CentER

Discussion Paper

No. 2006–101

PENSION SYSTEMS AND THE ALLOCATION OF MACROECONOMIC RISK

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October 2006

ISSN 0924-7815

Tilburg University

Pension systems and the allocation of macroeconomic risk*

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This revision: October 25, 2006

Abstract

This paper explores the optimal risk sharing arrangement between generations in an overlapping generations model with endogenous growth. We allow for nonseparable preferences, paying particular attention to the risk aversion of the old as well as overall "life-cycle" risk aversion. We provide a fairly tractable model, which can serve as a starting point to explore these issues in models with a larger number of periods of life, and show how it can be solved. We provide a general risk sharing condition, and discuss its implications. We explore the properties of the model quantitatively. Among the key findings are

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the following. First and for reasonable parameters, the old typically bear a larger burden of the risk in productivity surprises, if old-age risk-aversion is smaller than life risk aversion, and vice versa. Thus, it is not necessarily the case that the young ensure smooth consumption of the old. Second, consumption of the young and the old always move in the same direction, even for population growth shocks. This result is in contrast to the result of a fully-funded decentralized system without risk-sharing between generations. Third, persistent increases in longevity will lead to lower total consumption of the old (and thus certainly lower per-period consumption of the old) as well as the young as well as higher work effort of the young. The additional resources are instead used to increase growth and future output, resulting in higher consumption of future generations.

Keywords: social optimum, pension systems, risk sharing, overlapping generations.

JEL codes: E21, E61, E62, O40, H21, H55

1 Introduction

One can distinguish pension system along three dimensions (see Lindbeck and Persson (2003)): defined benefit versus defined contribution; funded versus pay as you go; and actuarial versus non-actuarial. The first dimension involves intergenerational risk sharing: how are macro-economic risks distributed over various generations? In particular, in defined-benefit systems, the retired generations are shielded from macro-economic risks. In defined-contribution systems, in contrast, these generations are bearing some of the investment risk. The second dimension relates to intergenerational redistribution: are there predictable transfers from the young, working generations to the retired generations? The third dimension, finally, involves the link between contributions and benefits on an individual level. A system is actuarially fair if the individual premium paid corresponds to the actuarial value of the additional pension benefits that are being accumulated.

This paper explores the first two dimensions of an optimal pension system. In exploring optimal pension systems, we distinguish between predictable intergenerational transfers and optimal risk sharing. Hence, in contrast to Merton (1983) and Krueger and Kubler (2002, 2005), we allow the government to enhance intergenerational risk sharing without necessarily implementing a pay-as-you-go system. Indeed, defined-benefit funded pension systems that link benefits to wage rates and absorb financial-market shocks by adjusting pension premia paid by the young can help to trade risks between the young who are long on human capital and the old who are long on financial capital. In contrast to defined-benefit pay-as-you-go systems, these funded systems enhance risk sharing without crowding out capital formation. We in effect analyze the optimal mix between defined-benefit and defined-contribution pensions as part of the optimal risk sharing scheme while at the same time determining the optimal mix between funded and pay-as-you-go financing as part of optimal redistribution between the old, young and future generations.

To investigate optimal intergenerational redistribution and optimal intergenerational risk sharing in a general equilibrium setting, we develop a stochastic overlapping generations model of a closed economy featuring endogenous growth. Optimal risk sharing under an ex-ante welfare criterion has been explored in a partial-equilibrium setting by Gordon and Varian (1988), Shiller (1999), and De Menil, Murtin and Sheshinski (2005). Our analysis builds on the general equilibrium model of Bohn (1998, 1999, 2003, 2005), who investigates optimal risk sharing under an ex-ante welfare criterion between overlapping generations in an exogenous growth model with capital accumulation. Regarding related contributions on this issue, see also Olovsson (2004), for optimal risk sharing under an ex-post criterion, see Blanchard and Weil (2001), Demange (2002) and Barbie, Hagedorn, and Kaul (2003, 2004) while for the role of agent heterogeneity, see e.g. Conessa and Krueger (1999). We extend Bohn's analysis by incorporating endogenous growth so that we can also explore the optimal response of long-term growth to various productivity and demographic shocks. Moreover, we integrate various elements that Bohn has studied in separate papers, such as endogenous labor supply (see Bohn (1998)), productivity shocks (in labor productivity, factor productivity, and depreciation, see Bohn (1998)) and demographic shocks (fertility and longevity, see Bohn (1999)). We explore also how government transfers between the generations should optimally respond to various shocks and clearly distinguish between predictable intergenerational transfers and optimal risk sharing.

Key to studying issues of risk sharing is the issue of risk aversion. For that reason, we pay particular attention to the distinction between intertemporal substitution and risk aversion, as formulated by Epstein and Zin (1989) or Weil (1990). In fact, as in Bohn (1998), we also allow for risk aversion of the young agents, which one may either read as a desire of the social planner towards insuring the yet-unborn or life-risk aversion of young agents before they are able to participate in market activities. The life risk aversion parameter will play a crucial role.

As in the classic overlapping-generations model of Diamond (1965), at any point in time two generations are alive. Only the older generation participates in the capital market and is thus subject to capital-market risks. The younger generations works and is subject to labor-income risks. These two overlapping generations can not trade risks in the capital market because the young cannot participate in the capital market before the shocks occur and thus cannot insure against the realization of uncertainty at their birth. An alternative interpretation is that the young cannot borrow against their human capital to invest in the capital market. The young who are long on human capital and the old who are long on physical capital thus have to rely on government intervention to trade and diversify risks. To illustrate, through the pension system, the government in effect can create new non-tradable assets that are not traded in financial markets and give the old a claim on labor income and the young a claim on capital income. By endowing the various generations with net positions in these assets, the government can in principle create an insurance equilibrium that would emerge if agents could freely trade ex ante. The government can trade not only risks between these two overlapping generations but also shift risks between current generations and future generations through capital accumulation. In this way, the government can engineer also implicit trades between non-overlapping generations. We thus consider risk sharing both within periods between overlapping generations and across time between non-overlapping generations.

We consider a social planning problem and characterize some necessary properties of optimal risk sharing arrangements across generations. Our economy features three imperfections calling for government intervention. First, the endogenous growth feature implies an externality in capital formation, which calls for an investment subsidy. Second, the intergenerational distribution may not be optimal from the point of view of a social planner, who thus may want to engage in intergenerational redistribution. Third, ex-ante trading in risks between generations is not possible because generations can participate in capital markets only after they are born and most shocks have already materialized. Shifting risks to the groups who can best bear them can thus create an ex-ante Pareto improvement and maintains incentives for risk-taking in general equilibrium. Improved risk sharing generates a Pareto improvement only in an ex-ante sense and not in an ex-post sense. Indeed, thinking about risk sharing before shocks actually hit helps to avoid divisive battles about intergenerational redistribution after the shocks have in fact materialized.

The rest of the paper is structured as follows. After Section 2 formulates the model, section 3 sets up the social planners problem and interprets the conditions for optimal intergenerational risk sharing. Section 4 solves the social planners problem. It investigates the solution to the steady state, showing that the calculation of the equilibrium boils down to solving one equation in labor supply. For the dynamics, it relies on using log-linearization techniques rather than projection algorithms as in Krueger and Kubler (2004) in order to derive insights into the linearized behaviour around the steady state. In particular, it is shown that the dynamics of the model is characterized by a single, endogenous state vector, which combines ex-ante utilities as well as the current capital stock. We thus provide a flexible framework that can be used and calibrated as a workhorse for exploring intergenerational risk sharing with endogenous growth. Section 5 explores how the social optimum can be decentralized through a pension system. Section 6 provides some quantitative explorations of the model and investigates the factors determining optimal intergenerational distribution and risk sharing. Section 7 concludes.

We obtain the following key results.

1. We provide a general characterization of the optimal risk sharing condition, see in particular propositions 1, 2 and 3. A detailed interpretation is provided subsequently to proposition 3.

2. An endogenous growth model is provided, which is fairly tractable, despite featuring endogenous labor supply, a number of different shocks and nonseparable preferences. In passing, we provide the conditions necessary for balanced growth under Epstein-Zin preferences, see section 3.2.1 and appendix A.2. In special cases, it is possible to characterize the steady state completely, while it generally requires solving a one-dimensional equation characterizing steady state labor supply, see section 4 and in particular equation (62) and appendices A.4.5 and A.4.7.

3. We use log-linearization techniques to characterize the response to shocks, see propositions 2 and 3 as well as appendices A.1 and A.4.8, including implications for risk premia, see appendix A.3. It should be understood that this is a "small-shock" approximation or an approximation to shocks with bounded support, see Samuelson (1970) and the discussion in Judd and Guu (2001). Despite the limitations of log-linear approximations, this provides a useful starting point for further explorations of risk-sharing with higher-order numerical techniques, or, alternatively, for searching for interesting special cases, which can be solved analytically.

4. The endogenous dynamics can be shown to solve a third-degree polynomial in a single state variable, with typically only one stationary solution. This is shown explicitely in appendix A.5. Thus, the model is suitable for exploring a rich set of features while keeping the amount of analytics moderate, providing a starting point and intuition for models featuring a larger number of periods of life.

5. A number of interesting insights emerge from the quantitative exploration of the model (see section 6). The usual caveat, that these statements are strictly valid only for the parameter ranges investigated (and should not be read as globally valid theorems) applies.

   (a) At a benchmark parameterization, the old bear a smaller burden of the risk in productivity surprises, if old-age risk-aversion is larger than life risk aversion, but a larger one, if the old-age risk aversion is smaller, see equation (53) and section 6.5 . Thus, whether young agents should insure the old agents against the random returns on stock markets due to the uncertainty in technological progress depends on how one views the risk aversion of old people versus the desire to ensure the yet-unborn against their "life-risk". Note how a fully-funded defined-contribution system, where the old save for retirement by holding equity, would lead to an equal sharing of the productivity risk between young and old in the case of full depreciation, while a defined-benefit system would impose the entire risk on the young. Neither solution is typically optimal from a social planners perspective, although the defined contribution private capital system seems to be optimal in some special circumstances with respect to productivity shocks that equally affect labor and capital income.

   (b) Consumption of the young and the old always move in the same direction, even for a population growth shock, where the size of the new young generation is larger than expected, see section 6.6 . This result is in contrast to the result of a fully-funded decentralized system without risk-sharing between generations: there, the old would receive higher consumption (due to the increase in return to capital), while the young would receive lower consumption (due to decreasing marginal returns in labor from the larger working population).

   (c) Persistent increases in longevity will lead to lower total consumption of the old (and thus certainly lower per-period consumption of the old) and the young as well as higher work effort of the young. The additional resources are used to increase growth and future output, resulting in higher consumption of future generations, see 6.7 . Thus, increases in life-expectancy require old agents not only to get by with less per period but also with less in total, in contrast to what a defined benefit system or standard insurance contracts might offer.

2 The model

2.1 Motivation and Overview

We aim at providing a model in which the key features can be studied, while remaining tractable. As a starting point, we follow Uhlig and Yanagawa (1996) and construct an overlapping generations model with capital and endogenous growth, effectively delivering an AK model. This eliminates capital as a state variable, thus simplifying the analysis. Since we want to study both risk aversion and intertemporal substitution, we keep them separate using Epstein-Zin preferences. As sources of risks, we allow for variations in factor productivities, depreciation, longevity, and population growth.

2.2 The environment

Consider a discrete time overlapping generations model with $t = 1, 2, \ldots$ in which all agents live for two periods, and where random shocks occur at the beginning of each period. For convenience, tables 1 and 2 summarize the symbols. Let $c_{t,y}, c_{t+1,o}$ denote the per-capita consumption of the representative agent born at the beginning of period t when young and when old, respectively. Agents work only during the first period of their lives. n_t denotes labor supply of the generation born in period t. ϖ_{t+1} stands for be the fraction of the second period during which the old are actually alive to enjoy consumption. ϖ_{t+1} is an exogenous variable known at the beginning of period $t+1$. It measures the remaining expected life-time of the old given the medical and demographic knowledge at the beginning of period $t+1$. We define $c_{t+1,o}$ as total consumption of the old in the second period, so that $c_{t+1,o}/\varpi_{t+1}$ represents consumption of the old per unit of time. Given some utility $q(\cdot)$ for old-age consumption per unit of time, total utility when old is

$$U_{t,t+1} = \varpi_{t+1} q \left(\frac{c_{t+1,o}}{\varpi_{t+1}} \right) \tag{1}$$

We wish to distinguish intertemporal elasticity of substitution from risk aversion. We therefore use some concave and differentiable function $x(\cdot)$ to increase the curvature of $U_{t,t+1}$ before taking expectations and undoing this curvature again. Furthermore, we allow for risk aversion to life-time risk (or, alternatively, for a preference of the social planner towards equality across generations). Overall preferences of the generation born at the beginning of time t (conditional on the information that is available during that period) are assumed to be given by

$$\begin{aligned} U_{t,t} &= U(c_{t,y}, n_t, c_{t+1,o}; \varpi_{t+1}) \\ &= z\left(u\left(c_{t,y}, n_t\right) + \beta x^{-1}\left(E_t\left[x\left(U_{t,t+1}\right)\right]\right)\right) \quad (2) \\ &= z\left(u\left(c_{t,y}, n_t\right) + \beta x^{-1}\left(E_t\left[x\left(\varpi_{t+1} q\left(\frac{c_{t+1,o}}{\varpi_{t+1}}\right)\right)\right]\right)\right), \quad (3) \end{aligned}$$

where $u(\cdot,\cdot)$ is an instantaneous utility function in consumption and work effort during the first period of life, and $z(\cdot), x(\cdot)$ and $q(\cdot)$ are strictly increasing and continuously differentiable functions such that $U(.)$ is strictly concave in $c_{t,y}, n_t$ and $c_{t+1,o}$.

As an important special case, consider

$$U_{t,t} = \frac{\left(\left(c_{t,y} v(n)\right)^{1-\eta} + \beta\left(E_t\left[\left(\varpi_{t+1}^{\eta} c_{t+1,o}^{1-\eta}\right)^{\frac{1-\nu}{1-\eta}}\right]\right)^{\frac{1-\eta}{1-\nu}}\right)^{\frac{1-\xi}{1-\eta}}}{1-\xi}. \quad (4)$$

where $\xi \geq 0$ and $\nu \geq \eta \geq 0$, see also equation (44). This specification and its consequences are discussed in greater detail in section 3.3.

For the generation which is old at the beginning of the planning period, $U_{0,1}$ are the preferences at the start of period 1, see equation (1), and

$$\begin{aligned} U_{0,0} &= z(u_0 + \beta x^{-1}\left(E_0\left[x\left(U_{0,1}\right)\right]\right) \quad (5) \\ &= z\left(u_0 + \beta x^{-1}\left(E_0\left[x\left(\varpi_1 q\left(\frac{c_{1,o}}{\varpi_1}\right)\right)\right]\right)\right), \end{aligned}$$

for some parameter u_0 are the preferences of the initially old, when considering them ex ante, i.e. behind the veil of ignorance before the uncertainty about the first period is resolved. We have incorporated $z(\cdot)$ and u_0 as a normalization in order to compare utilities of the initially old and agents living in two periods in the same units, when studying the social planners problem.

Let the population of the young at date t be Π_t, and denote the growth factor of the young population as

$$\pi_t = \frac{\Pi_t}{\Pi_{t-1}}. \tag{6}$$

We use capital letters to denote aggregate variables, so that $C_{t,y} = \Pi_t c_{t,y}$, and so on. With k_{t-1}, we denote the capital stock available for production per old person at date t. Thus, $K_{t-1} = \Pi_{t-1} k_{t-1}$ represents the aggregate capital stock in period t.

The aggregate production function is given by

$$Y_t = A_t K_{t-1} f(Z_t n_t), \tag{7}$$

for some positive, concave, strictly increasing and strictly concave function $f(\cdot)$. Here, A_t stands for a total factor productivity parameter and Z_t represents a labor-specific productivity parameter. Aggregate production Y_t is proportional to the aggregate capital stock. We are thus essentially assuming an AK model. Note that labor per young person n_t rather than aggregate labor $\Pi_t n_t$ appears as an argument of the function $f(\cdot)$. This aggregate AK production function can arise from a decentralized production economy with a production externality that is proportional to the capital stock per young person, K_{t-1}/Π_t, see section 5.

With y_t, we denote aggregate production, divided by the population of the old (i.e. $y_t = Y_t/\Pi_{t-1}$)

$$y_t = A_t k_{t-1} f(Z_t n_t). \tag{8}$$

Aggregate feasibility requires that

$$C_{t,y} + C_{t,o} + K_t = Y_t + (1 - \delta_t) K_{t-1}, \tag{9}$$

where we allow for time-variation in the depreciation rate δ_t. Expressed relative to the older population, this equation reads (after substituting (8) to eliminate y_t)

$$c_{t,y}\pi_t + c_{t,o} + k_t\pi_t = \left(A_t f(Z_t n_t) + 1 - \delta_t\right) k_{t-1}. \tag{10}$$

The vector $(Z_t, A_t, \delta_t, \pi_t, \varpi_t) \in \mathcal{S}$ is stochastic and iid, and where $\mathcal{S} = \mathbf{R}^5_{++}$. k_0 and Π_0 are given and non-random, and we normalize $\Pi_0 = 1$.

$$h_t = ((Z_t, A_t, \delta_t, \pi_t, \varpi_t), \ldots, (Z_1, A_1, \delta_1, \pi_1, \varpi_1)) \tag{11}$$

denotes the history up to and including t. Let $\mathcal{H}_t = \{h_t \mid \text{all } h_t\} = \mathcal{S}^t \subseteq \mathbf{R}^{5t}$ denote the set of all histories up to and including t and let $\mathcal{H} = \bigcup_{t=1}^{\infty} \mathcal{H}_t$ denote their union.

Let some initial level of capital k_{-1}^* be given.

Definition 1 *A* **feasible allocation** *is a mapping from the set of all histories into a vector of positive real numbers, $\Phi : \mathcal{H} \rightarrow \mathbf{R}_+^5$, such that $k_{-1} = k_{-1}^*$ and the vector $(c_{t,y}, c_{t,o}, n_t, k_t) = \Phi(h_t)$ satisfies (10) for all histories $h_t \in \mathcal{H}$.*

Associated with a feasible allocation are ex-post utilities $U_{t,t} = U_t(h_t)$, $U_{0,1} = U_0(h_1)$.

3 The social planners problem

The social planner maximizes the utility of the agents (2) subject to the feasibility constraint (10) given the exogenous stochastic process $(Z_t, A_t, \delta_t, \pi_t, \varpi_t)$. In this connection, one needs to take a stand on whether agents born at the same time but with different histories are different agents or not. If they are different, then "insurance" of young agents against shocks during their period of birth (i.e. redistributing from young agents born in "good" times to young agents born in "bad" times) will typically not constitute a Pareto-improvement. Indeed, this would be a redistribution from good-state agents to bad-state agents.

Definition 2 [PO 1] *A* **non-insurance Pareto optimum** *is a feasible allocation with associated "ex post" utilities $U_{t,t}$ (i.e. (2)) and $U_{0,1}$, see (1), such that no other feasible allocation exists that attains equally large utilities for all histories and generations and a strictly larger utility for at least one history and generation.*

When contemplating insurance, the social planner compares states with different population sizes. In particular, the social planner treats all agents born at the same date and under the same history in the same way and thus assigns the same weight per person across different states of nature. Hence, the "ex-ante" preferences of the social planner for generation $t > 0$ at the beginning of time amount to

$$U_{t,0} = E_0 \left[\Pi_t U_{t,t} \right]. \tag{12}$$

The "ex-ante" preferences of the social planner for the initial old are given by $U_{0,0}$ stated in (5) in view of our normalization $\Pi_0 = 1$.

Definition 3 [**PO 2**] *A* **Pareto optimum with insurance** *is a feasible allocation with associated "ex ante" utilities $U_{0,0}$ (see (5)) and $U_{t,0}$ ((12)), such that no other feasible allocation exists attaining equally large ex-ante utilities for all generations and strictly larger ex-ante utility for at least one generation.*

Clearly, any Pareto optimum with insurance is also a non-insurance Pareto optimum, but not vice versa. As an example, suppose that in some allocation A, a particular generation receives consumption $c - \mu$ if times are bad and $c + \mu$ if times are good, for some $\mu > 0$. Suppose that bad times and good times are equally likely. Suppose finally that some alternative allocation B exists, equivalent in all aspects, except that this generation then receives consumption c in both eventualities. In view of the concavity of the preferences, allocation A could not be a Pareto optimum with insurance, since allocation B provides insurance ex ante. However, allocation A might well be a non-insurance Pareto optimum, since the generation born in good times is considered to be different from the generation born in bad times. Thus, conditions necessary for a non-insurance Pareto optimum are also necessary for a Pareto optimum with insurance. Voluntary schemes can implement non-insurance Pareto optima (see e.g. Blanchard and Weil (2001) and Barbie, Hagedorn and Kaul (2003, 2004)). Insurance optima, in contrast, require compulsory participation to commit generations to the ex-post redistribution that is implied by ex-ante insurance.

Pareto optima with insurance are more interesting for our purposes because the issue of intergenerational risk sharing among the yet-unborn is at the heart of our analysis. One should bear in mind, however, that our analysis involves a choice regarding the welfare weights assigned to agents of the same period born under different histories: their relative weights correspond to the relative probabilities of their specific histories.

To provide a single objective for the social planner, we follow Bohn (2003) in formulating a weighted sum of the ex-ante utilities of the various generations with $(\omega_t)_{t=0}^{\infty}$ as a sequence of (nonstochastic) welfare weights.

Definition 4 (SP) *A* **social optimum** *for welfare weights* $(\omega_t)_{t=0}^{\infty}$ *is a feasible allocation solving*

$$\max_{\{\Phi|\Phi \textit{ feas. alloc.}\}} \sum_{t=0}^{\infty} \omega_t U_{t,0}. \tag{13}$$

This social optimum implements a particular Pareto optimum with insurance. Using (12), we can write the social objective function as

$$\max_{\{\Phi|\Phi \text{ feas. alloc.}\}} \left(\omega_0 U_{0,0} + E_0 \left[\sum_{t=1}^{\infty} \omega_t \Pi_t U_{t,t} \right] \right).$$

To focus on stationary solutions, we implement exponential discounting so that $\omega_t = \omega^t$.

