_2$. Thus, if a finite economy is close to a continuum economy in the sense that the consumption allocations are close in L_2-norm, then ex-ante utilies will be close as well. Since L_2-Riemann integration checks for closeness in L_2-norm of the integral (corresponding to the continuum economy) to some Riemann sum (corresponding to the finite economy), it follows that the L_2-Riemann integral or, equivalently, the Pettis-integral is the right tool to guarantee closeness in terms of ex-ante welfare. This argument applies more generally than just to the case of idiosynchratic risk. As a particular application, it shows that mutual insurance against individual income shocks provides agents in large finite economies with almost the same ex-ante welfare as the same insurance arrangement in a continuum economy where the idiosynchratic risk component averages out: this is an economically meaningful result.

The argument rests on the assumption of the L_2-continuity of the utility function $U(C)$. This continuity can be guarnateed for von-Neumann-Morgenstern utility functions under some additional assumptions:

Theorem 6. *Let $u: \mathbb{R}^+ \to \mathbb{R}$ be a utility function $u(c)$ in consumption $c \geq 0$, which is bounded from below, monotone increasing, continuous and concave. Then, expected utility $U(C) = E[u(C)]$ is uniformly continuous on $C \in L_2^+(\Omega, \Sigma, P)$.*

Proof: *Assume w.l.o.g. $u(0) = 0$. Choose some $\varepsilon > 0$. Fix some $x > 0$ and let $M = u(x)/x$. Fix α at $\alpha = \varepsilon/(2M(x+1))$. Note that u is equicontinuous on $\mathbb{R}^+$, so choose a δ to $\varepsilon/2$. Finally note that we can find $v > 0$, $v < 1$ so that $C, D \in L_2^+(\Omega, \Sigma, P)$ and $E[(C-D)^2] < v$ implies $Prob(|C-D| \geq \delta) < \alpha)$. Distinguish the four cases where $|C(\omega) - D(\omega)| < \delta$ or otherwise $|C(\omega)| < x$ or otherwise $|D(\omega)| < x$ or otherwise $|C(\omega)| \geq x$ as well as $|D(\omega)| \geq x$. Use $|u(a) - u(b)| \leq M|a-b| + Mx$ and the Cauchy-Schwartz inequality to find immediately*

$$|U(C) - U(D)| \leq (1-\alpha)\varepsilon/2 + \alpha M x + \alpha M v^{1/2} < \varepsilon$$

if $E[(C-D)^2] < v$. □

Finally, it may be useful to point out that one should not apply the law of large numbers blindfoldedly. The following illustration should serve as a counterexample and warning sign. Suppose that agents want to mutually insure themselves against random fluctations in their endowment $X(i)$. We assume that $X(i)$ is privately observed, but that the average endowment $\frac{1}{n}\sum_{i=1}^{n} X(i)$ is public information. For technical reasons, we also assume that the endowment of agents is always contained in some interval $X(i)(\omega) \in [X_{\min}, X_{\max}]$ with $2X_{\min} - X_{\max} > 2\varepsilon$ for some $\varepsilon > 0$.

Mutual insurance is achieved by agents agreeing to the following contract before their random endowment is realized. Each agent announces his or her endowment. Should the average of these announcements match the known average endowment, then each agent will recieve or will pay the appropriate difference to the average announcement. If not, every agent pays $X_{max} - X_{min} + \varepsilon$ and all payments are disposed of. The contract improves welfare ex ante if agents are risk averse. The contract is incentive compatible for any finite economy since it is in every agents best interest to tell the truth, if everybody else does. However, the contract ceases to be incentive compatible in the continuum economy, since an individual announcement or realization no longer matters for the economy-wide average. Thus, the continuum economy behaves very differently from any finite economy, even though for some suitable sequence of finite economies, the average endowment converges in L_2-norm to the average in the continuum economy computed as the L_2-Riemann integral.

6. Conclusions

This paper has shown how to obtain a law of large numbers for a continuum of uncorrelated random variables, avoiding the measurability problems raised by Judd (1985). This is accomplished by requiring convergence in mean square rather than convergence almost everywhere for the Riemann sums. It is shown that the L_2-Riemann integral introduced this way coincides with the Pettis integral, which is one of two standard integration concepts in the theory of vector-valued integration. It is shown how the other integration concept – the Bochner integral – is not suitable for the case of idiosynchratic risk and that the underlying probability space needs to be very large. The relationship of large finite economies to random continuum economies is discussed, pointing out the fundamental difficutly of a nonseparable commodity space. It is shown how convergence in mean square of the consumption allocations of the finite economies to the consumption allocation of the continuum economy suffices to guarantee convergence of ex-ante welfare, justifying the use of the L_2-Riemann integral to prove the law of large numbers. An example illustrating the dangers of "blindfolded" applications of the law of large numbers has been provided.

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