3.1 First-order conditions

Consider the problem of finding a social optimum for welfare weights $\omega_t = \omega^t$. Let $\omega^t \lambda_t$ be the Lagrange multiplier on the aggregate feasibility equation (9) so that $\Pi_{t-1}\omega^t\lambda_t$ is the Lagrange multiplier on the feasibility constraint (10). For ease of notation, assume that random variables can only take one of finitely many values, each with positive probability given any history. Let $\mathcal{H}_t$ be the set of all histories up to and including t, and let Prob(h_t) be its unconditional probability. The Lagrangian then amounts to

$$\begin{aligned} \max_{\{\Phi|\Phi \text{ feas. all.}\}} \{ \omega_0 U_{0,0} + \sum_{t=1}^{\infty} \sum_{h_t \in \mathcal{H}_\sqcup} \text{Prob}(h_t) \omega^t \left[\Pi_t U_{t,t}(h_t) \right. \\ \left. - \lambda_t(h_t) \Pi_{t-1} \left(c_{t,y}\pi_t + c_{t,o} + k_t \pi_t - \left(A_t f(Z_t n_t) + 1 - \delta_t \right) k_{t-1} \right) \right] \}, \end{aligned} \tag{14}$$

We have made the dependence of only λ_t and $U_{t,t}$ on h_t explicit. In fact, the population, their growth rates and all economic choices at date t, like $c_{t,y}$ or k_t, are similarly functions of h_t.

Dropping the argument h_t, the first-order conditions are

$$\left(\frac{\partial}{\partial c_{t,y}} : \right) \qquad \lambda_t = \frac{\partial U_{t,t}}{\partial c_{t,y}}, \tag{15}$$

$$\left(\frac{\partial}{\partial c_{t,o}}:\right) \qquad \lambda_t \,\mathrm{Prob}\,(h_t \mid h_{t-1}) = \frac{1}{\omega}\frac{\partial U_{t-1,t-1}}{\partial c_{t,o}}, \tag{16}$$

$$\left(\frac{\partial}{\partial n_t}:\right) \qquad \lambda_t A_t k_{t-1} f'(Z_t n_t) Z_t = -\pi_t \frac{\partial U_{t,t}}{\partial n_t}, \tag{17}$$

$$\left(\frac{\partial}{\partial k_t}:\right) \qquad \lambda_t = \omega E_t \left[\lambda_{t+1}\left(A_{t+1} f(Z_{t+1} n_{t+1}) + 1 - \delta_{t+1}\right)\right]. \tag{18}$$

In equation (16) we have been a bit more formal than usual. A more general treatment would involve treating this equation as an equality between measures or their Radon-Nikodym derivatives. The right hand side of that equation involves taking the derivative of a conditional expectation with respect to one of its arguments. To be more explicit (and after cancellations of the conditional probability terms on both sides of the equation), (16) can be rewritten as

$$\lambda_t = \frac{\beta}{\omega} d_t q'\left(\frac{c_{t,o}}{\varpi_t}\right) \tag{19}$$

where

$$d_t = z'(a_{t-1})\left(x^{-1}\right)'\left(E_{t-1}\left[x\left(\varpi_t q\left(\frac{c_{t,o}}{\varpi_t}\right)\right)\right]\right) x'\left(\varpi_t q\left(\frac{c_{t,o}}{\varpi_t}\right)\right) \tag{20}$$

is a "discounting correction" due to the separation of intertemporal substitution and risk aversion as well as allowing for life risk aversion: for the definition of $z'(a_{t-1})$, see equation (28) below.

Define w_t to be the marginal social productivity per unit of labor:

$$w_t \equiv A_t k_{t-1} f'(Z_t n_t) \frac{Z_t}{\pi_t}. \tag{21}$$

The first and third first-order condition together imply

$$\frac{\partial u(c_{t,y}, n_t)}{\partial c_{t,y}} w_t = -\frac{\partial u(c_{t,y}, n_t)}{\partial n_t}, \tag{22}$$

which is the familiar condition that the marginal rate of substitution between consumption and leisure should equal its social opportunity costs, i.e. the aggregate marginal rate of transformation.

Define the stochastic discount factor of the social planner as

$$m_{t+1} = \frac{\omega \lambda_{t+1}}{\lambda_t}, \tag{23}$$

and the social rate of return as

$$R_t = A_t f(Z_t n_t) + 1 - \delta_t. \tag{24}$$

We can then write the fourth first-order condition as a familiar asset price equation

$$1 = E_t[m_{t+1} R_{t+1}].$$

Substitute λ_t from (15), λ_{t+1} from (19), and $A_t f(Z_t n_t) + 1 - \delta_t$ from (24) into (18) to obtain

$$z'(a_t)\frac{\partial u}{\partial c_{t,y}} = \beta E_t \left[d_{t+1} q' \left(\frac{c_{t+1,o}}{\varpi_{t+1}} \right) R_{t+1} \right] \tag{25}$$

or explicitely

$$\begin{aligned} \frac{\partial u(c_{t,y}, n_t)}{\partial c_{t,y}} &= \beta \left(x^{-1}\right)' \left(E_t \left[x \left(\varpi_{t+1} q \left(\frac{c_{t+1,o}}{\varpi_{t+1}} \right) \right) \right] \right) \cdot \\ & E_t \left[x' \left(\varpi_{t+1} q \left(\frac{c_{t+1,o}}{\varpi_{t+1}} \right) \right) q' \left(\frac{c_{t+1,o}}{\varpi_{t+1}} \right) R_{t+1} \right]. \end{aligned} \tag{26}$$

(25) is the familiar asset pricing equation in terms of marginal utilities of generation t, adjusted with the discounting-correction term d_{t+1} and longevity risk ϖ_{t+1}.

3.2 Risk sharing

The first two first-order conditions imply

$$\lambda_t \text{Prob}\left(h_t \mid h_{t-1}\right) = \frac{\partial U_{t,t}}{\partial c_{t,y}} \text{Prob}\left(h_t \mid h_{t-1}\right) = \frac{1}{\omega} \frac{\partial U_{t-1,t-1}}{\partial c_{t,o}}, \tag{27}$$

which is a risk sharing or complete markets condition: young and old agents alive at the same time should evaluate risk using the same stochastic discount factor. Let a_t be the argument of the $z(\cdot)$-function in the specification of the utility function (2), i.e.

$$a_t = u\left(c_{t,y}, n_t\right) + \beta x^{-1} \left(E_t \left[x \left(\varpi_{t+1} q \left(\frac{c_{t+1,o}}{\varpi_{t+1}} \right) \right) \right] \right) \tag{28}$$

One can then write the risk sharing condition (27) as follows.

Proposition 1 **[The risk-sharing condition.]** *At the social planner optimum with constant discounting of the welfare of future generations,*

$$z'(a_t)\frac{\partial u(c_{t,y}n_t)}{\partial c_{t,y}} = z'(a_{t-1})\frac{\beta}{\omega}d_t q'\left(\frac{c_{t,o}}{\varpi_t}\right) \tag{29}$$

where d_t is defined in (20) and a_t is defined in equation (28).

Proof: *Direct.* •

In the special case of a linear $z(\cdot)$, the terms $z'(a_t)$ and $z'(a_{t-1})$ drop out and one obtains the familiar condition, equating marginal utility of consumption for the young to a constant factor times marginal utility of consumption for the old "state by state", and adjusted with the risk curvature term.

In the general case, log-linearization helps to deliver further insights. For the variables in (29), denote with an upper bar some benchmark value for each parameter, which we assume to be known as of date $t-1$ or possibly earlier, and which together satisfy equation (29). We use an inverted hat to denote the logarithmic deviation of a variable from this benchmark. Thus, for example,

$$c_{t,y} = \bar{c}_{t,y}\exp\left(\check{c}_{t,y}\right)$$

For any twice differentiable function $y = f(x)$, define the logarithmic derivative

$$\ell_f = \frac{f'(x)x}{f(x)} = \frac{\partial \log(f(\exp(\log(x))))}{\partial \log(x)}$$

with the latter equation valid only if $x > 0$ and $y > 0$. Further, define the negative logarithmic derivative of its first derivative per

$$\mu_f = -\frac{f''(x)x}{f'(x)}$$

For example, if $f(x) = cx^\alpha$, then $\ell_f = \alpha$ and $\mu_f = 1-\alpha$. Often, μ_f arises as an elasticity.

We shall keep a subindex t to denote the period for which the approximation applies. By assumption, the approximation is around a point known at $t-1$ or earlier. For functions with two arguments, we shall also note the argument(s) with respect to which the derivative is taken. For example, for some given utility function u, the intertemporal elasticity of substitution is given by the inverse of $\mu_{uc,c,t}$, whereas the cross-partial is given by

$$\mu_{uc,n,t} = -\frac{u_{nc}(\bar{c}_{t,y}, \bar{n}_t)\bar{n}_t}{u_c(\bar{c}_{t,y}, \bar{n}_t)}.$$

Furthermore, define the argument share

$$\alpha_t = \frac{u(\bar{c}_{t,y}, \bar{n}_t)}{\bar{a}_t}. \tag{30}$$

A complete, explicit list of all logarithmic derivatives can be found at the beginning of appendix A.1.

Proposition 2 [The loglinear risk-sharing condition.] *Optimal intergenerational risk sharing implies the following first-order approximation to (29) in log-deviations around the chosen benchmark,*

$$\begin{aligned} & \mu_{z,t}\check{a}_t + \mu_{uc,c,t}\check{c}_{t,y} + \mu_{uc,n,t}\check{n}_t \\ & = \quad \mu_{z,t-1}\check{a}_{t-1} + \check{d}_t + \mu_{q,t}\left(\check{c}_{t,o} - \check{\varpi}_t\right) \end{aligned} \tag{31}$$

where

$$\begin{aligned} \check{a}_t &= \alpha_t\left(\ell_{u,c,t}\check{c}_{t,y} + \ell_{u,n,t}\check{n}_t\right) \\ &\quad +(1-\alpha_t)\left((1-\ell_{q,t+1})E_t[\check{\varpi}_{t+1}] + \ell_{q,t+1}E_t[\check{c}_{t+1,o}]\right) \end{aligned} \tag{32}$$

$$\check{d}_t = \mu_{x,t}\left((1-\ell_{q,t})\left(\check{\varpi}_t - E_{t-1}[\check{\varpi}_t]\right) + \ell_{q,t}\left(\check{c}_{t,o} - E_{t-1}[\check{c}_{t,o}]\right)\right) \tag{33}$$

Proof: *Calculate or follow the calculations in appendix A.1.* •

Consider the standard case in which both $z(\cdot)$ and $x(\cdot)$ are linear (so that $\mu_{x,t} = \mu_{z,t} = \mu_{z,t-1} = 0$) and in which old and young agents feature the same risk aversion in consumption, i.e. $\mu_{q,t} = \mu_{uc,c,t}$). If additionally, labor is assumed to be constant (i.e. $\check{n}_t = 0$) or $u(c, n)$ is assumed to be separable in c and n (i.e. $\mu_{uc,n,t} = 0$), one obtains

$$\check{c}_{t,y} = \check{c}_{t,o} - \check{\varpi}_t, \tag{34}$$

In that case, the percentage changes of the consumption of the young as well as of the old per unit of time should be exactly the same. One may want to consider this as a natural benchmark for risk sharing between young and old.

If labor is not assumed to be constant and $u(c, n)$ is not separable, equation (31) becomes instead

$$\check{c}_{t,y} + \frac{\mu_{uc,n,t}}{\mu_{uc,c,t}} \check{n}_t = \check{c}_{t,o} - \check{\varpi}_t, \tag{35}$$

where one should note that $\mu_{uc,n,t}$ can have either sign. Consider the case where consumption and leisure are substitutes and hence, $\mu_{uc,n,t} < 0$. If circumstances are such that the social planner commands more work from the young, $\check{n}_t > 0$, the associated decline in leisure enjoyed by the young will make a marginal unit of consumption for the young relatively more valuable. In order to keep the total increase in marginal utility from consumption the same for both generations, the relative consumption decrease of the young should be smaller than the relative consumption decrease of the old per unit of time (i.e. $\check{c}_{t,y} > \check{c}_{t,o} - \check{\varpi}_t$) . The direction reverses if leisure and consumption are complements, $\mu_{u,t,nc} > 0$. Then, changes in leisure of the young (i.e. $-\check{n}_t$) are associated with changes in relative consumption of the young vis-a-vis the old, $\check{c}_{t,y} - (\check{c}_{t,o} - \check{\varpi}_t)$, which are of the the same sign. With complementarity between leisure and consumption, changes in leisure are associated with changes in consumption that further increase the impact of leisure on the overall utility level of the young. Intuitively, decreases in leisure reduce the marginal utility of the young so that the young not only obtain less leisure but also get less consumption relative to the old.

3.2.1 Scale Invariance and Balanced Growth

We wish to avoid effects from rescaling the units in which e.g. capital is measured. An alternative interpretation is that we seek a solution delivering a balanced growth path. Thus,

Definition 5 *Let the allocation*

$$\Phi(h_t) = (c_{t,y}, c_{t,o}, n_t, y_t, k_t)$$

be a solution to the social planners problem, given initial capital k^*_{-1}. *Let* $\alpha_t(h_t)$ *be the implied argument share, see equation (30). The preference* *specification*[1] *is said to be* **scale invariant***, if for all scalars* $\phi > 0$*,* $\tilde{\Phi}$ *is a solution to the planners problem with initial capital* ϕk^*_{-1}*, where*

$$\tilde{\Phi}(h_t) = (\phi c_{t,y}, \phi c_{t,o}, n_t, \phi y_t, \phi k_t)$$

and if additionally the implied argument share $\tilde{\alpha}_t$ *is unchanged (i.e.* $\tilde{\alpha}_t(h_t) = \alpha_t(h_t)$*) for all histories* h_t*.*

Scale invariance yields a number of implications (see the appendix):

$$\frac{\ell_{u,c,t}}{\ell_{u,n,t}} = -\frac{c_{t,y}}{w_t n_t}, \tag{36}$$

$$0 = -\mu_{uc,c,t} + \mu_{uc,n,t}\frac{\ell_{u,c,t}}{\ell_{u,n,t}} + 1, \tag{37}$$

$$\mu_{uc,c,t} = \mu_{q,t+1}, \tag{38}$$

$$\ell_{u,c,t} = \ell_{q,t+1}. \tag{39}$$

If in addition we impose that $\mu_{uc,c,t}$ is constant over time (i.e. $\eta \equiv \mu_{uc,c,t}$), we can derive a semi-closed form for the utility function $u(c,n)$, as is well-known from the literature, see e.g. King and Plosser (1989):

$$u(c_{t,y}, n_t) = \frac{(v(n_t)c_{t,y})^{1-\eta}}{1-\eta} \tag{40}$$

up to a constant, if $\eta \neq 1$ and

$$u(c_{t,y}, n_t) = \log c_{t,y} + v(n_t) \tag{41}$$

for $\eta = 1$.

3.3 Epstein-Zin Preferences

We now provide a particular parametric example, satisfying the general implications listed in subsection 3.2.1. This parametric example is therefore "special" only insofar the various elasticities and logarithmic derivatives have been assumed to be constant.

[1] Note that the production side is already made "scale invariant" by construction, i.e. our assumption about AK-type endogenous growth.

We use the preference specification of (40), i.e.

$$u(c,n) = \frac{(c\,v(n))^{1-\eta}}{1-\eta} \tag{42}$$

with $\eta \neq 1$. We restrict $v(n)$ to be strictly positive and such that $u(c,n)$ satisfies strict concavity and strict monotonicity. Define its logarithmic derivative,

$$\ell_v(n) = \frac{v'(n)n}{v(n)}. \tag{43}$$

Note that $\ell_v(n) < 0$. As before, we let

$$\ell_{v,t} = \ell_v(\bar{n}_t),$$

where $\bar{n}_t$ is some chosen benchmark.

The utility function is

$$U_{t,t} = \frac{\left((c_{t,y} v(n_t))^{1-\eta} + \beta \left(E_t \left[\left(\varpi_{t+1}^{\eta} c_{t+1,o}^{1-\eta} \right)^{\frac{1-\nu}{1-\eta}} \right] \right)^{\frac{1-\eta}{1-\nu}} \right)^{\frac{1-\xi}{1-\eta}}}{1-\xi}. \tag{44}$$

In terms of our general specification,

$$\begin{aligned} q(c) &= \frac{c^{1-\eta}}{1-\eta}, \\ x(q) &= \frac{((1-\eta)q)^{\frac{1-\nu}{1-\eta}}}{1-\nu}, \\ z(a) &= \frac{((1-\eta)a)^{\frac{1-\xi}{1-\eta}}}{1-\xi}. \end{aligned} \tag{45}$$

The argument share α_t is given per

$$\frac{1}{\alpha_t} = 1 + \beta \bar{\varpi}_{t+1}^{\eta} \left(\frac{\bar{c}_{t+1,o}}{\bar{c}_{t,y} v(\bar{n}_t)} \right)^{1-\eta}.$$

3.3.1 Preferences and the length of life.

If $\eta < 1$ and given a total amount $c_{t+1,o}$ of consumption when old, agents prefer a longer life with a fairly small amount of consumption per unit of time to a shorter life with a fairly large amount of consumption per unit of time. This is not so, however, if $\eta > 1$ with the preference specification above, and indeed, the utility function then even has the feature, that old agents strictly prefer a shorter life to a longer life. Put differently, committing suicide is optimal. This is obviously an undesirable feature. As a further consequence, note that with the preferences as defined above, the limit for $\eta \to 1$ is not well defined.

There are at least three ways to resolve this conundrum. One possibility is to modify preferences when old so as to generate strictly positive utility, whenever the agent is alive. This can be done e.g. by modifying

$$q(c) = \frac{(c + \underline{c})^{1-\eta} - \underline{c}^{1-\eta}}{1-\eta}, \tag{46}$$

for some "baseline" level of consumption $\underline{c} > 0$. With this, total utility while old is

$$\begin{aligned} \varpi q\left(\frac{c}{\varpi}\right) &= \varpi\left(\frac{\left(\frac{c}{\varpi} + \underline{c}\right)^{1-\eta} - \underline{c}^{1-\eta}}{1-\eta}\right) \\ &\to \varpi\left(\log\left(\frac{c}{\underline{c}\varpi} + 1\right)\right) \ (\eta \to 1) \end{aligned}$$

which exists and is always guaranteed to be positive, but has a nonconstant elasticity in c as consumption grows (holding $\underline{c}$ constant).

The second possibility is to let agents compare utility of consumption c while alive to some benchmark level of consumption $\underline{c}$ (possibly dependent on the date t), while alive, and otherwise assign zero utility,

$$q(c) = \frac{c^{1-\eta} - \underline{c}^{1-\eta}}{1-\eta} \tag{47}$$

so that the total utility for old agents is given by

$$\begin{aligned} \varpi q\left(\frac{c}{\varpi}\right) &= \varpi\left(\frac{\left(\frac{c}{\varpi}\right)^{1-\eta} - \underline{c}^{1-\eta}}{1-\eta}\right) \\ &\to \varpi\left(\log\left(\frac{c}{\underline{c}\varpi}\right)\right) \ (\eta \to 1) \end{aligned}$$

which exists and has a constant elasticitiy as c grows, but may no longer be guaranteed to be always positive.

The third possibility is to restrict $\eta < 1$ for the original preference specification. At the expense of generality but at the gain of some simplification for the analytics, we shall proceed with this restriction.

In either of these scenarios, the social planner effectively contemplates insurance against the utility loss (or gain!), when dead. Put differently, if there is a longevity shock, the old are happier simply due to living longer. Hence, the social planner may seek to redistribute the gains in utility of the old also to the young.

These insurance motives, while present in this paper, merit deeper philosphical thought. For suppose some medical treatment could be found, which extends the life of the old, but is rather costly. How much should it be worth to society and who should pay for it? This depends rather crucially on the additional utility generated due to being alive. For the third possibility, i.e. our original specification and with $\eta < 1$, the additional utility generated is bounded.

With $\eta > 1$ and the specification (46), the utility gain is unbounded, as $\underline{c} \rightarrow 0$, i.e., death can be made to be arbitrarily unattractive. In that case, the cost limit ultimately is the entire GDP. To see this, suppose, total average consumption of the old is given by $c = c(\varpi)$, reflecting the cost of improving longevity and the opportunity costs of transferring consumption from the young to the old. The first-order condition with respect to ϖ is given by

$$0 = \frac{\left(\frac{c(\varpi)}{\varpi}\right)^{1-\eta} - \underline{c}^{1-\eta}}{1-\eta} - \left(\frac{c(\varpi)}{\varpi}\right)^{1-\eta}(1-\ell_c) \tag{48}$$

where

$$\ell_c = \frac{c'(\varpi)\varpi}{c(\varpi)}$$

For any value $c(\varpi)/\varpi > 0$ and ℓ_c, there is some $\underline{c} > 0$, so that this derivative is positive, i.e., unless one obtains direction observations on choices involving the length of life, there is always some specification for the "fear of death", parameterized by $\underline{c}$, which would justify additional spending on life-prolonging measures, no matter how costly.

Ultimately, from an economic perspective, this raises the need for measuring the aversion of agents against death in data, and using it to calibrate these preferences. There are a number of activities, where agents clearly trade off the risk of dying against some utility-enhancing activity. For example, driving cars at a higher speed entails a higher risk of dying, and smoking causes people to die at an earlier age. By carefully measuring the utility gain (and measuring marginal changes, e.g. due to the change in the price for cigarettes or due to higher safety of cars), one may be able to calculate an economic value for death. This provides appropriates limits for the amount of resources which should be spent on extending lifes. The difficult discussions, when economic reasoning of this sort combined with ethic and moral judgements provide guidelines for the share of GDP to be spent on health care or, more drastically, turning off life-support measures for a terminally ill patient has only begun, and will surely intensify in the future, as the technological possibilities advance, see e.g. Murphy and Topel (2002) or Hall and Jones (2004).

3.3.2 Optimal risk sharing

Proposition 3 [Risk-Sharing with Epstein-Zin preferences.] *With the preferences given by (42) and (44) and up to a first-order approximation in the log-deviations around a chosen benchmark, optimal intergenerational risk sharing implies*

$$\begin{aligned}(\xi-\eta)\check{a}_t^* + \eta\check{c}_{t,y} - (1-\eta)\ell_{v,t}\check{n}_t & \\ = \ (\xi-\eta)\check{a}_{t-1}^* + \check{d}_t + \eta(\check{c}_{t,o} - \check{\varpi}_t), & \end{aligned} \tag{49}$$

where

$$\check{a}_t^* = \alpha_t\left(\check{c}_{t,y} + \ell_{v,t}\check{n}_t\right) + (1-\alpha_t)\left(\frac{\eta}{1-\eta}E_t[\check{\varpi}_{t+1}] + E_t[\check{c}_{t+1,o}]\right). \tag{50}$$

$$\check{d}_t = (\nu-\eta)\left(\frac{\eta}{1-\eta}\left(\check{\varpi}_t - E_{t-1}[\check{\varpi}_t]\right) + \left(\check{c}_{t,o} - E_{t-1}[\check{c}_{t,o}]\right)\right) \tag{51}$$

where

$$\check{a}_t^* = \left(\frac{\check{a}_t}{1-\eta}\right).$$

Proof: *Note first, that for any function $f(x) = x^\alpha$, we have $\ell_f = \alpha$ and $\mu_f = 1-\alpha$. Leaving away the time subscript t except for $\mu_{uc,n,t}$, $\ell_{u,n,t}$ and*

$\ell_{v,t}$, since everything else is now constant, calculate

$$\begin{aligned}
\mu_z &= \frac{\xi - \eta}{1 - \eta}, \\
\mu_{uc,c} &= \eta, \\
\mu_{uc,n,t} &= -(1 - \eta)\ell_{v,t}, \\
\mu_x &= \frac{\nu - \eta}{1 - \eta}, \\
\mu_q &= \eta
\end{aligned}$$

and

$$\begin{aligned}
\ell_{u,c} &= 1 - \eta, \\
\ell_{u,n,t} &= (1 - \eta)\ell_{v,t}, \\
\ell_x &= \frac{1 - \nu}{1 - \eta}, \\
\ell_q &= 1 - \eta.
\end{aligned}$$

Substitute into equation (31). •

Note that we assume $\eta < 1$, as discussed in section 3.3.1. Thus, leisure and first-period consumption are complements and $\mu_{u,nc,t} < 0$.

To provide some intution for the risk sharing condition, it may be useful to consider the following six special cases.

1. Assume: no Epstein-Zin, constant labor. Result: Perfect correlation of per-period consumption,

$$\check{c}_{t,y} = \check{c}_{t,o} - \check{\varpi}_t$$

2. Assume: additionally, endogenous labor. Result: Since $\eta < 1$, i.e., since leisure and consumption of the young are complements. leisure moves in the same direction as consumption of the young relative to per-period consumption of the old. To see this, rewrite

$$\check{c}_{t,y} - \frac{1 - \eta}{\eta}\ell_{v,t}\check{n}_t = \check{c}_{t,o} - \check{\varpi}_t$$

as

$$\frac{1-\eta}{\eta}((-1)*\ell_{v,t})((-1)*\check{n}_t)=\check{c}_{t,y}-(\check{c}_{t,o}-\check{\varpi}_t)$$

The intuition was discussed subsequent to proposition 2.

3. Assume: the old are risk-averse, $\nu > \eta$, while there is no additional life risk aversion, $\xi = \eta > 0$. Benchmark uses information up to $t-1$. Result: The old bear less of the risk. Longevity enters separately from old-age consumption.

$$\check{c}_{t,y}-\frac{1-\eta}{\eta}\ell_{v,t}\check{n}_t=\frac{\frac{\nu}{\eta}-1}{1-\eta}\check{\varpi}_t+\frac{\nu}{\eta}(\check{c}_{t,o}-\check{\varpi}_t)$$

4. Assume: the old are risk-averse, $\nu > 0$. The young are risk neutral, $\xi = \nu = 0$. Assume constant labor. Result: The young bear all the risk,

$$0=\check{c}_{t,o}-E_{t-1}[\check{c}_{t,o}]$$

5. Assume: infinite elasticity of intertemporal substitution, $\eta = 0$, constant labor, equal risk aversion of young and old, $\nu = \xi > 0$. Evaluate risk sharing based on information up to $t-1$. Result: the life-time consumption of young reacts as much as consumption of old,

$$\alpha_t\check{c}_{t,y}+(1-\alpha_t)E_t[\check{c}_{t+1,o}]=\check{c}_{t,o}-E_{t-1}[\check{c}_{t,o}]$$

6. Assume: infinite elasticity of intertemporal substitution, no risk aversion of the old $\eta = \nu = 0$, constant labor, but risk aversion of young, $\xi > 0$, Evaluate the risk sharing condition based on information up to and including $t-1$. Result: The old bear all the risk,

$$0=\alpha_t\check{c}_{t,y}+(1-\alpha_t)E_t[\check{c}_{t+1,o}]$$

The parameter ξ is most sensibly interpreted as life risk aversion. Suppose $\check{c}_{t,o} = E_{t-1}[\check{c}_{t,o}]$, i.e., old-age consumption changes are predictable when these agents are young, and suppose that $\check{\varpi}_t \equiv 0$. Then, the right-hand side of equation (49) can be written as

$$(\xi-\eta)\alpha_{t-1}\left(\check{c}_{t-1,y}+\ell_{v,t}\check{n}_{t-1}\right)+\left((\xi-\eta)(1-\alpha_{t-1})+\eta\right)\check{c}_{t,o}.$$

The relevant scaling parameter on the rate of change for consumption of the old $\check{c}_{t,o}$ is given by a combination of the curvature parameter η of the utility of the young and life risk aversion ξ. The curvature parameter of the young comes in because, with a foreseeable shock, the intertemporal elasticity of substitution between the two periods of life rather than the post-birth old-age risk aversion ν matters.

If relative consumption changes are the same, $\check{c}_{s,o} = \check{c}_{s-1,y}$, for $s = t$ and $s = t-1$, and if labor stays constant, $\check{n}_t = \check{n}_{t-1} = 0$. then $\check{a}^*_s = \check{c}_{s,y} = \check{c}_{s+1,o}$ for $s = t$ and $s = t-1$. and the risk-sharing condition (49) for a foreseeable shock becomes

$$\check{c}_{t,y} = \check{c}_{t,o}. \tag{52}$$

as in (34). I.e., young and old should share consumption risk equally (in proportion to their benchmark level), when faced with a foreseeable shock. Their relevant risk aversion is now the life risk aversion ξ, which is the same for both generations.

With unanticipated changes, by contrast, i.e. if $E_{t-1}[\check{c}_{t,o}] = \check{c}_{t,y} = 0$, and assuming fixed employment and longevity, as well as $\check{c}_{t,y} = E_t[\check{c}_{t+1,o}]$, the risk-sharing condition (49) becomes

$$\xi \check{c}_{t,y} = \nu \check{c}_{t,o}. \tag{53}$$

Whereas old-age risk aversion ν is relevant for consumption of the old in this case, life risk aversion ξ applies to the young. Indeed, in contrast to the young who can change all the arguments of their utility function, the old can adjust only their old-age consumption in response to an unanticipated shock. Eqation (53) is a version of one of the results announced in the introduction: the old bear a proportionally larger share of the consumptions risk, iff their risk-aversion is lower than the corresponding life risk aversion. In section **??** we shall see that this result continues to hold approximately for reasonable parameters, when preferences towards leisure are introduced.

4 Solving the social planners problem and parametric choices

Assume from now on that preferences are given by equation (44).

To turn the endogenous variables into a stationary system, define

$$\begin{aligned}
\tilde{c}_{t,y} &\equiv \frac{c_{t,y}}{k_{t-1}}, \\
\tilde{c}_{t,o} &\equiv \frac{c_{t,o}}{k_{t-1}}, \\
\tilde{w}_t &\equiv \frac{w_t}{k_{t-1}}, \\
\tilde{y}_t &\equiv \frac{y_t}{k_{t-1}}, \\
\tilde{k}_t &\equiv \frac{k_t}{k_{t-1}}, \\
\tilde{x}_t &\equiv \frac{x_t}{k_{t-1}},
\end{aligned}$$

except that we do not detrend n_t and that we define

$$\begin{aligned}
\tilde{a}_t &\equiv \frac{a_t}{k_{t-1}^{1-\eta}}, \\
\tilde{s}_t &\equiv s_t k_t^{\nu-\xi},
\end{aligned}$$

where

$$s_t \equiv ((1-\eta)a_t)^{\frac{\xi-\eta}{1-\eta}} x_t^{\eta-\nu}. \tag{54}$$

The exponent of the normalization variable differs here in order to achieve the appropriate scaling for these utility variables. Moreover note that we scale s_t by $k_t^{\nu-\xi}$ rather than by $k_{t-1}^{\nu-\xi}$ because that way, $\tilde{s}_{t-1}$ (rather than $\tilde{s}_{t-1}$ and $\tilde{k}_{t-1}$) turns out to be the only remaining endogenous state variable.

One can then solve for a steady state in these detrended variables as well calculate the dynamics around this steady state using loglinearization. The details are available in the appendix. Here we shall highlight only some key results.

4.1 Parametric choices

To explicitely calculate the steady state and provide quantitative results, we introduce a number of parametric assumptions regarding preferences, technologies and shocks.

For preferences regarding leisure, we assume specifically, that

$$v(n) = \bar{v}\frac{(1-n)^{1-\chi}}{1-\chi}, \tag{55}$$

with $0 \leq \chi < 1$. Consequently,

$$\begin{aligned} \ell_v(n) &= -(1-\chi)\frac{n}{1-n}, \\ \mu_v(n) &= -\chi\frac{n}{1-n}. \end{aligned}$$

For the production function $f(x)$, we assume

$$f(x) = \left(\theta + (1-\theta)x^{1-\frac{1}{\psi}}\right)^{\frac{1}{1-\frac{1}{\psi}}}, \tag{56}$$

with $0 \leq \theta < 1$ and $\psi > 0$ (where one should note that we usually use $x = Zn$ as argument), and thus

$$\begin{aligned} \ell_f(x) &= \frac{(1-\theta)x^{1-\frac{1}{\psi}}}{\theta + (1-\theta)x^{1-\frac{1}{\psi}}}, \\ \mu_f(x) &= \frac{1}{\psi}(1-\ell_f(x)). \end{aligned}$$

For $\psi \to 1$, this becomes[2]

$$f(x) = x^{1-\theta} \tag{57}$$

with

$$\begin{aligned} \ell_f(x) &= 1-\theta, \\ \mu_f(x) &= \theta. \end{aligned}$$

For the stochastic part, let

$$\zeta_t = \begin{bmatrix} \varpi_t \\ A_t \\ Z_t \\ \pi_t \\ \delta_t \end{bmatrix}$$

[2]This is shown in the appendix.

be the vector of exogenous parameters. We assume a steady state $\bar{\zeta}$ exists and that

$$\hat{\zeta}_t = \log(\zeta_t) - \log(\bar{\zeta})$$

follows an AR(1) process,

$$\hat{\zeta}_t = N\hat{\zeta}_{t-1} + \epsilon_t \tag{58}$$

for some 5×5 matrix N with nonexplosive roots, where

$$\epsilon_t = \begin{bmatrix} \epsilon_{\varpi,t} \\ \epsilon_{A,t} \\ \epsilon_{Z,t} \\ \epsilon_{\pi,t} \\ \epsilon_{\delta,t} \end{bmatrix}$$

is the vector of innovations for each exogenous parameter with

$$\epsilon_t \sim \mathcal{N}(0, \Sigma).$$

4.2 The steady state

For general production functions and preferences, the steady state equations yield the following relationships between the detrended variables,

$$\frac{\bar{\pi}\bar{c}_y}{\bar{y}} = \frac{\ell_f(\bar{Z}\bar{n})}{-\ell_v(\bar{n})}, \tag{59}$$

$$\frac{\bar{c}_o}{\bar{\pi}\bar{c}_y} = \kappa_3 \bar{k}^{\frac{\xi-\eta}{\eta}} v(\bar{n})^{\frac{\eta-1}{\eta}}, \tag{60}$$

$$\bar{k}^{\xi} = \kappa_4 \bar{R} = \kappa_4 \left(\bar{A} f(\bar{Z}\bar{n}) + 1 - \bar{\delta}\right), \tag{61}$$

where

$$\kappa_3 = \frac{1}{\bar{\pi}} \left(\frac{\beta}{\omega}\right)^{\frac{1}{\eta}} \bar{\varpi} \exp\left(\frac{\nu - \eta}{\eta(1-\nu)} \sigma_x^2 / 2\right),$$

and

$$\kappa_4 = \omega \exp\left(\sigma_{R-\nu c}^2 / 2\right),$$

where $\sigma_{R-\nu c}^2$ denotes the conditional variance of $\log R_{t+1} + \varsigma \log(\varpi_{t+1}) - \nu \log(c_{t+1,o})$ and σ_x^2 is the conditional variance of $\varsigma \log \varpi_{t+1} + (1-\nu) \log c_{t+1,o}$ where $\varsigma = \frac{\eta}{1-\eta}(1-\nu)$. These equations provide insight into the fraction of output spent on consumption of the young and the old, as well as the endogenous growth factor $\bar{k}$.

Substituting these equations into the equation on feasibility

$$\frac{\bar{\pi}\bar{c}_y}{\bar{y}}\left(1+\frac{\bar{c}_o}{\bar{\pi}\bar{c}_y}\right)=1+\frac{1-\bar{\delta}-\bar{\pi}\bar{k}}{\bar{A}f(\bar{Z}\bar{n})}$$

leads to a single nonlinear equation in $\bar{n}$,

$$\left(\frac{\ell_f(\bar{Z}\bar{n})}{-\ell_v(n)}\right)\left(1+\kappa_3\left(\kappa_4\left(\bar{A}f(\bar{Z}\bar{n})+1-\bar{\delta}\right)\right)^{\frac{\xi-\eta}{\xi\eta}}v(\bar{n})^{\frac{\eta-1}{\eta}}\right) \tag{62}$$
$$=1+\frac{1-\bar{\delta}-\bar{\pi}\left(\kappa_4\left(\bar{A}f(\bar{Z}\bar{n})+1-\bar{\delta}\right)\right)^{\frac{1}{\xi}}}{\bar{A}f(\bar{Z}\bar{n})}.$$

With the parametric choice for preferences (i.e. $v(n)=\bar{v}\frac{(1-n)^{1-\chi}}{1-\chi}$) and the production function (i.e. $f(x)=\left(\theta+(1-\theta)x^{1-\frac{1}{\psi}}\right)^{\frac{1}{1-\frac{1}{\psi}}}$), the Appendix analyzes a special case in which the steady-state solution for employment can be calculated in closed form.

4.3 The dynamics

After detrending with k_{t-1}, the model no longer contains a state variable, except

$$\tilde{s}_t=s_t k_t^{\nu-\xi}.$$

This variable arises solely from the intertemporal optimization of the social planner, and disappears if e.g. $\eta=\xi$, i.e., if life-time risk aversion equals the inverse of the intertemporal elasticity of substitution. One can think of $\tilde{s}_t$ as representing a utility promise to future generations, or as a device which allows the social planner to share a risk across many generations, if there is additional life-time risk aversion.

Using loglinearization and assuming a recursive law of motion, in which $\hat{s}_{t-1}$ is the only state variable, one can show that the dynamics can be reduced to solving a polynomial of third degree,

$$0=\theta_s+\theta_{s,s}\varphi_{s,s}+\theta_{s,ss}\varphi_{s,s}^2+\theta_{s,sss}\varphi_{s,s}^3 \tag{63}$$

for some coefficients $\theta_s, \theta_{s,s}, \theta_{s,ss}$ and $\theta_{s,sss}$. The explicit calculations can be found in appendix A.5.

Equation (63) generally has three roots, $\varphi_{s,s,i}, i = 1, 2, 3$, and for which closed-form solutions are available. If only one of these roots is stable, i.e. less than one in absolute value, then this is the root we use. If more than one stable root exists, then an additional state variable is needed as generally would be necessary to solve the system. If there are no stable roots, the system is explosive and our baseline assumption that there is a stationary solution to the social planners problem in the detrended variables unjustified.

When computing results below, we use the "toolkit" implementation, see Uhlig (1999), and allow for up to five rather than one endogenous state variable mostly for accounting reasons, see the appendix.

5 Decentralization and generational accounting

The solution of the social planner can be implemented in a decentralized economy using lump sum taxes and transfers on young and old agents and possibly a return subsidy to old agents, when privately holding capital in order to equate the social and the private rate of return.

We assume that production takes place by a competitive sector of firms, renting capital and hiring labor from competitive households, and experiencing an externality in production. More precisely, assume that production by firm j is given by

$$Y_{t,j} = A_t K_{t-1,j} f\left(Z_t \left(\frac{K_{t-1}}{\Pi_t} \right) \frac{N_{t,j}}{K_{t-1,j}} \right) \tag{64}$$

where $N_{t,j}$ is the amount of labor hired by firm j and $K_{t-1,j}$ is the amount of capital rented by firm j. The term $\left(\frac{K_{t-1}}{\Pi_t}\right)$ is an externality, enhancing labor productivity in proportion to the capital available per young person. Assuming that firms hire workers at their marginal product and rent capital at its marginal product on competitive markets, one can easily show that the capital-labor ratio across all firms is the same, and that the aggregrate production function becomes (7).

The private rate of return to investing in capital is given by

$$R_t^{\text{priv}} = R_t^{\text{soc}} - A_t f'(Z_t n_t) Z_t n_t \tag{65}$$

where we recall the social rate of return from equation (24) as

$$R_t^{\text{soc}} = A_t f(Z_t n_t) + 1 - \delta_t. \tag{66}$$

Thus, the private rate of return is diminished by the externality in accumulating capital, which is measured by the total wage payments per unit of capital,

$$A_t f'(Z_t n_t) Z_t n_t = \frac{w_t \Pi_t n_t}{K_{t-1}} = \frac{w_t \pi_t n_t}{k_{t-1}}$$

Note that wages are given by (21),i.e.

$$w_t = A_t k_{t-1} f'(Z_t n_t) \frac{Z_t}{\pi_t}. \tag{67}$$

We assume that the budget constraint of the individual household is given by

$$c_{t,y} + s_t + \tau_{t,y} = w_t n_t \tag{68}$$

$$c_{t+1,o} + \tau_{t+1,o} = (1 + \sigma_{t+1}) R_{t+1}^{\text{priv}} s_t \tag{69}$$

where s_t is private savings at date t, $\tau_{t,y}$ is a lump sum tax when young, $\tau_{t+1,o}$ is a lump-sum tax, when old, and σ_{t+1} is a return subsidy.

We shall focus on two extreme scenarios. In the first scenario - call it the case of "private capital" - all capital is held privately and the government budget balances period by period,

$$s_t = k_t$$

Define the tax share

$$\tau_{t,priv} = \frac{\tau_{t,y}}{w_t n_t} = 1 - \frac{c_{t,y}}{w_t n_t} - \frac{k_t}{w_t n_t} \tag{70}$$

in order to express the magnitude of the lump-sum taxes more intuitively in proportion of the wage earnings of the young. One may want to view the average value of this share as reflecting the desire of the social planner towards redistribution, while its fluctuations may be viewed as fluctuating insurance payments of the young to the old in response to realizations of macroeconomic risks.

In this scenario, it is necessary to subsidize the returns to capital in proportion to capital held by the individual agent, so that the total proportional subsidy amounts to giving the entire production including the capital stock net of depreciation to the capital-holding old agents, and in turn lump-sum taxing the old agents so that the government budget constraint balances,

$$\begin{aligned}\sigma_{t+1,o} &= \frac{R_{t+1}^{\text{soc}}}{R_{t+1}^{\text{priv}}} - 1 \\ &= \frac{w_{t+1}\pi_{t+1}n_{t+1}}{R_{t+1}^{\text{priv}}k_t} \\ \tau_{t+1,o} &= \sigma_{t+1,o}R_{t+1}^{\text{priv}}k_t - \pi_{t+1}\tau_{t+1,y} \\ &= \pi_{t+1}(c_{t+1,y} + k_{t+1})\end{aligned}$$

One interpretation of this last equation is that the generational account balances, i.e. the generational account of the old equals minus the generational account of the young.

In the second scenario - call it the case of "public capital" - all capital is held by pension funds,

$$s_t = 0$$

Thus, the lump sum taxes to be paid by the young are payments to pension funds, which in turn finance old-age consumption. Note that this is a mixture of a pay-as-you-go system and a fully funded system. In that case, the payments by the young to the pension system are

$$\tau_{t,pub} = \frac{\tau_{t,y}}{w_t n_t} = 1 - \frac{c_{t,y}}{w_t n_t} \tag{71}$$

We examine both.

A "defined benefit" system can be viewed as a system, where the old do not bear any of the risk (except perhaps longevity) whereas the old bear the entire risk of random returns in a "defined contribution" system. The analysis here instead considers the degree of optimal risk sharing between young and old. The relationship between the solution to the social planners problem, as investigated here, and these more specific "real world" pension system shall be investigated in future research.

6 Quantitative Results

This section explores the quantitative properties of the model and numerically calculates the reaction of the various quantities to the relevant shocks. We use a "hat" on variables to denote the logarithmic deviation from the expected balanced growth path with $\tilde{s}_t$ constant, see appendix A.4.8 for details.

6.1 Benchmark parameterization

As a benchmark parameterization, pick $\omega = \beta = 0.4$, $\psi = 1$, $\eta = 0.5$, $\bar{\delta} = 1$ and $\xi = \nu = 2$. Set $\chi = .5$, $\theta = 1/3$, $\bar{\varpi} = 1$, $\bar{\pi} = 1$, $\bar{Z} = 1$ and let $\bar{Z}$ such that there is no growth in steady state, $\bar{k} = 1$, requiring $\bar{A} = 2.825$.

The resulting benchmark equilibrium has $\bar{n} = 0.831$, $\bar{c}_y/\bar{y} = 0.27$, $\bar{c}_o/\bar{c}_y = 1.22$, $\bar{R} = 2.5$, which corresponds to an annualized interest rate of 3.1%, assuming that one period lasts 30 years.

The lump sum taxes relative to the wage bill of the young is $\tau_{priv} = -0.65\%$, i.e. in the case of privately held capital, the young agent would receive a fairly neglible subsidy (financed out of a lump-sum tax on the "rich old", who also finance their own return subsidy). The lump-sum payment to the pension fund in case of "publicly held capital" is $\tau_{pub} = 59\%$, i.e. the young would pay somewhat more than half of his wage earnings into the fund.

For the exogenous parameters, we have assumed them to be iid, except for longevity ϖ_t, which we assume to be a random walk. For the latter, it seems plausible that medical progress is permanent. For the other variables, note that e.g. TFP has a unit root due to the endogenous growth feature of our model, even though the TFP parameter A_t (and the labor productivity parameter Z_t) is iid. Also, note that π_t denotes population growth, so that the log of population follows a random walk, if π_t is iid.

6.2 Comparative statics

The sensitivity of these results to three parameters, ξ, δ and $\omega = \beta$ can be seen in table 3. The last row in that table lists the feedback coefficient on the endogenous state $\hat{s}_t$ for the dynamic solution. While the first three columns have risk aversion of young and old the same, the last three columns set the life risk aversion to the intertemporal elasticity of substitution of the young, i.e., do not modify the ex-post utility of the young by a further risk-transformation so that the social intertemporal substitution elasticity coincides with the corresponding private elasticity. Note that the last three columns are fairly similar to the first three columns except for this feedback coefficient and a substantial change for τ_{priv} in case of no depreciation.

Table 3 shows which factors determine optimal intergenerational distribution. In particular, the sign of the variable $\bar{\tau}_{priv}$ indicates to what extent the optimal pension system is funded. A positive value for $\bar{\tau}_{priv}$ indicates that the pension system is in part pay-as-you-go financed. In that case, the generational account for the young is negative (and for the old is positive) in the absence of shocks. There is systematic redistribution from the young to the old.

In the benchmark calculation, systematic redistribution is limited. The benchmark social optimum thus calls for a small amount of systematic redistribution from the old to the young in the decentralized economy.

The second column of Table 3 shows that the systematic redistribution towards the young is increased if the current old become richer compared to the young on account of a lower depreciation rate (more capital income compared to labor income). Note also that the growth rate $\bar{k}$ increases. The higher return on capital (as a result of lower depreciation rate) allows for a higher growth rate. The young save more because of two reasons: a higher return (substitution effect) and higher income (since the old transfer more income to the young). Another way of interpreting the increased systematic redistribution from the old to the young (an inverse PAYG system) is that the pension system is overfunded. Intuitively, funding increases because the return on capital increases and the older generation becomes richer.

A comparison of the second and fifth column of table 3 shows that the additional systematic redistribution in favor of the young due to a lower depreciation rate becomes more substantial if ex-ante risk aversion ξ decreases. The reason is that there is a lower social preference for similar utility levels across generations. Hence, making the future generations better off compared to the current generations (as a result of higher return on capital and thus growth) becomes more attractive and systematic redistribution from the old to the young is increased further. The old keep less of their additional income as a result of higher capital income due to a lower depreciation rate.

Obviously, this systematic redistribution between young and old would be heavily affected by letting the discount factor of the social planner ω and the discount factor of the private agent β differ. As an alternative possibility for comparing steady states, one could calibrate ω and β in such a way that laissez faire in intergenerational distribution is optimal in the absence of shocks, i.e. $\tau_{priv} = 0$. Then, the decentralized economy with private capital yields the correct intergenerational distribution, assuming that the old pay their own investment subsidy.

6.3 Endogenous dynamics

In table 3 and focussing on the last low in the last three columns, note that there is no feedback on the endogenous state (i.e. $\varphi_{s,s}$), if the lifetime risk aversion ξ equals the intertemporal elasticity of substitution η, i.e., if the social intertemporal substitution elasticity coincides with the private intertemporal substitution elasticity.

We are particularly interested in exploring the risk-sharing features as the risk aversion of the old, ν, is varied vis-a-vis the life risk aversion of the young, ξ. Figure 7 shows the dependence of the feedback coefficient $\varphi_{s,s}$ as these two risk-aversion parameters ν and ξ are varied between $\eta = 0.5$ and the value 4 at the upper end. As one can see, the endogenous dynamics depends practically entirely on ξ alone within this two-dimensional variation.

The impact of the state variable $\varphi_{s,s}$ is determined by the divergence between ξ and η and the importance of endogenous labor supply (i.e. the size of χ; if $\chi = 1$, labor supply is exogenous). The more difficult it becomes to substitute across generations compared to intertemporal substitution within generations as measured η, the more shocks are spread out across various generations as indicated by a larger feedback impact of the state variable.

Shifting risks between the old and the young when the shocks hit is sufficient if $\xi = \eta$. In that case, the young affect their saving behavior in the socially optimal way to redistribute between generations and the government does not have to perform any additional redistribution next period.

6.4 Intergenerational risk sharing

Table 4 contains the corresponding feedback coefficients to three shocks: total factor productivity Z, longevity ϖ and population growth π. There, $\varphi_{growth,\cdot}$ refers to the absolute change in the growth rate (rather than relative to the growth factor $\bar{k}$). If e.g. $\varphi_{growth,Z} = 1.28$, then this means that the capital stock will grow by an additional 1.28%, if TFP increases by 1%. Likewise, $\phi_{\tau,priv,Z} = -0.35$ says that the lump sum tax to be paid in case of privately held capital is lowered by 0.35% of the current wage bill, in case TFP increases by 1%.

To shed light on risk sharing, we have included three more quantities. First and second, we have calculated the reaction coefficient for the total tax collection from the young, normalized by the unchanged wage earnings, each for the case of private as well as public capital. In the case of private capital, this is

$$\begin{aligned}\tau_{priv,tot} &= \frac{\pi_t \tau_{t,y}}{\bar{\pi}\bar{w}\bar{n}} \\ &= \frac{\pi_t \left(w_t n_t - c_{t,y} - k_t \right)}{\bar{\pi}\bar{w}\bar{n}}\end{aligned} \tag{72}$$

and its percent change is given by

$$\hat{\tau}_{priv,tot} = \hat{\tau}_{priv} + \bar{\tau}_{priv}(\hat{n} + \hat{w} + \hat{\pi})$$

Note that this coincides with the reaction coefficient for $\hat{\tau}_{priv}$, if $\bar{\tau} = 0$. Likewise,

$$\tau_{pub,tot} = \frac{\pi_t \left(w_t n_t - c_{t,y} \right)}{\bar{\pi}\bar{w}\bar{n}} \tag{73}$$

with

$$\hat{\tau}_{pub,tot} = \hat{\tau}_{pub} + \bar{\tau}_{pub}(\hat{n} + \hat{w} + \hat{\pi})$$

The total $\tau_{priv,tot}$ is the total contribution of the young to a pension system, where capital is held privately. In particular, if it is identical to zero and unaffected by shocks, then this means that the young do not contribute to insuring the old, and that the old have to bear the entire risk to the returns of their capital alone. Put differently, this number indicates additional intergenerational risk sharing compared to a decentralized equilibrium in which the old can save only through capital market (and the old finance their own investment subsidy). τ_{priv} indicates how the government can in fact create new assets to allow trade of risks between generations.

Third, we have also calculated the change in consumption of the old relative to the total change in consumption, i.e., the feedback coefficient for the (loglinearized) quantity

$$rat_t = \frac{c_{o,t}}{c_{tot,t}} = \frac{c_{o,t}}{c_{o,t} + \pi_t c_{y,t}} \tag{74}$$

In log-deviations,

$$\hat{rat}_t = \frac{1}{1 + \frac{\bar{c}_o}{\bar{\pi}\bar{c}_y}} \left(\hat{c}_{o,t} - \hat{c}_{y,t} - \hat{\pi}_t \right) \tag{75}$$

If this quantity is unchanged, i.e., if the feedback is zero, then this means that consumption of the old increases in proportion with overall consumption resources. Generally, $\hat{rat} > 0$ in one of three cases. It happens, if both consumptions rise, but $c_{t,o}$ rises relatively more. It happens, if both consumptions fall, but $c_{t,o}$ falls relatively less. Finally, it happens if $c_{t,o}$ rises and $c_{t,y}$ falls. These three cases ought to be kept in mind when evaluating the results.

The dependence of the shock-reaction of these three quantities as well as labor supply on the two risk aversion parameters ν and ξ are shown in figures 8 to 11, both as a three-dimensional mesh as well as a two-dimensional contour plot.

For the impulse responses, we have used the benchmark parameterization, except for setting $\xi = 1$ rather than $\xi = 2$. This does not change the steady state, but makes for a differential reaction of consumption of the young vis-a-vis consumption of the old, see the discussion below, and therefore for more differentiated impulse response figures. They are shown in figures 1 to 6. The upper left corner shows the response of normalized consumption of the young, old and the capital stock, whereas the lower left corner provides the corresponding "level" variables, i.e. without dividing by aggregate capital. The upper right corner shows the responses of labor, output (normalized) and growth while the lower right corner shows the responses of the lump-sum taxes to be paid by the young, expressed in percent of the steady state wage bill. Likewise, the response of depreciation is expressed in percent of the capital stock, not in percent of the steady state depreciation rate (which is assumed to be zero, anyways). All figures also show the response of the "shocked" variable, i.e. longevity, TFP, labor productivity, population and depreciation as well as the response of the state variable s, as one proceeds from figure 1 to 6.

The lower-left corner pictures show the persistence effect of shocks, due to the endogenous growth feature of the model. However, the effects are usually not an instantaneous adjustment to some new level, but the response is "smeared out" over a few generations, due to the endogenous dynamics of the state variable s_t. The endogenous dynamics itself is plotted in figure 6.

The responses to a shock in TFP are practically the same (up to scaling) to a shock to labor productivity, because we have essentially assumed a Cobb-Douglas production function. For the benchmark calibration and in reaction, the consumption of the young rises somewhat more than the consumption of the old, which can also be seen by a reduction the young are supposed to make to the pension system, i.e. a lowering of their lump-sum taxes. In response to a shock to population growth, consumption falls: essentially, the young now have to make-do with less capital-per-capita than before. Due to the endogenous growth formulation of the model, this effect persists.

6.5 Technology shocks

In the simplest risk sharing case, consumption of the young and old rise by the same percentage, see the first case of the interpretation of the general risk-sharing equation and equation (34). There, this case is obtained if there is no endogenous labor supply and risk-aversion and intertemporal substitution are assumed to be the same for the young and the old, and where there are no life risks. The case of equal risk sharing can arise in the full version of the model as well. Indeed, for the case of a technology shock Z and for the case that $\xi = \nu$, and with full depreciation $\bar{\delta} = 1$, consumption of the young as well as the old rises one-for-one with productivity, and there is no shift in old-age consumption relative to total consumption, as the first block of table 4 shows. The tax payments both for the case of private capital as well as public capital do not react as both labor income (collected by the young) and capital income (collected by the old) rise proportionally with productivity. There is a reaction of the *total* tax collection from the young in case of publicly held capital simply because their wages rise, and because the initial tax rate in case of publicly held capital is not zero.

Whether or not the consumption of the old or the consumption of the young reacts more strongly depends on the ratio between life risk aversion versus risk-aversion when old. The upper-left corner of the contour plot figure 9 shows a straight line of no reaction in the old-consumption-to-total-consumption-ratio for the case $\nu = \xi$. If ν is larger than ξ, i.e., if the old are relatively more risk averse, then the consumption of the old reacts less strongly, and most of the risk is thus born by the young. Indeed, the young will then consume more goods and consume more leisure (as the income effect dominates the substitution effect), which is why labor declines in response, see the upper-right corner of the same contour plot.

To implement the higher sensitivity of the consumption of the young, when the risk-aversion of the young is relatively low, actually requires a additional subsidy from the old to the young in the case of privately held capital or a lower total tax payment in the case of publicly held capital if a positive production shock materializes. This is paid out of the increased return to capital in the hands of the old, as e.g. the upper-left hand corner of figure 10 shows. It is easier to see the intuition for a case of a less-than-expected growth in technology, resulting in lower capital and labor income. In that case, the young have to make up for the low returns on equity by working harder and giving up some of their wage income. In other words, the young bear most of the productivity risk by transferring additional resources to the old in bad times and by collecting additional resources from the old in bad times. Risks are traded in just the opposite way if the old are relatively less risk averse.

The reaction of total consumption is modified by the rate of depreciation: with only partial depreciation, as in the second and fifth column of table 4, some consumption smoothing is possible out of capital so production shocks result in less volatile consumption. At the same time, a positive production results in less additional growth (compared to a full one-percent advantage in growth due to the additional one-percent growth in TFP).

6.6 Population growth shocks

Note that the consumption of the young and the old always move in the same direction for all shocks, as can be seen both from the impulse responses as well as table 4. This is a key difference between optimal risk sharing and decentralized pension systems. For example, in the absence of insurance, a population growth shock will enhance the return to capital, due to the abundance of labor. At the same time, the shock induces lower wages, lower consumption of the young, and higher consumption of the old. This is not the case with optimal risk sharing: both the consumption of the young and the old decrease.

The response of old versus young consumption depends again on the relationship between old-age risk aversion versus life risk aversion. The young need to be actually slightly more risk averse for the relative consumption changes of the old and young to be the same, see the first three columns and the rows for c_o and c_{tot} in table 4.

Labor of the young declines with the population shock: this should be an unsurprising consequence of the substitution effect, as the marginal product of labor falls. Note, however, how the income effect (which may lead young agents to work harder) is offset here due to the risk-sharing arrangement with the currently old.

Note also that the difference between the tax rate τ_{priv}, when not accounting for the changes in wages, population and labor, and the total tax collection $\tau_{priv,tot}$ is fairly minor in all cases, i.e., even if the steady state values for these tax rates are nonzero. The same holds true for the case of publicly held capital. This result comes about, because the additional population growth is almost nearly completely offset by the decline in wages and the change in hours.

6.7 Shocks to Longevity

The reaction to a longevity-shock shows a number of interesting and possibly counterintuitive features. If the young are sufficiently risk-neutral, they are asked to work harder individually, in order to somehow generate additional resources. However, as table 4 shows, these resources do not go to the *currently* old - total consumption for the currently old (and therefore, certainly per-period consumption of the currently old) actually falls! Instead, because the longevity shock is persistent, the additional resources are planted into higher capital for the future, and growth picks up. I.e., the same generation that now is asked to work harder is also the first generation that gets to enjoy the fruits of this additional labor in the form of higher old-age consumption, see also the impulse response in figure 1. The drop in (total) consumption for the currently old is not quite as dramatic as the drop in consumption of the currently young, and thus, consumption is actually shifted in their favor in terms of their share of total consumption.

The intuition for the additional redistribution in favor of future generations in response to increased longevity is as follows. Longer longevity raises the marginal utility of old-age consumption and therefore increases the return on private saving. The higher implicit return on saving makes it more attractive to redistribute towards future generations, especially if the social intertemporal substitution elasticity $1/\xi$ is large. The comparison between the first three columns and the last three columns of Table 4 shows indeed that the growth response to longevity is larger if ξ is smaller. We find a larger growth response to an increase in longevity than Bohn (2003) for two reasons. First of all, our calibration implies a relatively large intertemporal substitution elasticity. Second, in view of the endogenous growth feature of our model, the marginal product of capital does not decline with capital accumulation. Hence, shifting resources over time and between generations can be attractive in response to shocks.

Increased longevity raises not only saving but also work effort. Indeed, the higher marginal utility of consumption on account of a longer expected life makes work more attractive. Note also that the difference between the tax rate τ_{priv}, when not accounting for the changes in wages, population and labor, and the total tax collection $\tau_{priv,tot}$ is fairly substantial, if the steady state values for the tax rates are nonzero, most notably in the fifth column of table 4. The same holds true for the case of publicly held capital. This result comes about, because the fairly change in labor supply by the young triggers additional tax collections in that scenario.

7 Conclusions

We have developed a stochastic endogenous growth with overlapping generations to explore optimal intergenerational risk sharing and redistribution to explore how the pension system can implement optimal intergenerational risk sharing and redistribution between old, young and future generations. Our endogenous growth model is fairly tractable, despite featuring endogenous labor supply, a number of different shocks and nonseparable preferences. Depending on risk aversion of the various generations, the pension system can help to diversify the financial-market risks faced by older generations and the labor-market and human-capital risks faced by younger generations.

A number of interesting insights emerge from the quantitative exploration of the model, as discussed in detail in section 6: Neither a defined benefit system or a defined contribution system is typically optimal from a social planners perspective. In particular, per capita consumption of the young and the old always move in the same direction, even for positive population growth shocks. This result is in contrast to the response of a fully-funded decentralized system to such shocks: there, the old would receive higher per-capita consumption (due to the increase in return to capital), while the young would receive lower per-capita consumption (due to decreasing marginal returns in labor from the larger working population). In contrast to what a defined benefit system offers, higher life-expectancy optimally requires old agents to get by with less resources. Indeed, persistent increases in longevity will lead to lower total consumption of the old (and thus certainly lower per-period consumption of the old) and the young as well as higher work effort of the young. The additional resources are used to increase growth and future output, resulting in higher consumption of future generations.

In future work, we plan to use the model to explore how the intergenerational risk sharing properties of the pension system affect the equity premium. In particular, in a laissez fair equilibrium without intergenerational risk sharing, the equity premium may be quite high as the risk-averse retired generations cannot share capital income risk with other generations. In the presence of a defined-benefit pension system, however, the old can shed some of this risk to younger generations. In effect, the young issue bonds to the older generations and invest the proceeds in the capital market. In this way, the young have become residual claimants of the pension funds and thus share in the investment risk. By thus spreading investment risk more widely over the population and allowing the young to in effect borrow against their human capital to invest in the stock market (see Constantinides, Donaldson and Mehra (2002)), the equity premium can fall (and the risk-free rate can rise) in general equilibrium. Indeed, defined-benefit pension funds may help individuals to implement an optimal life investment plan, which typically involves individuals to heavily borrow in the beginning of their life to invest in the stock market (see Bodie, Merton and Samuelson (1992), Davis, Kubler, and Willen (2002), Jagannathan, and Kocherlakota (1996), and Teulings and de Vries (2006)). These pension plans thus in effect allow the young to participate in the stock market. If households can choose between low-risk (and low-return) and high-risk (and high-return) investments, better intergenerational risk sharing can boost growth by maintaining incentives for risk taking in high-return investments (see Obstfeld (1994) in the context of international risk sharing). At the same time, however, better risk sharing may decrease precautionary saving, thereby reducing capital accumulation and growth. We plan to estimate the potential effects of the pension system in calibrated models.

In allowing intergenerational risk sharing through government intervention, we have assumed that the young cannot participate in capital markets at all to share risks (before the uncertainty during their life time has unfolded). One interpretation is that human capital is not tradable and that the young therefore cannot borrow at all against their human capital to invest in financial capital (see also Constantinides, Donaldson and Mehra (2002)). In practice, however, the young may be able to participate in equity-market risk that materializes during their working career, either by borrowing, by investing all their saving in the risk-bearing capital, or by buying call options. Indeed, capital markets allow in principle for risk-sharing between overlapping generations, especially if the young can borrow. In this regard, our calculations, which assumed only two discrete periods of life, are likely to overstate the potential risk-sharing benefits from defined-benefit pension plans. In future work, we would like to explore how sensitive our results are with respect to alternative assumptions about the extent (including the frequency) which the young can participate in capital markets.

Our analysis has assumed that governments can implement optimal intergenerational risk sharing by committing future generations to a risk sharing contract. For the question of implementability, additional aspects seem crucial to us. First, government intervention may not only help to share market risks but also give rise to new risks. These additional political risks must be traded off against the possible gains in sharing market risks. Second, the government may face serious problems in committing future generations to an optimal complete risk-sharing contract. For one, in a democracy in which current generations have the voting power, the government faces a serious commitment problem; future generations can always opt out. For another, the government faces substantial fundamental uncertainty so that complete contracts are excessively costly. With substantial fundamental uncertainty, discretion rather than rules become optimal.

Optimal risk sharing is sensitive to the magnitude of ex-ante risk aversion ξ versus ex-post risk aversion ν. The latter type of risk aversion may exceed ex-ante risk aversion in the presence of habit formation (see also Bohn (2003)). Habit formation, however, gives rise to new phenomena. Another explanation for high levels of risk aversion is 'standard-of-living' utility in which people are sensitive to their utility level compared to that of others. Exploring the sensitivity of our results to alternative specifications of preferences is an important subject for future research.

We have relied on numerical solutions. In some special cases, however, we can solve the model analytically. This may provide a useful benchmark for understanding the features of the optimal solution in other, more complicated cases. We plan to investigate the analytical solutions in the future.

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A Details and additionally useful results

A.1 The calculations for proposition 2

The explicit expressions for the logarithmic derivatives are

$$
\begin{aligned}
\ell_{u,c,t} &= \frac{u_c(\bar{c}_{t,y}, \bar{n}_t)\bar{c}_{t,y}}{u(\bar{c}_{t,y}, \bar{n}_t)}, \\
\ell_{u,n,t} &= \frac{u_n(\bar{c}_{t,y}, \bar{n}_t)\bar{n}_t}{u(\bar{c}_{t,y}, \bar{n}_t)}, \\
\ell_{x,t+1} &= \frac{x'(\bar{\varpi}_{t+1} q(\frac{\bar{c}_{t+1,o}}{\bar{\varpi}_{t+1}}))\bar{\varpi}_{t+1} q(\frac{\bar{c}_{t+1,o}}{\bar{\varpi}_{t+1}})}{x(\bar{\varpi}_{t+1} q(\frac{\bar{c}_{t+1,o}}{\bar{\varpi}_{t+1}}))}, \\
\ell_{q,t+1} &= \frac{q'(\frac{\bar{c}_{t+1,o}}{\bar{\varpi}_{t+1}})\frac{\bar{c}_{t+1,o}}{\bar{\varpi}_{t+1}}}{q(\frac{\bar{c}_{t+1,o}}{\bar{\varpi}_{t+1}})},
\end{aligned}
$$

Furthermore, define

$$
\begin{aligned}
\mu_{z,t} &= -\frac{z''(\bar{a}_t)\bar{a}_t}{z'(\bar{a}_t)}, \\
\mu_{uc,c,t} &= -\frac{u_{cc}(\bar{c}_{t,y}, \bar{n}_t)\bar{c}_{t,y}}{u_c(\bar{c}_{t,y}, \bar{n}_t)}, \\
\mu_{q,t} &= -\frac{q''(\frac{\bar{c}_{t,o}}{\bar{\varpi}_t})\frac{\bar{c}_{t,o}}{\bar{\varpi}_t}}{q'(\frac{\bar{c}_{t,o}}{\bar{\varpi}_t})}, \\
\mu_{x,t} &= -\frac{x''(\bar{\varpi}_t q(\frac{\bar{c}_{t,o}}{\bar{\varpi}_t}))\bar{\varpi}_t q(\frac{\bar{c}_{t,o}}{\bar{\varpi}_t})}{x'(\bar{\varpi}_t q(\frac{\bar{c}_{t,o}}{\bar{\varpi}_t}))}, \\
\mu_{uc,n,t} &= -\frac{u_{nc}(\bar{c}_{t,y}, \bar{n}_t)\bar{n}_t}{u_c(\bar{c}_{t,y}, \bar{n}_t)}.
\end{aligned}
$$

Recall (30),

$$\alpha_t = \frac{u(\bar{c}_{t,y}, \bar{n}_t)}{\bar{a}_t}. \tag{76}$$

Note that all these logarithmic derivatives are known as of $t-1$ by assumption. The time subscript therefore does not indicate "information", but rather indicates the argument.

To derive the equations in proposition 2, one needs to take a first-order Taylor expansion in the logarithmic deviations. These calculations can often be performed rather speedily by combining several "rules", which are easily verified:

[Rule 1:] To take the Taylor expansion for a single variable, note

$$x_t = \bar{x}_t \exp \check{x}_t \approx \bar{x}_t (1 + \hat{x}_t)$$

[Rule 2:] If $z_t = A x_t y_t$, where A is some constant, and the same is true for the benchmark values, then

$$\check{z}_t = \check{x}_t + \check{y}_t.$$

[Rule 3:] If $z_t = x_t + y_t$ and the same is true for the benchmark values, then

$$\bar{z}_t \check{z}_t = \bar{x}_t \check{x}_t + \bar{y}_t \check{y}_t.$$

[Rule 4:] Suppose, some variable z_t satisfies $z_t = f(x_t)$ as well as $\bar{z}_t = f(\bar{x}_t)$. Then,

$$\check{z}_t = \ell_f \check{x}_t.$$

where ℓ_f is the logarithmic derivative of $f(\cdot)$ at $\bar{x}_t$,

$$\ell_f = \frac{f'(\bar{x}_t)\bar{x}_t}{f(\bar{x}_t)}$$

The extension to two variables, $z_t = f(x_t, y_t)$ is

$$\check{z}_t = \ell_{f,x} \check{x}_t + \ell_{f,y} \check{y}_t,$$

where

$$\ell_{f,x} = \frac{f_x(\bar{x}_t, \bar{y}_t)\bar{x}_t}{f(\bar{x}_t, \bar{y}_t)}, \ \ell_{f,y} = \frac{f_y(\bar{x}_t, \bar{y}_t)\bar{y}_t}{f(\bar{x}_t, \bar{y}_t)}.$$

Note furthermore, that

$$\ell_{x^{-1},t} = \ell_{x,t+1}^{-1} \tag{77}$$

and that

$$\mu_{x^{-1},t} = -\frac{\mu_{x,t+1}}{\ell_{x,t+1}} \tag{78}$$

Define the variables

$$\begin{aligned}
z'_t &\equiv z'(a_t) \\
u_t &\equiv u(c_{t,y}, n_t) \\
u'_{c,t} &\equiv \frac{\partial u(c_{t,y} n_t)}{\partial c_{t,y}} \\
x_t &\equiv x\left(\varpi_t q\left(\frac{c_{t,o}}{\varpi_t}\right)\right) \\
x'_t &\equiv x'\left(\varpi_t q\left(\frac{c_{t,o}}{\varpi_t}\right)\right) \\
e_t &\equiv E_t[x_{t+1}] \\
\iota_t &\equiv x^{-1}(e_t) \\
\iota'_{t-1} &\equiv (x^{-1})'(e_{t-1}) \\
q'_t &\equiv q'\left(\frac{c_{t,o}}{\varpi_t}\right)
\end{aligned}$$

Note that $\bar{e}_t = x\left(\bar{\varpi}_{t+1} q\left(\frac{\bar{c}_{t+1,o}}{\bar{\varpi}_{t+1}}\right)\right)$ and $\bar{e}_{t-1} = x\left(\bar{\varpi}_t q\left(\frac{\bar{c}_{t,o}}{\bar{\varpi}_t}\right)\right)$, since we assume that the benchmark levels are known at date $t-1$ already.

With this, the risk-sharing condition (29), i.e.

$$z'(a_t)\frac{\partial u(c_{t,y} n_t)}{\partial c_{t,y}} = z'(a_{t-1})\frac{\beta}{\omega} d_t q'\left(\frac{c_{t,o}}{\varpi_t}\right)$$

can be written as

$$z'_t u'_{c,t} = z'_{t-1}\frac{\omega}{\beta} d_t q'_t. \tag{79}$$

The argument definition

$$a_t = u(c_{t,y}, n_t) + \beta x^{-1}\left(E_t\left[x\left(\varpi_{t+1} q\left(\frac{c_{t+1,o}}{\varpi_{t+1}}\right)\right)\right]\right) \tag{80}$$

can likewise be written as

$$a_t = u_t + \beta \iota_t \tag{81}$$

Finally, the definition for the discounting correction (20), i.e.

$$d_t = \left(x^{-1}\right)' \left(E_{t-1} \left[x \left(\varpi_t q \left(\frac{c_{t,o}}{\varpi_t} \right) \right) \right] \right) x' \left(\varpi_t q \left(\frac{c_{t,o}}{\varpi_t} \right) \right) \tag{82}$$

can be written as

$$d_t = \iota'_{t-1} x'_t \tag{83}$$

1. Taking logarithms of equation (79) delivers

$$\check{z}'_t + \check{u}'_{c,t} = \check{z}'_{t-1} + \check{d}_t + \check{q}'_t$$

Application of rule 4 to these items immediately delivers (84), i.e.

$$\begin{aligned} &\mu_{z,t}\check{a}_t + \mu_{uc,c,t}\check{c}_{t,y} + \mu_{uc,n,t}\check{n}_t \\ &= \ \mu_{z,t-1}\check{a}_{t-1} + \check{d}_t + \mu_{q,t}\left(\check{c}_{t,o} - \check{\varpi}_t\right) \end{aligned} \tag{84}$$

2. To loglinearize (81), use rule 3 to obtain

$$\check{a}_t = \alpha_t \check{u}_t + (1 - \alpha_t)\check{\iota}_t.$$

With rule 4, extended to two arguments, and applied to $u_t = u(c, n)$,

$$\check{u}_t = \ell_{u,c,t}\check{c}_{t,y} + \ell_{u,n,t}\check{n}_t.$$

Also with rule 4,

$$\check{\iota}_t = \ell_{\iota,t}\check{e}_t$$

Interchanging expectation and differentiation, rule 4 delivers

$$\check{e}_t = \ell_{x,t+1}\left((1 - \ell_{q,t+1})E_t[\check{\varpi}_{t+1}] + \ell_{q,t+1}E_t[\check{c}_{t+1,o}]\right). \tag{85}$$

Combining and exploiting (77), one obtains equation (33), i.e.

$$\begin{aligned} \check{a}_t \ = \ & \alpha_t\left(\ell_{u,c,t}\check{c}_{t,y} + \ell_{u,n,t}\check{n}_t\right) \\ & +(1 - \alpha_t)((1 - \ell_{q,t+1})E_t[\check{\varpi}_{t+1}] + \ell_{q,t+1}E_t[\check{c}_{t+1,o}]). \end{aligned} \tag{86}$$

3. Taking logarithms of (83) delivers

$$\check{d}_t = \check{\iota}'_{t-1} + \check{x}'_t$$

For $\check{x}'_t$, rule 4 delivers

$$\check{x}'_t = -\mu_{x,t}\left((1-\ell_{q,t})\check{\varpi}_t + \ell_{q,t}\check{c}_{t,o}\right)$$

For $\check{\iota}'_{t-1}$, repeated application of rule 4 delivers

$$\begin{aligned}\check{\iota}'_{t-1} &= -\mu_{x^{-1},t-1}\check{e}_t \\ &= -\mu_{x^{-1},t-1}E_{t-1}\left[\ell_{x,t}\left((1-\ell_{q,t})\check{\varpi}_t + \ell_{q,t}\check{c}_{t,o}\right)\right]\end{aligned}$$

Combining and exploiting (78) delivers equation (33), i.e.

$$\check{d}_t = \mu_{x,t}\left((1-\ell_{q,t})\left(\check{\varpi}_t - E_{t-1}[\check{\varpi}_t]\right) + \ell_{q,t}\left(\check{c}_{t,o} - E_{t-1}[\check{c}_{t,o}]\right)\right)$$

A.2 Scale Invariance and Balanced Growth

Scale invariance has a number of implications.

1. To save on notation, define $\check{c}_{t,y} = \phi c_{t,y}$, $\check{c}_{t,o} = \phi c_{t,o}$ and $\check{w}_t = \phi w_t$: these are part of the allocation $\check{\Phi}$. Thus, equation (22) can be rewritten as

$$\frac{\ell_{u,c,t}}{\ell_{u,n,t}} = -\frac{c_{t,y}}{w_t n_t} \tag{87}$$

Note that the right hand side - which can be interpreted as the income share spend on consumption of the young - does not depend on ϕ. Hence, the ratio of the two logarithmic derivatives $\ell_{u,c,t} = \ell_{u,c,t}(\check{c}_{t,y}, n_t)$ and $\ell_{u,n,t} = \ell_{u,n,t}(\check{c}_{t,y}, n_t)$ is independent of the argument $\check{c}_{t,y}$.

2. Suppose that the income share spend on consumption of the young is constant

$$\frac{c_{t,y}}{w_t n_t} \equiv \sigma_y, \tag{88}$$

then

$$\ell_{u,c,t} = \sigma_y \ell_{u,n,t}. \tag{89}$$

We note that this condition will hold along a balanced growth path, where indeed the income share should be constant.

3. Replace $c_{t,y}$ with $\phi c_{t,y}$ in (22), and take the logarithmic derivative with respect to ϕ to obtain

$$0 = -\mu_{uc,c,t} - \mu_{uc,n,t}\frac{c_{t,y}}{w_t n_t} + 1, \tag{90}$$

or

$$0 = -\mu_{uc,c,t} + \mu_{uc,n,t}\frac{\ell_{u,c,t}}{\ell_{u,n,t}} + 1 \tag{91}$$

across all values $\tilde{c}_{t,y}$.

4. Suppose additionally, that $\mu_{uc,c,t}$ is constant over time and $\eta \equiv \mu_{uc,c,t}$. We can derive a semi-closed form for the utility function $u(c, n)$, as is well-known from the literature, see e.g. King and Plosser (1989). We provide the derivation here for the sake of completeness. Consider first the case $\eta \neq 1$. Since $\mu_{uc,c,t}$ is the (negative) logarithmic derivative of u_c with respect to c, this implies log-linearity for u_c,

$$\log u_c(\breve{c}_{t,y}, n_t) = (1-\eta)\log v(n_t) - \eta \log \breve{c}_{t,y},$$

where the "constant" term $(1-\eta)\log v(n_t)$, may depend on n_t (and seems to have been written in a rather complicated fashion, as this will turn out to be convenient below). Rewrite as

$$u_c(\breve{c}_{t,y}, n_t) = v(n_t)^{1-\eta}\breve{c}_{t,y}^{-\eta},$$

and integrate to obtain

$$u(\breve{c}_{t,y}, n_t) = \frac{(v(n_t)\breve{c}_{t,y})^{1-\eta}}{1-\eta} + \kappa(n_t), \tag{92}$$

where $\kappa(n_t)$ is an additional constant, possibly depending on n_t. Differentiating with respect to n_t and comparing to (22) shows that $\kappa(n_t) \equiv \kappa$ independent of n_t as follows. Rewrite (22) for the allocation $\breve{\Phi}$ as

$$\begin{aligned} \phi w_t &= -\frac{u_n(\phi c_{t,y}, n_t)}{u_c(\phi c_{t,y}, n_t)} \\ &= -\frac{v'(n_t)}{v(n_t)}\phi c_{t,y} - \frac{\kappa'(n_t)}{v(n_t)^{1-\eta}(\phi c_{t,y})^{-\eta}}. \end{aligned}$$

Divide by ϕ to see that the left hand side and the first term on the right hand side do not depend on ϕ, whereas the last one does, unless either $\eta = 1$ or $\kappa'(n_t) = 0$. Since we assumed $\eta \neq 1$, $\kappa(n_t)$ must be a constant, independent of n_t. The case $\eta = 1$ similarly yields

$$u(\breve{c}_{t,y}, n_t) = \log \breve{c}_{t,y} + v(n_t). \tag{93}$$

5. Consider a nonstochastic case. Equation (26) reads

$$u_c(\phi c_{t,y}, n_t) = \beta \left(x^{-1}\right)' \left(x \left(\varpi_{t+1} q \left(\frac{\phi c_{t+1,o}}{\varpi_{t+1}} \right) \right) \right) \tag{94}$$
$$x' \left(\varpi_{t+1} q \left(\frac{\phi c_{t+1,o}}{\varpi_{t+1}} \right) \right) q' \left(\frac{\phi c_{t+1,o}}{\varpi_{t+1}} \right) R_{t+1}.$$

Take the logarithmic derivative with respect to ϕ and exploit (78) to find

$$\mu_{uc,c,t} = \mu_{q,t+1} \tag{95}$$

6. Again for the nonstochastic case, the constant share condition $\tilde{\alpha}_t = \alpha_t$ can be rewritten as

$$\left(\frac{1}{\alpha_t} - 1 \right) u(\phi c_{t,y}, n_t) = \beta q(\phi c_{t+1,o})$$

where α_t does not depend on ϕ. Taking the logarithmic derivative with respect to ϕ, we establish

$$\ell_{u,c,t} = \ell_{q,t+1} \tag{96}$$

A.3 The equity premium

To relate the risk-aversion parameters to observables, one possibility is that observed allocations are (nearly) optimal as far as intergenerational risk sharing is concerned, and to calibrate the risk-aversion parameters to observed market prices of risk. In our context, this means to observe market prices of risk from young agents trading assets which pay off when they are old.

Given an allocation which solves the social planners problem, suppose that the social planer contemplates transferring resources between periods using some asset a with return R^a_{t+1}. Since the allocation is already optimal, the social planner must wish not to execute this reallocation. Hence, it has to be the case that the asset pricing condition (26) not only holds for R_{t+1}, but for the return R^a_{t+1} as well, which might, of course, coincide with R_{t+1}. Recall the definition of the stochastic discount factor m_{t+1},

$$\begin{aligned} m_{t+1} &= \beta \left(x^{-1}\right)' \left(E_t \left[x \left(\varpi_{t+1} q \left(\frac{c_{t+1,o}}{\varpi_{t+1}} \right) \right) \right] \right) \\ & \quad x' \left(\varpi_{t+1} q \left(\frac{c_{t+1,o}}{\varpi_{t+1}} \right) \right) q' \left(\frac{c_{t+1,o}}{\varpi_{t+1}} \right) \end{aligned}$$

The asset pricing equation (26) then implies

$$E_t \left[m_{t+1} R^a_{t+1} \right] = E_t \left[m_{t+1} R_{t+1} \right]. \tag{97}$$

Decompose R^a_{t+1}, $c_{t+1,o}$, ϖ_{t+1}, $m_{s,t+1}$ into their predicted part and surprise part,

$$\begin{aligned} \log(R^a_{t+1}) &= \log(\bar{R}^a_{t+1}) + \check{R}^a_{t+1}, \\ \log(c_{t+1,o}) &= \log(\bar{c}_{t+1,o}) + \check{c}_{t+1}, \\ \log(\varpi_{t+1}) &= \log(\bar{\varpi}_{t+1}) + \check{\varpi}_{t+1}, \\ \log(m_{t+1}) &= \log(\bar{m}_{t+1}) + \check{m}_{t+1}, \end{aligned}$$

with

$$0 = E_t[\check{R}^a_{t+1}] = E_t[\check{c}_{t+1}] = E_t[\check{\varpi}_{t+1}] = E_t[\check{m}_{t+1}]$$

We assume that the surprise parts are jointly normally distributed, conditional on information up to and including t. Denote the standard deviations by e.g. $\sigma_{R,t+1}$ and $\sigma_{m,t+1}$ and correlations denoted by e.g. $\rho_{m,R,t+1}$. Define the Sharpe ratio, defined as the difference of the log expected returns divided by the standard deviation of the log return,

$$\mathbf{SR}_t = \frac{\log\left(E_t[R_{t+1}]\right) - \log\left(R^f_t\right)}{\sigma_{R,t+1}}. \tag{98}$$

Using a standard calculation, see Lettau and Uhlig (2002), one can show that

$$\mathbf{SR}_t = -\rho_{m,R,t+1}\sigma_{m,t+1} \tag{99}$$

It is easy to see that, up to a log-linear approximation,

$$\begin{aligned}\check{m}_{t+1} &= -\mu_{x,t+1}(\check{\varpi}_{t+1} + \ell_{q,t+1}(\check{c}_{t+1,o} - \check{\varpi}_{t+1})) - \mu_{q,t+1}(\check{c}_{t+1,o} - \check{\varpi}_{t+1}) \\ &= (\mu_{x,t+1}(\ell_{q,t+1} - 1) + \mu_{q,t+1})\check{\varpi}_{t+1} - (\mu_{x,t+1}\ell_{q,t+1} + \mu_{q,t+1})\check{c}_{t+1,o}\end{aligned}$$

Using the same calculation as for (99), one can then show that

$$\begin{aligned}\mathbf{SR}_t &= -(\mu_{x,t+1}(\ell_{q,t+1} - 1) + \mu_{q,t+1})\rho_{\varpi,R,t+1}\sigma_{\varpi,t+1} \\ &\quad +(\mu_{x,t+1}\ell_{q,t+1} + \mu_{q,t+1})\rho_{c,o,R,t+1}\sigma_{c,o,t+1}\end{aligned} \tag{100}$$

For the specific functional form of subsection 3.3, we thus obtain

$$\mathbf{SR}_t = \frac{\eta - \nu}{1 - \eta}\rho_{\varpi,R,t+1}\sigma_{\varpi,t+1} + \nu\rho_{c,o,R,t+1}\sigma_{c,o,t+1} \tag{101}$$

These equations are informative about measuring curvature parameters of the utility specification. In particular, we see that if there is no surprise longevity risk, $\check{\varpi}_{t+1} \equiv 0$, then the risk premium is proportional to the risk aversion parameter ν, the standard deviation $\sigma_{c,o,t+1}$ of old-age consumption $c_{o,t+1}$ and the correlation $\rho_{c,o,R,t+1}$ of old-age consumption with the asset return.

In solving a social planners problem and in characterizing the speed of capital accumulation, the appropriate risk premium and thus, the appropriate "scarcity" of capital to generate the required average return, ought to be taken into account.

A.4 Solving the social planners problem

A.4.1 Collecting the equations

For reference, the social planners problem solves the following set of equations:

$$\begin{aligned}y_t &= A_t k_{t-1} f(Z_t n_t), \\ c_{t,y}\pi_t + c_{t,o} + k_t\pi_t &= (A_t f(Z_t n_t) + 1 - \delta_t)\, k_{t-1}, \\ w_t &= A_t k_{t-1} f'(Z_t n_t)\frac{Z_t}{\pi_t}, \\ \frac{\partial u(c_{t,y}, n_t)}{\partial c_{t,y}} w_t &= -\frac{\partial u(c_{t,y}, n_t)}{\partial n_t},\end{aligned}$$

$$
\begin{aligned}
R_t &= A_t f(Z_t n_t) + 1 - \delta_t, \\
\frac{\partial u(c_{t,y}, n_t)}{\partial c_{t,y}} &= \beta (x^{-1})' \left(E_t \left[x \left(\varpi_{t+1} q \left(\frac{c_{t+1,o}}{\varpi_{t+1}} \right) \right) \right] \right) \cdot \\
& \quad E_t \left[x' \left(\varpi_{t+1} q \left(\frac{c_{t+1,o}}{\varpi_{t+1}} \right) \right) q' \left(\frac{c_{t+1,o}}{\varpi_{t+1}} \right) R_{t+1} \right], \\
z'(a_t) \frac{\partial u(c_{t,y} n_t)}{\partial c_{t,y}} &= \frac{\beta}{\omega} z'(a_{t-1}) (x^{-1})' \left(E_{t-1} \left[x \left(\varpi_t q \left(\frac{c_{t,o}}{\varpi_t} \right) \right) \right] \right) \cdot \\
& \quad x' \left(\varpi_t q \left(\frac{c_{t,o}}{\varpi_t} \right) \right) q' \left(\frac{c_{t,o}}{\varpi_t} \right) \\
a_t &= u(c_{t,y}, n_t) + \beta x^{-1} \left(E_t \left[x \left(\varpi_{t+1} q \left(\frac{c_{t+1,o}}{\varpi_{t+1}} \right) \right) \right] \right).
\end{aligned}
$$

We assume that preferences are given by (44). Define

$$
\varsigma = \frac{\eta}{1-\eta}(1-\nu) \tag{102}
$$

and note that

$$
\frac{1-\eta}{1-\nu}\varsigma = \eta
$$

which turns out to be useful in some calculations below. Note that

$$
\begin{aligned}
x \left(\varpi_t q \left(\frac{c_{t,o}}{\varpi_t} \right) \right) &= \frac{1}{1-\nu} \varpi_t^{\varsigma} c_{t,o}^{1-\nu} \\
x' \left(\varpi_t q \left(\frac{c_{t,o}}{\varpi_t} \right) \right) q' \left(\frac{c_{t,o}}{\varpi_t} \right) &= \varpi_t^{\varsigma} c_{t,o}^{-\nu}
\end{aligned}
$$

Thus, the equations above become

$$
\begin{aligned}
y_t &= A_t k_{t-1} f(Z_t n_t), \\
c_{t,y} \pi_t + c_{t,o} + k_t \pi_t &= (A_t f(Z_t n_t) + 1 - \delta_t) k_{t-1}, \\
w_t &= A_t k_{t-1} f'(Z_t n_t) \frac{Z_t}{\pi_t}, \\
\frac{w_t}{c_{t,y}} &= -\frac{v'(n_t)}{v(n_t)}, \\
R_t &= A_t f(Z_t n_t) + 1 - \delta_t, \\
c_{t,y}^{-\eta} v(n_t)^{1-\eta} &= \beta x_t^{\nu-\eta} E_t[\varpi_{t+1}^{\varsigma} c_{t+1,o}^{-\nu} R_{t+1}],
\end{aligned}
$$

$$
\begin{aligned}
\frac{\beta}{\omega} s_{t-1}^{-1} \varpi_t^{\varsigma} c_{t,o}^{-\nu} &= ((1-\eta) a_t)^{\frac{\eta-\xi}{1-\eta}} c_{t,y}^{-\eta} v(n_t)^{1-\eta}, \\
(1-\eta) a_t &= (c_{t,y} v(n_t))^{1-\eta} + \beta x_t^{1-\eta}, \\
x_t &= \left(E_t \left[\varpi_{t+1}^{\varsigma} c_{t+1,o}^{1-\nu} \right] \right)^{\frac{1}{1-\nu}}, \\
s_t &= ((1-\eta) a_t)^{\frac{\xi-\eta}{1-\eta}} x_t^{\eta-\nu},
\end{aligned}
$$

where the last two lines define x_t and s_t.

A.4.2 Normalization

To turn this into a stationary system, the growing variables need to be divided by the beginning-of-period capital level k_{t-1}. Thus, let

$$
\begin{aligned}
\tilde{c}_{t,y} &\equiv \frac{c_{t,y}}{k_{t-1}}, \\
\tilde{c}_{t,o} &\equiv \frac{c_{t,o}}{k_{t-1}}, \\
\tilde{w}_t &\equiv \frac{w_t}{k_{t-1}}, \\
\tilde{y}_t &\equiv \frac{y_t}{k_{t-1}}, \\
\tilde{k}_t &\equiv \frac{k_t}{k_{t-1}}, \\
\tilde{x}_t &\equiv \frac{x_t}{k_{t-1}},
\end{aligned}
$$

except that we define

$$
\begin{aligned}
\tilde{a}_t &\equiv \frac{a_t}{k_{t-1}^{1-\eta}}, \\
\tilde{s}_t &\equiv s_t k_t^{\nu-\xi}.
\end{aligned}
$$

The exponent differs here in order to achieve the appropriate scaling for these utility variables. Moreover, we scale s_t by $k_t^{\nu-\xi}$ rather than by $k_{t-1}^{\nu-\xi}$ because that way, $\tilde{s}_{t-1}$ (rather than $\tilde{s}_{t-1}$ and $\tilde{k}_{t-1}$) turns out to be the only remaining endogenous state variable.

Rewrite the system as

$$\begin{aligned}
\tilde{y}_t &= A_t f(Z_t n_t), \\
\tilde{c}_{t,y}\pi_t + \tilde{c}_{t,o} + \tilde{k}_t\pi_t &= A_t f(Z_t n_t) + 1 - \delta_t, \\
\tilde{w}_t &= A_t f'(Z_t n_t)\frac{Z_t}{\pi_t}, \\
\frac{\tilde{w}_t}{\tilde{c}_{t,y}} &= -\frac{v'(n_t)}{v(n_t)}, \\
R_t &= A_t f(Z_t n_t) + 1 - \delta_t, \\
\tilde{c}_{t,y}^{-\eta} v(n_t)^{1-\eta} &= \beta \tilde{x}_t^{\nu-\eta}\tilde{k}_t^{-\nu} E_t[\varpi_{t+1}^{\varsigma}\tilde{c}_{t+1,o}^{-\nu} R_{t+1}], \\
\frac{\beta}{\omega}\tilde{s}_{t-1}^{-1}\varpi_t^{\varsigma}\tilde{c}_{t,o}^{-\nu} &= ((1-\eta)\tilde{a}_t)^{\frac{\eta-\xi}{1-\eta}}\,\tilde{c}_{t,y}^{-\eta} v(n_t)^{1-\eta}, \\
(1-\eta)\tilde{a}_t &= (\tilde{c}_{t,y} v(n_t))^{1-\eta} + \beta\tilde{x}_t^{1-\eta}, \\
\tilde{x}_t &= \tilde{k}_t \left(E_t\left[\varpi_{t+1}^{\varsigma}\tilde{c}_{t+1,o}^{1-\nu}\right]\right)^{\frac{1}{1-\nu}}, \\
\tilde{s}_t &= ((1-\eta)\tilde{a}_t)^{\frac{\xi-\eta}{1-\eta}}\,\tilde{x}_t^{\eta-\nu}\tilde{k}_t^{\nu-\xi}.
\end{aligned}$$

Note that indeed the only endogenous state variable remaining is $\tilde{s}_{t-1}$.

A.4.3 Stochastic assumptions

To make further progress, we need some additional, tractable assumptions. Let

$$\zeta_t = \begin{bmatrix} \varpi_t \\ A_t \\ Z_t \\ \pi_t \\ \delta_t \end{bmatrix} \tag{103}$$

be the vector of exogenous parameters. We assume that a steady state $\bar{\zeta}$ exists and that

$$\hat{\zeta}_t = \log(\zeta_t) - \log(\bar{\zeta})$$

follows an AR(1) process,

$$\hat{\zeta}_t = N\hat{\zeta}_{t-1} + \epsilon_t \tag{104}$$

for some 5×5 matrix N with nonexplosive roots, where

$$\epsilon_t = \begin{bmatrix} \epsilon_{\varpi,t} \\ \epsilon_{A,t} \\ \epsilon_{Z,t} \\ \epsilon_{\pi,t} \\ \epsilon_{\delta,t} \end{bmatrix}$$

is the vector of innovations for each exogenous parameter and

$$\epsilon_t \sim \mathcal{N}(0, \Sigma).$$

A.4.4 Steady state

The steady state for the "tilde"-variables - which we shall now also use as our benchmark values - is given by the equations

$$\begin{aligned}
\bar{y} &= \bar{A}f(\bar{Z}\bar{n}), && (105)\\
\bar{c}_y\bar{\pi} + \bar{c}_o + \bar{k}\bar{\pi} &= \bar{A}f(\bar{Z}\bar{n}) + 1 - \bar{\delta}, && (106)\\
\bar{w} &= \bar{A}f'(\bar{Z}\bar{n})\frac{\bar{Z}}{\bar{\pi}}, && (107)\\
\frac{\bar{w}}{\bar{c}_y} &= -\frac{v'(\bar{n})}{v(\bar{n})}, && (108)\\
\bar{R} &= \bar{A}f(\bar{Z}\bar{n}) + 1 - \bar{\delta}, && (109)\\
\bar{c}_y^{-\eta}v(\bar{n})^{1-\eta} &= \beta\bar{x}^{\nu-\eta}\bar{k}^{-\nu}\bar{\varpi}^{\varsigma}\bar{c}_o^{-\nu}\bar{R}\exp\left(\sigma^2_{R-\nu c}/2\right), && (110)\\
\frac{\beta}{\omega}\bar{s}^{-1}\bar{\varpi}^{\varsigma}\bar{c}_o^{-\nu} &= ((1-\eta)\bar{a})^{\frac{\eta-\xi}{1-\eta}}\,\bar{c}_y^{-\eta}v(\bar{n})^{1-\eta}, && (111)\\
(1-\eta)\bar{a} &= (\bar{c}_y v(\bar{n}))^{1-\eta} + \beta\bar{x}^{1-\eta}, && (112)\\
\bar{x} &= \bar{k}(\exp\left(\sigma^2_x/2\right)\bar{\varpi}^{\varsigma}\bar{c}_o^{1-\nu})^{\frac{1}{1-\nu}}, && (113)\\
\bar{s} &= ((1-\eta)\bar{a})^{\frac{\xi-\eta}{1-\eta}}\,\bar{x}^{\eta-\nu}\bar{k}^{\nu-\xi}, && (114)
\end{aligned}$$

where $\sigma^2_{R-\nu c}$ denotes the conditional variance of $\log R_{t+1} + \varsigma\log(\varpi_{t+1}) - \nu\log(c_{t+1,o})$ and σ^2_x is the conditional variance of $\varsigma\log\varpi_{t+1} + (1-\nu)\log c_{t+1,o}$.

We have included the risk terms in the intertemporal equations, thus including a precautionary motive for saving. We implicitely assumed (as remains to be shown in the log-linearized version), that the variances are constant over time. This will be justified below, when solving for the linear recursive law of motion for the loglinearized system, see subsection A.4.10.

Finally, define the argument share

$$\alpha = \frac{(\bar{c}_y v(\bar{n}))^{1-\eta}}{(1-\eta)\bar{a}}$$

for $\eta \neq 1$ and

$$\alpha = \frac{1}{1+\beta}$$

for $\eta = 1$.

A.4.5 Solving for the steady state

Labor supply (108) can be written as

$$\frac{\bar{w}\bar{n}}{\bar{c}_y} = -\ell_v(\bar{n}), \tag{115}$$

which says that the inverse of the share of wage income spent on consumption when young is tied to $\ell_v(\bar{n})$. Labor demand (107) together with (105) implies

$$\frac{\bar{w}\bar{n}\bar{\pi}}{\bar{y}} = \ell_f(\bar{Z}\bar{n}). \tag{116}$$

The labor share in production is thus closely relared to the productivity-weighted labor input $\bar{Z}\bar{n}$. Combining equations (115) and (116), one can eliminate the wage rate $\bar{w}$

$$\frac{\bar{\pi}\bar{c}_y}{\bar{y}} = \frac{\ell_f(\bar{Z}\bar{n})}{-\ell_v(\bar{n})}. \tag{117}$$

We also assume that variances and covariances are known: the underlying fixed point problem will be discussed below. Equation (113) implies

$$\bar{x} = \kappa_1 \bar{k}\bar{c}_o, \tag{118}$$

where

$$\kappa_1 = \exp\left(\frac{\sigma_x^2/2}{1-\nu}\right)\bar{\varpi}^{\frac{\varsigma}{1-\nu}}.$$

The asset pricing equation (110), which can be viewed as describing optimal saving, can be rewritten with (118) as

$$\bar{c}_y^{-\eta} v(\bar{n})^{1-\eta} = \kappa_2 \bar{c}_o^{-\eta} \bar{k}^{-\eta} \bar{R}, \tag{119}$$

where

$$\begin{aligned} \kappa_2 &= \beta \kappa_1^{\nu-\eta} \varpi^{\zeta} \exp\left(\sigma_{R-\nu c}^2/2\right) \\ &= \beta \bar{\varpi}^{\eta} \exp\left(\frac{\nu-\eta}{1-\nu}\sigma_x^2/2\right) \exp\left(\sigma_{R-\nu c}^2/2\right). \end{aligned}$$

The risk-sharing condition (111) together with the definitions (114) and (118) implies

$$\bar{c}_y^{-\eta} v(\bar{n})^{1-\eta} = \left(\frac{\beta}{\omega}\kappa_1^{\nu-\eta}\bar{\varpi}^{\zeta}\right) \bar{c}_o^{-\eta} \bar{k}^{\xi-\eta}, \tag{120}$$

which can be rewritten as

$$\frac{\bar{c}_o}{\bar{\pi}\bar{c}_y} = \kappa_3 \bar{k}^{\frac{\xi-\eta}{\eta}} v(\bar{n})^{\frac{\eta-1}{\eta}}, \tag{121}$$

where

$$\begin{aligned} \kappa_3 &= \frac{1}{\bar{\pi}}\left(\frac{\beta}{\omega}\kappa_1^{\nu-\eta}\bar{\varpi}^{\zeta}\right)^{\frac{1}{\eta}} \\ &= \frac{1}{\bar{\pi}}\left(\frac{\beta}{\omega}\right)^{\frac{1}{\eta}} \bar{\varpi} \exp\left(\frac{\nu-\eta}{\eta(1-\nu)}\sigma_x^2/2\right). \end{aligned}$$

While (119) arises from an intertemporal savings decision, equation (120) arises from the risk-sharing condition. The difference between these two equations thus partially stems from the different weights given to an agent currently alive or alive in the future due to population growth and social planner discounting. Equations (119) and (120) with (109) together imply

$$\bar{k}^{\xi} = \kappa_4 \bar{R} = \kappa_4 \left(\bar{A} f(\bar{Z}\bar{n}) + 1 - \bar{\delta}\right), \tag{122}$$

where

$$\kappa_4 = \frac{\omega \kappa_2}{\beta \kappa_1^{\nu-\eta} \varpi^{\zeta}} = \omega \exp\left(\sigma_{R-\nu c}^2/2\right),$$

which sheds light on the relationship between growth of the economy $\bar{k}$ and required labor $\bar{n}$ versus the discount factor of the social planner ω and a term $\exp(\sigma^2_{R-\nu c})$ related to the risk premium. Note also, that

$$\kappa_3 = \frac{1}{\bar{\pi}} \left(\frac{\kappa_2}{\kappa_4} \right)^{\frac{1}{\eta}}.$$

The right-hand side of (106) equals $\bar{R}$. Use (105) to rewrite (106) as

$$\frac{\bar{\pi}\bar{c}_y}{\bar{y}} \left(1 + \frac{\bar{c}_o}{\bar{\pi}\bar{c}_y} \right) = 1 + \frac{1 - \bar{\delta} - \bar{\pi}\bar{k}}{\bar{A}f(\bar{Z}\bar{n})}. \tag{123}$$

Combining equations in $\frac{\bar{\pi}\bar{c}_y}{\bar{y}}$, $\frac{\bar{c}_o}{\bar{\pi}\bar{c}_y}$, $\bar{k}$, and $\bar{n}$, we have

$$\begin{aligned}
\frac{\bar{\pi}\bar{c}_y}{\bar{y}} \left(1 + \frac{\bar{c}_o}{\bar{\pi}\bar{c}_y} \right) &= 1 + \frac{1 - \bar{\delta} - \bar{\pi}\bar{k}}{\bar{A}f(\bar{Z}\bar{n})}, &(124)\\
\frac{\bar{\pi}\bar{c}_y}{\bar{y}} &= \frac{\ell_f(\bar{Z}\bar{n})}{-\ell_v(\bar{n})}, &(125)\\
\frac{\bar{c}_o}{\bar{\pi}\bar{c}_y} &= \kappa_3 \bar{k}^{\frac{\xi-\eta}{\eta}} v(\bar{n})^{\frac{\eta-1}{\eta}}, &(126)\\
\bar{k}^{\xi} &= \kappa_4 \bar{R} = \kappa_4 \left(\bar{A}f(\bar{Z}\bar{n}) + 1 - \bar{\delta} \right). &(127)
\end{aligned}$$

Substituting the second and third equation into the first to eliminate $\frac{\bar{\pi}\bar{c}_y}{\bar{y}}$ and $\frac{\bar{c}_o}{\bar{\pi}\bar{c}_y}$, one obtains

$$\left(\frac{\ell_f(\bar{Z}\bar{n})}{-\ell_v(\bar{n})} \right) \left(1 + \kappa_3 \bar{k}^{\frac{\xi-\eta}{\eta}} v(\bar{n})^{\frac{\eta-1}{\eta}} \right) = 1 + \frac{1 - \bar{\delta} - \bar{\pi}\bar{k}}{\bar{A}f(\bar{Z}\bar{n})}.$$

Finally, use the fourth equation (127) to eliminate $\bar{k}$ to obtain

$$\left(\frac{\ell_f(\bar{Z}\bar{n})}{-\ell_v(n)} \right) \left(1 + \kappa_3 \left(\kappa_4 \left(\bar{A}f(\bar{Z}\bar{n}) + 1 - \bar{\delta} \right) \right)^{\frac{\xi-\eta}{\xi\eta}} v(\bar{n})^{\frac{\eta-1}{\eta}} \right)$$

$$= 1 + \frac{1 - \bar{\delta} - \bar{\pi} \left(\kappa_4 \left(\bar{A}f(\bar{Z}\bar{n}) + 1 - \bar{\delta} \right) \right)^{\frac{1}{\xi}}}{\bar{A}f(\bar{Z}\bar{n})}. \tag{128}$$

This is a single and nonlinear equation in $\bar{n}$. Solving it requires a specification for $v(\cdot)$ and $f(\cdot)$. A solution may perhaps be given in special cases, but numerical methods must be used generally. There may be multiple solutions $\bar{n} > 0$, indicating a multiplicity of steady states. Given a solution to this equation, all other steady state variables can then be calculated.

A.4.6 A parameterization

We shall assume that

$$v(n) = \bar{v}\frac{(1-n)^{1-\chi}}{1-\chi}$$

with $0 \leq \chi < 1$, so that

$$\begin{aligned} \ell_v(n) &= -(1-\chi)\frac{n}{1-n}, \\ \mu_v(n) &= -\chi\frac{n}{1-n}. \end{aligned}$$

Thus, (115) can be rewritten as

$$\frac{\bar{w}(1-\bar{n})}{\bar{c}_y} = 1-\chi. \tag{129}$$

The "expenditure ratio" of leisure over consumption when young is thus given by $1-\chi$.

Further, we assume that

$$f(x) = \left(\theta + (1-\theta)x^{1-\frac{1}{\psi}}\right)^{\frac{1}{1-\frac{1}{\psi}}} \tag{130}$$

with $0 \leq \theta < 1$ and $\psi > 0$ (where one should note that we usually use $x = Zn$ as argument), and thus

$$\begin{aligned} \ell_f(x) &= \frac{(1-\theta)x^{1-\frac{1}{\psi}}}{\theta + (1-\theta)x^{1-\frac{1}{\psi}}}, \\ \mu_f(x) &= \frac{1}{\psi}(1-\ell_f(x)). \end{aligned}$$

For $\psi \to 1$, this becomes

$$f(x) = x^{1-\theta} \tag{131}$$

Proof: *Let $\epsilon = 1 - \frac{1}{\psi}$. Note that*

$$\begin{aligned} \log f(x;\epsilon) &= \frac{1}{\epsilon}\log\left(1 + (1-\theta)\left(\exp\left(\epsilon \log x\right) - 1\right)\right) \\ &\approx \frac{1}{\epsilon}(1-\theta)\left(\exp\left(\epsilon\log x\right) - 1\right) \\ &\approx \frac{1}{\epsilon}(1-\theta)\left(\epsilon \log x\right) \\ &= (1-\theta)\log x, \end{aligned}$$

which delivers the claim. •

Either directly or per $\psi \to 1$, one then gets

$$\begin{aligned} \ell_f(x) &= 1-\theta, \\ \mu_f(x) &= \theta. \end{aligned}$$

A.4.7 A special case

We now analyze the case $\psi = 1, \xi = \eta \to 1, \bar{\delta} = 1$ per successively investigating the implications of each additional restriction.

For the special case $\psi = 1$, replace in equation (125) to get

$$\frac{\bar{\pi}\bar{c}_y}{\bar{y}} = \left(\frac{1}{\bar{n}} - 1\right) \frac{(1-\theta)}{(1-\chi)},$$

and in equation (128) to find

$$\left(\frac{1}{\bar{n}} - 1\right) \frac{(1-\theta)}{(1-\chi)} \left(1 + \kappa_3 \left(\kappa_4 \left(\bar{A}(\bar{Z}\bar{n})^{1-\theta} + 1 - \bar{\delta}\right)\right)^{\frac{\xi-\eta}{\xi\eta}} \left(\bar{v}\frac{(1-\bar{n})^{1-\chi}}{1-\chi}\right)^{\frac{\eta-1}{\eta}}\right)$$
$$= 1 + \frac{1 - \bar{\delta} - \bar{\pi}\left(\kappa_4 \left(\bar{A}(\bar{Z}\bar{n})^{1-\theta} + 1 - \bar{\delta}\right)\right)^{\frac{1}{\xi}}}{\bar{A}(\bar{Z}\bar{n})^{1-\theta}}. \qquad (132)$$

For $\xi = \eta \to 1$, the ratio of old-age consumption to young-age consumption is given by (126) as

$$\frac{\bar{c}_o}{\bar{c}_y} = \bar{\pi}\kappa_3 = \frac{\beta}{\omega}\exp\left(-\sigma_x^2/2\right). \qquad (133)$$

Furthermore for $\xi = \eta \to 1$, one obtains

$$\left(\frac{1}{\bar{n}} - 1\right) \frac{(1-\theta)}{(1-\chi)} (1+\kappa_3) = (1 - \bar{\pi}\kappa_4)\left(1 + \frac{1-\bar{\delta}}{\bar{A}\bar{Z}^{1-\theta}\bar{n}^{1-\theta}}\right). \qquad (134)$$

With complete depreciation (i.e. $\bar{\delta} = 1$), this becomes

$$\bar{n} = \left(1 + \kappa_5 \frac{(1-\chi)}{(1-\theta)}\right)^{-1}, \qquad (135)$$

where

$$\frac{\bar{\pi}\bar{c}_y}{\bar{y}} = \kappa_5 = \frac{1 - \bar{\pi}\kappa_4}{1 + \kappa_3}. \tag{136}$$

A closed-form solution is thus available in this case. Ignoring the variance and risk premium terms or assuming a nonstochastic steady state, we can write κ_5 as

$$\frac{\bar{\pi}\bar{c}_y}{\bar{y}} = \kappa_5 = \frac{1 - \bar{\pi}\omega}{1 + \beta(\bar{\pi}\omega)^{-1}}, \tag{137}$$

so that (by substitution into (136))

$$\bar{n} = \left(1 + \frac{(1 - \bar{\pi}\omega)}{(1 + \beta(\bar{\pi}\omega)^{-1})}\frac{(1 - \chi)}{(1 - \theta)}\right)^{-1}, \tag{138}$$

where one can now investigate the impact of a variety of parameters on the steady state.

A particularly simple case is $\beta = \omega$ and $\bar{\pi} = 1$. Ignoring the variance and risk premium terms, one has

$$\begin{aligned} \frac{\bar{c}_y}{\bar{y}} = \frac{\bar{c}_o}{\bar{y}} &= \frac{1 - \beta}{2} \\ \bar{n} &= \left(1 + \frac{(1 - \beta)}{2}\frac{(1 - \chi)}{(1 - \theta)}\right)^{-1} \end{aligned} \tag{139}$$

with the gross-investment-to-GDP ratio $\frac{\bar{\pi}\bar{k}}{\bar{y}}$ given by β.

Given a value of $\bar{n}$, equation (127) shows that the growth factor $\bar{k}$ of the economy and the level of total factor productivity $\bar{A}$ are closely related. Varying the latter will not affect $\bar{n}$ under full depreciation in case $\xi = \eta \to 1$, as shown above. Thus, to obtain an economy with a growth factor $\bar{k}$, we can set

$$\bar{A} = \frac{\bar{k}}{\omega(\bar{Z}\bar{n})^{\theta}},$$

where we once again ignore the variance and risk premium terms.

These calculations also help in a somewhat more general case. For suppose alternatively, that $\bar{\delta} = 1$, but that $\xi \neq \eta \neq 1$. Use the solution $\bar{n}$ of equation (135), but without imposing $\eta = 1$ for calculating κ_5, i.e. using (136) to "back out" the level parameter $\bar{v}$ in the disutility of the young consistent with this steady state

$$\bar{v} = \left(\kappa_4 \left(\bar{A}(\bar{Z}\bar{n})^{1-\theta} + 1 - \bar{\delta}\right)\right)^{\frac{\xi-\eta}{\xi(1-\eta)}} \left(\frac{(1-\bar{n})^{1-\chi}}{1-\chi}\right)^{-1}.$$

This may be useful in order to investigate local comparative statics around a known steady state or to find an initial point for numerically calculating the steady state.

A.4.8 Loglinearization

We now use hats on variables to denote the loglinearization of the detrended variables around the detrended steady state, e.g.

$$\hat{c}_{t,y} = \log(\tilde{c}_{t,y}) - \log(\bar{c}_{t,y}).$$

Furthermore and from here onwards and in slight abuse of notation, let ℓ_f and ℓ_v be the logarithmic derivatives of $f(\cdot)$ and $v(\cdot)$, and μ_f and μ_v the (negative) elasticity of $f'(\cdot)$ and $v'(\cdot)$, all evaluated at steady-state employment $\bar{n}$ for $v(\cdot)$ resp. $\bar{Z}\bar{n}$ for $f(\cdot)$. The loglinearization is given by

$$\begin{aligned}
\hat{y}_t &= \hat{A}_t + \ell_f(\hat{Z}_t + \hat{n}_t), && (140)\\
&\bar{c}_y\bar{\pi}(\hat{c}_{t,y} + \hat{\pi}_t) + \bar{c}_o\hat{c}_{t,o} + \bar{k}\bar{\pi}(\hat{k}_t + \hat{\pi}_t) && (141)\\
&= \bar{y}(\hat{A}_t + \ell_f(\hat{Z}_t + \hat{n}_t)) - \bar{\delta}\hat{\delta}_t, && (142)\\
\hat{w}_t &= \hat{A}_t - \mu_f\left(\hat{Z}_t + \hat{n}_t\right) + \hat{Z}_t - \hat{\pi}_t, && (143)\\
\hat{w}_t - \hat{c}_{t,y} &= -(\mu_v + \ell_v)\hat{n}_t, && (144)\\
\bar{R}\hat{R}_t &= \bar{y}(\hat{A}_t + \ell_f(\hat{Z}_t + \hat{n}_t)) - \bar{\delta}\hat{\delta}_t, && (145)\\
-\eta\hat{c}_{t,y} + (1-\eta)\ell_v\hat{n}_t &= -\eta\hat{k}_t + E_t\left[\eta\hat{\varpi}_{t+1} - \eta\hat{c}_{t+1,o} + \hat{R}_{t+1}\right], && (146)\\
\hat{s}_{t-1} + \nu\hat{c}_{t,o} - \varsigma\hat{\varpi}_t &= (\xi-\eta)\left(\frac{\hat{a}_t}{1-\eta}\right) + \eta\hat{c}_{t,y} - (1-\eta)\ell_v\hat{n}_t, && (147)\\
\left(\frac{\hat{a}_t}{1-\eta}\right) &= \alpha\left(\hat{c}_{t,y} + \ell_v\hat{n}_t\right) && (148)\\
&\quad +(1-\alpha)\left(\hat{k}_t + E_t\left[\frac{\eta}{1-\eta}\hat{\varpi}_{t+1} + \hat{c}_{t+1,o}\right]\right), && (149)
\end{aligned}$$

$$\hat{s}_t = (\xi - \eta)\left(\left(\frac{\hat{a}_t}{1-\eta}\right) - \hat{k}_t\right) \quad (150)$$

$$+(\eta - \nu)E_t\left[\frac{\eta}{1-\eta}\hat{\varpi}_{t+1} + \hat{c}_{t+1,o}\right], \quad (151)$$

where we replaced $\hat{x}_t$ everywhere with

$$\hat{x}_t = \hat{k}_t + E_t\left[\frac{\eta}{1-\eta}\hat{\varpi}_{t+1} + \hat{c}_{t+1,o}\right]. \quad (152)$$

A.4.9 Preparation for MATLAB implementation

In order to implement this system of equations, it is more convenient to use $\hat{\delta}^*_t = \bar{\delta}\hat{\delta}_t$ - since this is already in percent anyways and to handle $\bar{\delta} = 0$ as limit - as well as $\hat{a}^*_t = \left(\frac{\hat{a}_t}{1-\eta}\right)$. Write the system as

$$0 = \hat{A}_t + \ell_f(\hat{Z}_t + \hat{n}_t) - \hat{y}_t, \quad (153)$$

$$0 = \bar{R}\hat{R}_t - \bar{c}_y\bar{\pi}(\hat{c}_{t,y} + \hat{\pi}_t) - \bar{c}_o\hat{c}_{t,o} - \bar{k}\bar{\pi}(\hat{k}_t + \hat{\pi}_t), \quad (154)$$

$$0 = \hat{A}_t - \mu_f\left(\hat{Z}_t + \hat{n}_t\right) + \hat{Z}_t - \hat{\pi}_t - \hat{w}_t, \quad (155)$$

$$0 = -(\mu_v + \ell_v)\hat{n}_t - \hat{w}_t + \hat{c}_{t,y}, \quad (156)$$

$$0 = \bar{y}\hat{y}_t - \hat{\delta}^*_t - \bar{R}\hat{R}_t, \quad (157)$$

$$0 = (\xi - \eta)\hat{a}^*_t + \eta\hat{c}_{t,y} - (1-\eta)\ell_v\hat{n}_t \quad (158)$$

$$-\hat{s}_{t-1} - \nu\hat{c}_{t,o} + \varsigma\hat{\varpi}_t),$$

$$0 = -\eta\hat{k}_t + E_t\left[\eta\hat{\varpi}_{t+1} - \eta\hat{c}_{t+1,o} + \hat{R}_{t+1}\right] \quad (159)$$

$$+\eta\hat{c}_{t,y} - (1-\eta)\ell_v\hat{n}_t,$$

$$0 = \alpha\left(\hat{c}_{t,y} + \ell_v\hat{n}_t\right) - \hat{a}^*_t \quad (160)$$

$$+(1-\alpha)\left(\hat{k}_t + E_t\left[\frac{\eta}{1-\eta}\hat{\varpi}_{t+1} + \hat{c}_{t+1,o}\right]\right)$$

$$0 = (\xi - \eta)\left(\hat{a}^*_t - \hat{k}_t\right) - \hat{s}_t \quad (161)$$

$$+(\eta - \nu)E_t\left[\frac{\eta}{1-\eta}\hat{\varpi}_{t+1} + \hat{c}_{t+1,o}\right].$$

In order to incorporate the generational account perspective, we also add the equations

$$\begin{aligned} 0 &= \hat{\tau}_{t,priv} + \frac{\bar{c}_y}{\bar{w}\bar{n}}(\hat{c}_{y,t} - \hat{w}_t - \hat{n}_t) + \frac{\bar{k}}{\bar{w}\bar{n}}(\hat{k} - \hat{w}_t - \hat{n}_t) \\ 0 &= \hat{\tau}_{t,pub} + \frac{\bar{c}_y}{\bar{w}\bar{n}}(\hat{c}_{y,t} - \hat{w}_t - \hat{n}_t) \end{aligned}$$

to capture the movements in the ratios of lump sum taxes to wage earnings to be paid by the young, expressed in percent (rather than in percent deviation from the steady state of this ratio).

As state variables we choose $\hat{s}_t$, $\hat{c}_{t,y}$, $\hat{c}_{t,o}$, $\hat{a}_t^*$ and k. The first three come in due to the insights from some analytic calculations towards deriving the law of motion in closed form, available as a technical appendix in a working paper version, and are to having three rather than one equation containing expectations. The variable $\hat{a}_t^*$ has been added in order to avoid potential difficulties of a purely algebraic nature in the special case $\xi = \eta$. Finally, k has been added as a state to recalculate the impulse responses for the non-normalized variables, if so desired.

A.4.10 The recursive law of motion

We wish to solve for the linear recursive law of motion,

$$\begin{aligned} \hat{s}_t &= \varphi_{s,s}\hat{s}_{t-1} + \varphi_{s,\zeta}\zeta_t \\ \hat{n}_t &= \varphi_{n,s}\hat{s}_{t-1} + \varphi_{n,\zeta}\zeta_t \\ \hat{c}_{t,y} &= \varphi_{cy,s}\hat{s}_{t-1} + \varphi_{cy,\zeta}\zeta_t \\ \hat{c}_{t,o} &= \varphi_{co,s}\hat{s}_{t-1} + \varphi_{co,\zeta}\zeta_t \\ \hat{R}_t &= \varphi_{R,s}\hat{s}_{t-1} + \varphi_{R,\zeta}\zeta_t \end{aligned} \tag{162}$$

etc.., i.e. we wish to solve for the coefficients $\varphi_{(\cdot,s)} \in \mathbf{R}$ and $\varphi_{(\cdot,\zeta)} \in \mathbf{R}^5$ such that the linear recursive law of motion satisfies the loglinearized equations. Note that the linear recursive law of motion implies that the conditional variances and covariances are constant and given by e.g.

$$\begin{aligned} \sigma_{c,o}^2 &= \varphi_{co,\zeta}\Sigma\varphi'_{co,\zeta} \\ \rho_{c,o,R}\sigma_{c,o}\sigma_R &= \varphi_{co,\zeta}\Sigma\varphi'_{R,\zeta} \end{aligned}$$

In particular, the Sharpe ratio (100) can now be calculated.

In principle, this involves the calculation of a fixed point: the steady state requires knowledge of these variances and covariances, which can be calculated, given the linear recursive law of motion. But the latter is a solution to system of equations, whose coefficients depend on the steady state. An iterative procedure typically works well. As a first step, assume these variances, covariances and the Sharpe ratio to be zero, in which case one obtains the nonstochastic steady state. Use it to generate the loglinear approximation and solve it for the recursive law of motion. Calculate the implied variances and covariances, and use them to recalculate the steady state, etc..

This procedure was used in e.g. Canton (1997, 2002) in a different context. The procedure typically converges fast. In fact, typically a single step often suffices for all practical purposes. For that, use the nonstochastic steady state to generate a linear recursive law of motion, and use the latter to calculate variances and covariances.

A.5 A closed-form solution

We shall now provide a closed-form solution for the recursive law of motion, given the loglinearized system. The procedure follows the methodology explained in Uhlig (1999). We proceed in three steps. The first two steps concentrate entirely on calculating the deterministic law of motion. In the first step, we reduce the loglinearized equations to a system of three equations in $\hat{s}_t$, $\hat{c}_{t,y}$ and $\hat{c}_{t,o}$ and their leads and lags. Plugging in the recursive law of motion, we obtain a system of three equations in the three coefficients $\varphi_{cy,s}$, $\varphi_{co,s}$ and $\varphi_{s,s}$, which we reduce to a quadratic equation in $\varphi_{s,s}$ and solve in the second step. With the solution to thee deterministic part at hand, we proceed to calculate the coefficients on the stochastic part by solving a linear system of equations in the third step.

A.5.1 Step 1: Reducing the system

Due to linearity, the solution to the deterministic part is obtained by solving the loglinearized system under the assumption that all shocks are equal to zero, i.e.

$$\hat{y}_t = \ell_f \hat{n}_t \tag{163}$$

$$
\begin{aligned}
\bar{c}_y \bar{\pi} \hat{c}_{t,y} + \bar{c}_o \hat{c}_{t,o} + \bar{k}\bar{\pi}\hat{k}_t &= \bar{A} f(\bar{Z}\bar{n}) \ell_f \hat{n}_t && (164)\\
\hat{w}_t &= -\mu_f \hat{n}_t && (165)\\
\hat{w}_t - \hat{c}_{t,y} &= -(\mu_v + \ell_v)\hat{n}_t && (166)\\
\bar{R}\hat{R}_t &= \bar{A} f(\bar{Z}\bar{n}) \ell_f \hat{n}_t && (167)\\
-\eta \hat{c}_{t,y} + (1-\eta)\ell_v \hat{n}_t &= -\eta \hat{k}_t - \eta \hat{c}_{t+1,o} + \hat{R}_{t+1} && (168)\\
\hat{s}_{t-1} + \nu \hat{c}_{t,o} &= (\xi - \eta)\left(\frac{\hat{a}_t}{1-\eta}\right) + \eta \hat{c}_{t,y} - (1-\eta)\ell_v \hat{n}_t && (169)\\
\left(\frac{\hat{a}_t}{1-\eta}\right) &= \alpha\left(\hat{c}_{t,y} + \ell_v \hat{n}_t\right) + (1-\alpha)\left(\hat{k}_t + \hat{c}_{t+1,o}\right) && (170)\\
\hat{s}_t &= (\xi - \eta)\left(\left(\frac{\hat{a}_t}{1-\eta}\right) - \hat{k}_t\right) + (\eta - \nu)\hat{c}_{t+1,o} && (171)
\end{aligned}
$$

The first equation (163) is not needed for the reduction. Use the third and the forth equation (165), (166) to express $\hat{n}_t$ in terms of $\hat{c}_{t,y}$. Use that in the second equation (164) to express $\hat{k}_t$ in terms of $\hat{c}_{t,y}$ and $\hat{c}_{t,o}$, in the fifth equation (167) to express $\hat{R}_t$ in terms of $\hat{c}_{t,y}$ and in the seventh equation (169) to express $\hat{a}_t/(1-\eta)$ in terms of $\hat{c}_{t,y}$, $\hat{c}_{t,o}$ and $\hat{c}_{t+1,o}$,

$$
\begin{aligned}
\hat{n}_t &= \theta_{n,y}\hat{c}_{t,y}\\
\hat{k}_t &= \theta_{k,y}\hat{c}_{t,y} + \theta_{k,o}\hat{c}_{t,o}\\
\hat{R}_t &= \theta_{R,y}\hat{c}_{t,y}\\
\frac{\hat{a}_t}{1-\eta} &= \theta_{a,y}\hat{c}_{t,y} + \theta_{a,o}\hat{c}_{t,o} + \theta_{a,Eo}\hat{c}_{t+1,o}
\end{aligned}
$$

where

$$
\begin{aligned}
\theta_{n,y} &= \frac{1}{\mu_v + \ell_v - \mu_f}\\
\theta_{k,y} &= \frac{\bar{A} f(\bar{Z}\bar{n})\ell_f}{\bar{k}\bar{\pi}}\theta_{n,y} - \frac{\bar{c}_y}{\bar{k}}\\
\theta_{k,o} &= -\frac{\bar{c}_o}{\bar{k}\bar{\pi}}\\
\theta_{R,y} &= \frac{\bar{A} f(\bar{Z}\bar{n})\ell_f}{\bar{R}}\theta_{n,y}\\
\theta_{a,y} &= \alpha(1 + \ell_v\theta_{n,y}) + (1-\alpha)\theta_{k,y}
\end{aligned}
$$

$$\begin{aligned}\theta_{a,o} &= (1-\alpha)\theta_{k,o} \\ \theta_{a,Eo} &= 1-\alpha\end{aligned}$$

Use these results in the remaining sixth, seventh and ninth equation (168) (169) and (171) to obtain

$$\begin{array}{rclcccccccc} 0 &=& & & \theta_{1,y}\hat{c}_{t,y} &+& \theta_{1,o}\hat{c}_{t,o} &+& \theta_{1,Ey}\hat{c}_{t+1,y} &+& \theta_{1,Eo}\hat{c}_{t+1,o} \\ s_{t-1} &=& & & \theta_{2,y}\hat{c}_{t,y} &+& \theta_{2,o}\hat{c}_{t,o} & & &+& \theta_{2,Eo}\hat{c}_{t+1,o} \\ 0 &=& -s_t &+& \theta_{3,y}\hat{c}_{t,y} &+& \theta_{3,o}\hat{c}_{t,o} & & &+& \theta_{3,Eo}\hat{c}_{t+1,o} \end{array} \tag{172}$$

where

$$\begin{aligned}\theta_{1,y} &= \eta-(1-\eta)\ell_v\theta_{n,y}-\eta\theta_{k,y} \\ \theta_{1,o} &= -\eta\theta_{k,o} \\ \theta_{1,Ey} &= \theta_{R,y} \\ \theta_{1,Eo} &= -\eta \\ \theta_{2,y} &= (\xi-\eta)\theta_{a,y}+\eta-(1-\eta)\ell_v\theta_{n,y} \\ \theta_{2,o} &= (\xi-\eta)\theta_{a,o}-\nu \\ \theta_{2,Eo} &= (\xi-\eta)\theta_{a,Eo} \\ \theta_{3,y} &= (\xi-\eta)(\theta_{a,y}-\theta_{k,y}) \\ \theta_{3,o} &= (\xi-\eta)(\theta_{a,o}-\theta_{k,o}) \\ \theta_{3,Eo} &= (\xi-\eta)\theta_{a,Eo}+\eta-\nu\end{aligned}$$

A.5.2 Step 2: A system of three coefficient equations

Use the deterministic part of the recursive law of motion

$$\begin{aligned}\hat{s}_t &= \varphi_{s,s}\hat{s}_{t-1} \\ \hat{c}_{t,y} &= \varphi_{cy,s}\hat{s}_{t-1} \\ \hat{c}_{t,o} &= \varphi_{co,s}\hat{s}_{t-1}\end{aligned}$$

to replace all variables except s_{t-1}. For variables dated $t+1$, this requires "plugging in twice". Comparing coefficients (or, equivalently, dividing by

s_{t-1}) yields

$$\begin{array}{lcccccccccc} 0 & = & & & \theta_{1,y}\varphi_{cy,s} & + & \theta_{1,o}\varphi_{co,s} & + & \theta_{1,Ey}\varphi_{s,s}\varphi_{cy,s} & + & \theta_{1,Eo}\varphi_{s,s}\varphi_{co,s} \\ 1 & = & & & \theta_{2,y}\varphi_{cy,s} & + & \theta_{2,o}\varphi_{co,s} & & & + & \theta_{2,Eo}\varphi_{s,s}\varphi_{co,s} \\ 0 & = & -\varphi_{s,s} & + & \theta_{3,y}\varphi_{cy,s} & + & \theta_{3,o}\varphi_{co,s} & & & + & \theta_{3,Eo}\varphi_{s,s}\varphi_{co,s} \end{array} \tag{173}$$

Multiply the second equation with $\theta_{3,Eo}$ and the third with $-\theta_{2,Eo}$, add and solve for $\varphi_{cy,s}$,

$$\varphi_{cy,s} = \theta_{cy} + \theta_{cy,s}\varphi_{s,s} + \theta_{cy,co}\varphi_{co,s} \tag{174}$$

where

$$\begin{array}{rcl} \theta_{cy} & = & \dfrac{\theta_{3,Eo}}{\theta_{3,Eo}\theta_{2,y} - \theta_{2,Eo}\theta_{3,y}} \\ \theta_{cy,s} & = & \dfrac{-\theta_{2,Eo}}{\theta_{3,Eo}\theta_{2,y} - \theta_{2,Eo}\theta_{3,y}} \\ \theta_{cy,co} & = & \dfrac{\theta_{2,Eo}\theta_{3,o} - \theta_{3,Eo}\theta_{2,o}}{\theta_{3,Eo}\theta_{2,y} - \theta_{2,Eo}\theta_{3,y}} \end{array}$$

Use that in the first two equations to replace $\varphi_{cy,s}$,

$$\begin{array}{lcccccccccc} 0 & = & \theta_4 & + & \theta_{4,s}\varphi_{s,s} & + & \theta_{4,o}\varphi_{co,s} & + & \theta_{4,ss}\varphi_{s,s}^2 & + & \theta_{4,so}\varphi_{s,s}\varphi_{co,s} \\ 0 & = & \theta_5 & + & \theta_{5,s}\varphi_{s,s} & + & \theta_{5,o}\varphi_{co,s} & & & + & \theta_{5,so}\varphi_{s,s}\varphi_{co,s} \end{array} \tag{175}$$

where

$$\begin{array}{rcccc} \theta_4 & = & \theta_{1,y}\theta_{cy} & & \\ \theta_{4,s} & = & \theta_{1,y}\theta_{cy,s} & & \\ \theta_{4,o} & = & \theta_{1,y}\theta_{cy,co} & + & \theta_{1,co} \\ \theta_{4,ss} & = & \theta_{1,Ey}\theta_{cy,s} & & \\ \theta_{4,so} & = & \theta_{1,Ey}\theta_{cy,co} & + & \theta_{1,Eo} \\ \theta_5 & = & \theta_{2,y}\theta_{cy} & - & 1 \\ \theta_{5,s} & = & \theta_{2,y}\theta_{cy,s} & & \\ \theta_{5,o} & = & \theta_{2,y}\theta_{cy,co} & + & \theta_{2,o} \\ \theta_{5,so} & = & & & \theta_{2,Eo} \end{array}$$

Multiply the first of these two equations with $\theta_{5,so}$, the second with $-\theta_{4,so}$, add and solve for $\varphi_{co,s}$,

$$\varphi_{co,s} = \theta_{co} + \theta_{co,s}\varphi_{s,s} + \theta_{co,ss}\varphi_{s,s}^2 \tag{176}$$

where

$$\begin{aligned}\theta_{co} &= \frac{\theta_{4,so}\theta_5 - \theta_{5,so}\theta_4}{\theta_{5,so}\theta_{4,o} - \theta_{4,so}\theta_{5,o}} \\ \theta_{co,s} &= \frac{\theta_{4,so}\theta_{5,s} - \theta_{5,so}\theta_{4,s}}{\theta_{5,so}\theta_{4,o} - \theta_{4,so}\theta_{5,o}} \\ \theta_{co,ss} &= \frac{-\theta_{5,so}\theta_{4,ss}}{\theta_{5,so}\theta_{4,o} - \theta_{4,so}\theta_{5,o}}\end{aligned}$$

Use this to replace $\varphi_{co,s}$ in either of the two equations of (175). We use the second equation and obtain

$$0 = \theta_s + \theta_{s,s}\varphi_{s,s} + \theta_{s,ss}\varphi_{s,s}^2 + \theta_{s,sss}\varphi_{s,s}^3 \tag{177}$$

where

$$\begin{array}{lclclcl}\theta_s &=& \theta_5 &+& \theta_{5,o}\theta_{co} && \\ \theta_{s,s} &=& \theta_{5,s} &+& \theta_{5,o}\theta_{co,s} &+& \theta_{5,so}\theta_{co} \\ \theta_{s,ss} &=& && \theta_{5,o}\theta_{co,ss} &+& \theta_{5,so}\theta_{co,s} \\ \theta_{s,sss} &=& && && \theta_{5,so}\theta_{co,ss}\end{array}$$

Equation (177) is a polynomial of third degree, which generally has three roots, $\varphi_{s,s,i}, i = 1, 2, 3$, and for which closed-form solutions are available. If only one of these roots is stable, i.e. less than one in absolute value, then this is the root we use. If there are more than one stable root, then an additional state variable is needed as generally would be necessary to solve the system (175). If there are no stable roots, the system is explosive and our baseline assumption that there is a stationary solution to the social planners problem in the detrended variables unjustified.

Experimentation with reasonable parameter choices has only delivered the case of a single stable root. We shall therefore concentrate on that case from here on. Thus, let $\varphi_{s,s}$ be that solution.

The other two key coefficients $\varphi_{co,s}$ and $\varphi_{cy,s}$ can now be found from equations (176) and (174).

For the remaining variables, we have

$$\begin{array}{lclclcl}\varphi_{n,s} &=& \theta_{n,y}\varphi_{cy,s} &&&& \\ \varphi_{y,s} &=& \ell_f\theta_{n,y}\varphi_{cy,s} &&&& \\ \varphi_{k,s} &=& \theta_{k,y}\varphi_{cy,s} &+& \theta_{k,o}\varphi_{co,s} && \\ \varphi_{w,s} &=& -\mu_f\theta_{n,y}\varphi_{cy,s} &&&& \\ \varphi_{R,s} &=& \theta_{R,y}\varphi_{cy,s} &&&& \\ \varphi_{a,s} &=& \theta_{a,y}\varphi_{cy,s} &+& \theta_{a,o}\varphi_{co,s} &+& \theta_{a,Eo}\varphi_{s,s}\varphi_{co,s}\end{array} \tag{178}$$

where $\varphi_{a,s}$ is the feedback coefficient for $\frac{\hat{a}_t}{1-\eta}$.

We summarize these feedback coefficients per

$$\varphi_{\cdot,s} = \begin{bmatrix} \varphi_{s,s} \\ \varphi_{cy,s} \\ \varphi_{co,s} \\ \varphi_{n,s} \\ \varphi_{y,s} \\ \varphi_{k,s} \\ \varphi_{w,s} \\ \varphi_{R,s} \\ \varphi_{a,s} \end{bmatrix} \tag{179}$$

A.5.3 Step 3: Solving for the coefficients on exogenous variables.

Solving for the exogenous variables is now a matter of solving a linear system of equations. In the equations (140) to (151) replace each endogenous variable dated t with the feedback rule given (162). The variable $\hat{s}_{t-1}$ stays as is. The variables dated $t+1$ show up in expectations, and are replaced with the feedback rules as e.g. in

$$\begin{aligned} E_t[c_{t+1,o}] &= E_t[\varphi_{co,s}s_t + \varphi_{co,\zeta}\zeta_{t+1}] \\ &= \varphi_{co,s}\varphi_{s,s}s_{t-1} + (\varphi_{co,s}\varphi_{co,\zeta} + \varphi_{co,\zeta}N)\,\zeta_t \end{aligned}$$

where $\varphi_{s,s}$ and $\varphi_{co,s}$ are now known, while we still seek to solve for $\varphi_{co,\zeta}$.

The resulting system contains coefficients on the variable s_{t-1}, which we already know to hold from the calculations above. Let $\varphi_{\cdot,\zeta}$ be the matrix of the to-be-solved-for feedback coefficients on the exogenous variables ζ, given by

$$\varphi_{\cdot,\zeta} = \begin{bmatrix} \varphi_{s,\zeta} \\ \varphi_{cy,\zeta} \\ \varphi_{co,\zeta} \\ \varphi_{n,\zeta} \\ \varphi_{y,\zeta} \\ \varphi_{k,\zeta} \\ \varphi_{w,\zeta} \\ \varphi_{R,\zeta} \\ \varphi_{a,\zeta} \end{bmatrix} \tag{180}$$

where $\varphi_{a,\zeta}$ is the feedback coefficient for $\frac{\hat{a}_t}{1-\eta}$. Take the feedback coefficients $\varphi_{\cdot,s}$ on $\hat{s}_{t-1}$ as given via the calculations above, and compare coefficients on the entries in ζ_t. By carefully examining the system or, alternatively, exploiting the matrix algebra provided in Uhlig (1999), the remaining system can be written in the form

$$V\mathbf{vec}(\varphi_{\cdot,\zeta}) = W$$

for some matrices V and W and the columnwise vectorization $\mathbf{vec}(\varphi_{\cdot,\zeta})$ of the matrix of coefficients $\varphi_{\cdot,\zeta}$. If $N = 0$, i.e., if the exogenous variables are iid, then this can be written more conveniently as

$$(C\varphi_{\cdot,\zeta}) = D$$

In either case, one obtains a linear system in the entries of $\varphi_{\cdot,\zeta}$, which can be solved under the usual conditions for invertibility. We shall skip the tedious details on explicitely stating V and W or C and D.

A.5.4 Impulse responses

Define the vector of endogenous variables

$$\psi_t = \begin{bmatrix} \hat{s}_t \\ \hat{c}_{y,t} \\ \hat{c}_{o,t} \\ \hat{n}_t \\ \hat{y}_t \\ \hat{k}_t \\ \hat{w}_t \\ \hat{R}_t \\ \hat{a}_t/(1-\eta) \end{bmatrix}$$

With the solution above, one can now determine the effect of a shock ϵ_0 recursively per

$$\begin{array}{rclcrclcl} \zeta_0 & = & \epsilon_0 & , & \psi_0 & = & & & \varphi_{\cdot,\zeta}\zeta_0 \\ \zeta_1 & = & N\zeta_0 & , & \psi_1 & = & \varphi_{\cdot,s}(\psi_0)_s & + & \varphi_{\cdot,\zeta}\zeta_1 \\ \zeta_2 & = & N\zeta_1 & , & \psi_2 & = & \varphi_{\cdot,s}(\psi_1)_s & + & \varphi_{\cdot,\zeta}\zeta_2 \end{array} \quad (181)$$

etc., where e.g. $(\psi_0)_s$ is the first entry (corresponding to s_0) of ψ_0.

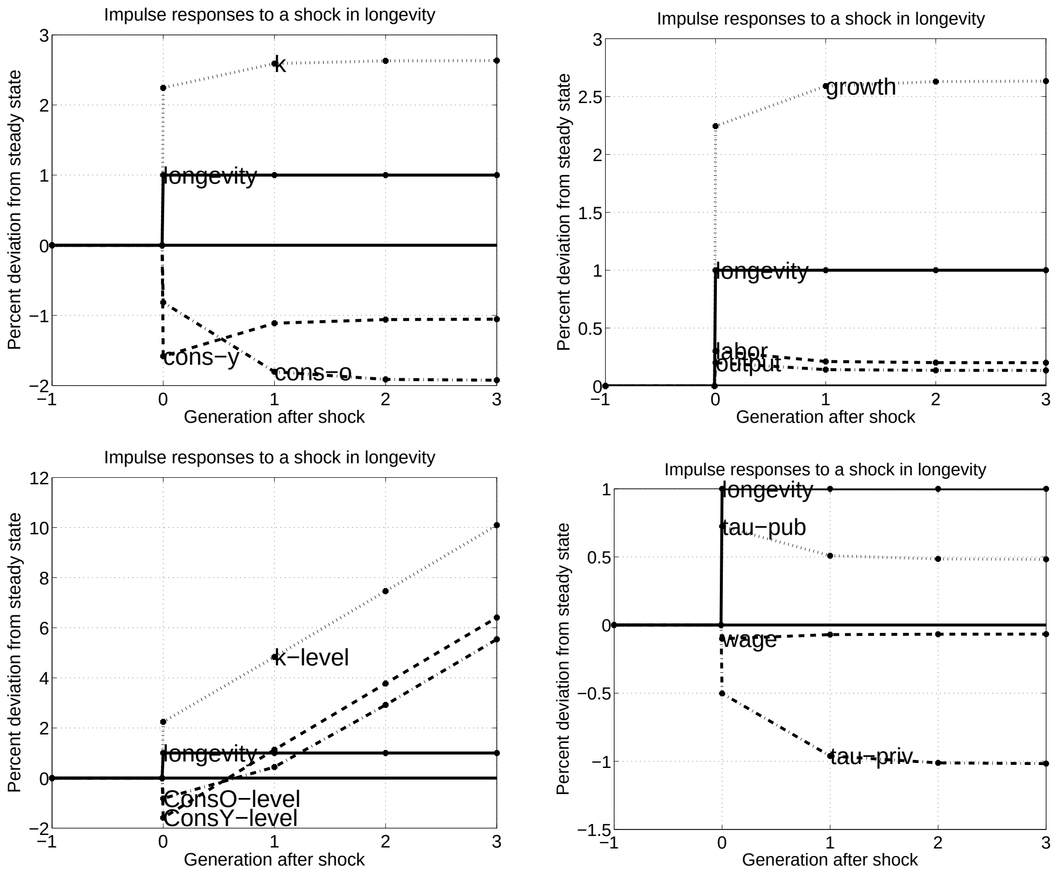


Figure 1: *Impulse responses for a one percent shock to longevity, given the benchmark parameterization*

Tables and Figures

Notation	meaning	first occurance
A_t	total factor productivity	(7)
a_t	argument of $z(\cdot)$.	(28)
$C_{t,y}$	aggregate cons. of young at t	(9)
$c_{t,y}$	cons. of gen. born in t, when young	(2)
$\tilde{c}_{t,y}$	rescaled consumption	below definition 5
$C_{t,o}$	aggregate cons. of old at t	(9)
$c_{t+1,o}$	cons. of gen. born in t, when old	(2)
$\tilde{c}_{t,o}$	rescaled consumption	below definition 5
d_t	discounting correction	(20)
$f(\cdot)$	production function	(7)
$\mathcal{H}_t$	set of all possib. histories at t	below (11)
$\mathcal{H}$	set of all possib. histories	below (11)
h_t	history up to t	(11)
K_{t-1}	aggr. capital, used in prod. at t	(7)
k_{t-1}	capital in prod. at t, per old	(8)
$\ell_{u,c,t}$	logarithm. deriv. of u w.r.t. c	above (31)
$\ell_{u,n,t}$	logarithm. deriv. of u w.r.t. n	above (31)
$\ell_{x,t}$	logarithm. deriv. of x	above (31)
$\ell_{q,t}$	logarithm. deriv. of q	above (31)
ℓ_v	logarithm. deriv. of v	(43)
m_{t+1}	discount factor of soc. plan.	(23)
n_t	labor supply of gen. born in t, when young	(2)
$q(\cdot)$	utility function when old	(2)
R_t	social rate of return	(24)
$\mathcal{S}$	state space for date-t-variables	below (10)
$\mathbf{SR}_t$	Sharpe ratio	(98)
$U(\cdot)$	overall utility	(2)
$U_{t,t}$	overall utility of generation t, cond. on t-info	(2)
$u(\cdot,\cdot)$	instantaneous utility when young	(2)
u_o	parameter for initially old	(5)
$U_t(h_t)$	ex-post utility	below definition 1
$U_{t,0}$	ex-ante expected utility at beginning of time	(12)
$x(\cdot)$	risk aversion transformation	(2)
Y_t	aggregate output at t	(7)
y_t	output at t, per old	(8)
$z(\cdot)$	overall ex-ante risk-aversion function	(2)
Z_t	labor-specific productivity t	(7)

Table 1: Summary table for the notation and symbols used in this paper, part 1.

Notation	meaning	first occurance
α_t	argument share	above (31)
β	discount factor of agent	(2)
χ	pref. parameter for leisure	(55)
δ_t	depreciation rate	(9)
$\mu_{z,t}$	(negative) elasticity of $z'(\cdot)$	above (31)
$\mu_{uc,c,t}$	(negative) elasticity of $u_c(\cdot,\cdot)$ w.r.t. c	above (31)
$\mu_{uc,n,t}$	(negative) elasticity of $u_c(\cdot,\cdot)$w.r.t. n	above (31)
$\mu_{x,t}$	(negative) elasticity of $x'(\cdot)$	above (31)
$\mu_{q,t}$	(negative) elasticity of $q'(\cdot)$	above (31)
ϕ	scale factor	in definition 5
$\Phi(\cdot)$	allocation	in definition 1
$\tilde{\Phi}$	rescaled allocation	in definition 5
γ_t	extra (policy) shock	below (10)
η	inverse of intertemp. elast. of subst.	above (40), (42)
ι	x^{-1}	just below (78)
$\kappa(n_t)$	integration constant	(40)
κ	integration constant	below (40)
λ_t	Lagrange multiplicator on feasibility	(14)
ν	risk avers. wrt old-age risk	(44)
Π_t	young population	(6)
π_t	young population growth factor	(6)
σ_y	income share	(88)
σ_t	return subsidy	(69)
$\tau_{t,y}$	lump sum tax on young	(68)
$\tau_{t,o}$	lump sum tax on old	(69)
$\tau_{t,priv}$	tax, if priv. cap.	(70)
$\tau_{t,pub}$	tax, if publ. cap.	(71)
ξ	life-time risk aversion parameter	(44)
ω_t	welfare weight	in definition 4
ω	welfare weight factor	below definition 4
ρ	correlation	before (98)
ψ	production function param.	(56)
θ	production function param.	(56)
ϖ_t	expected life time of the old in period t	(2)
ς	exponent of ϖ	(102)
ζ	vector of exog. var.	(103)

Table 2: Summary table for the notation and symbols used in this paper, part 2.

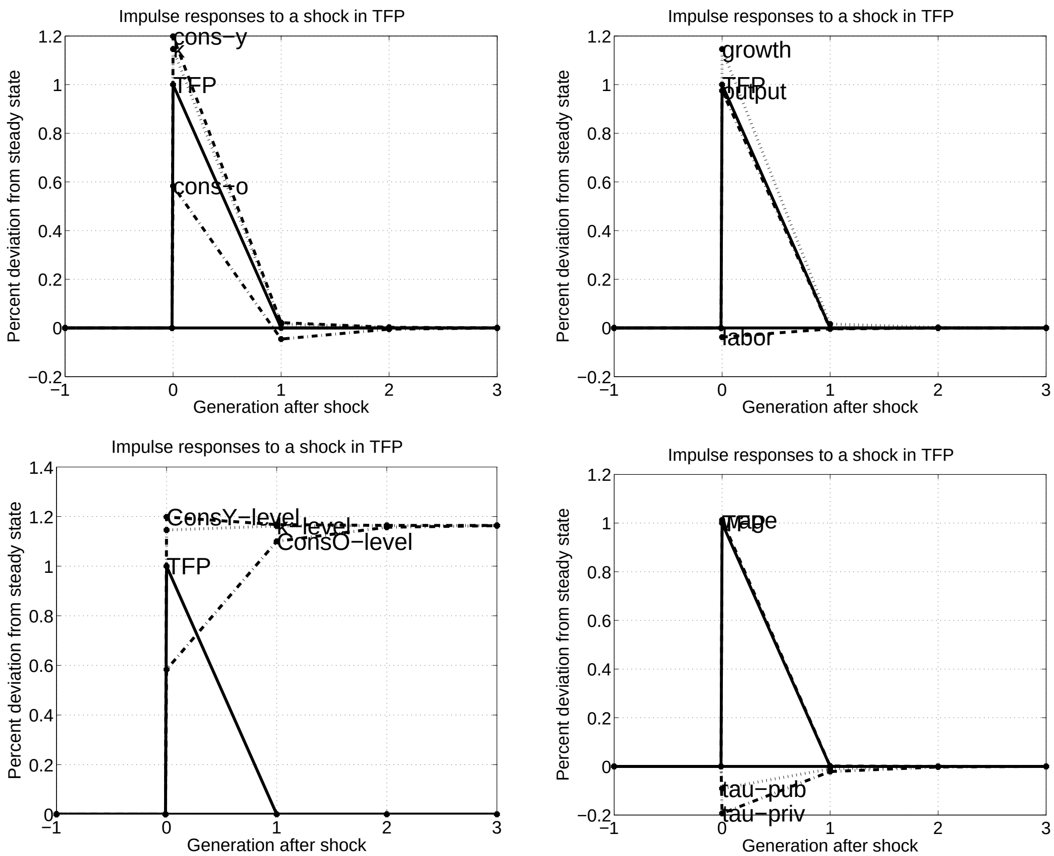


Figure 2: *Impulse responses for a one percent shock to total factor productivity, given the benchmark parameterization*

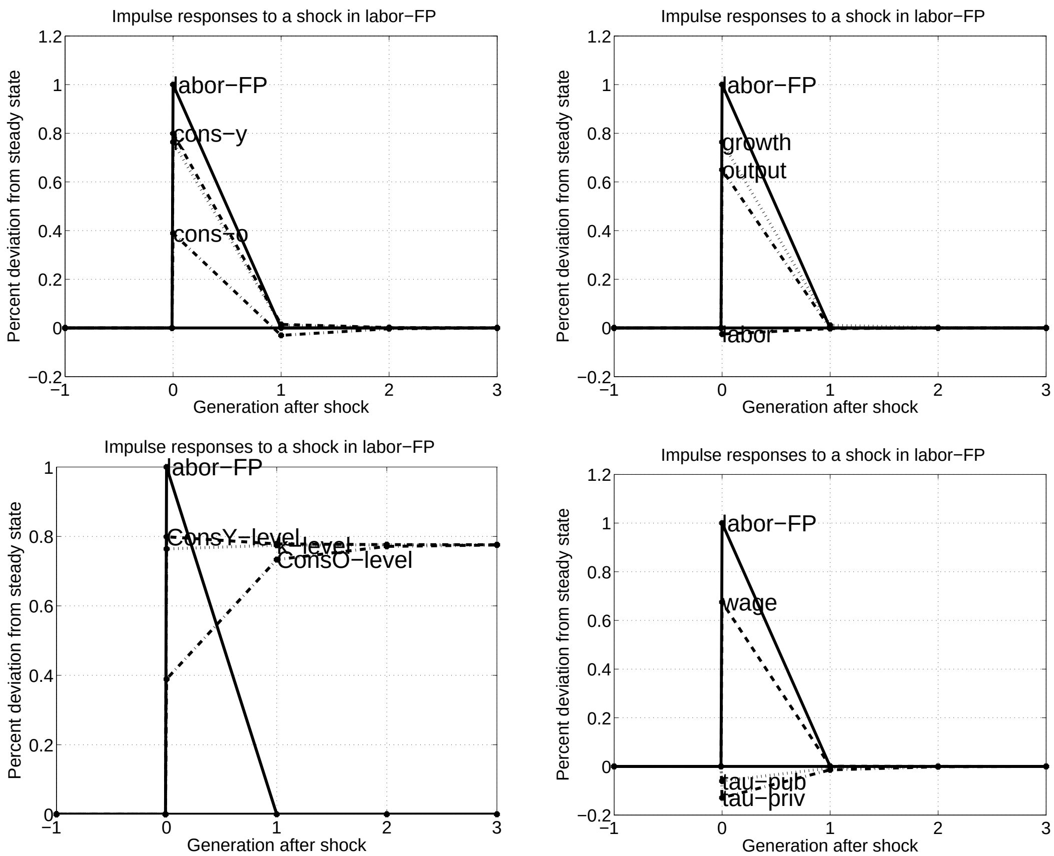


Figure 3: *Impulse responses for a one percent shock to labor productivity, given the benchmark parameterization*

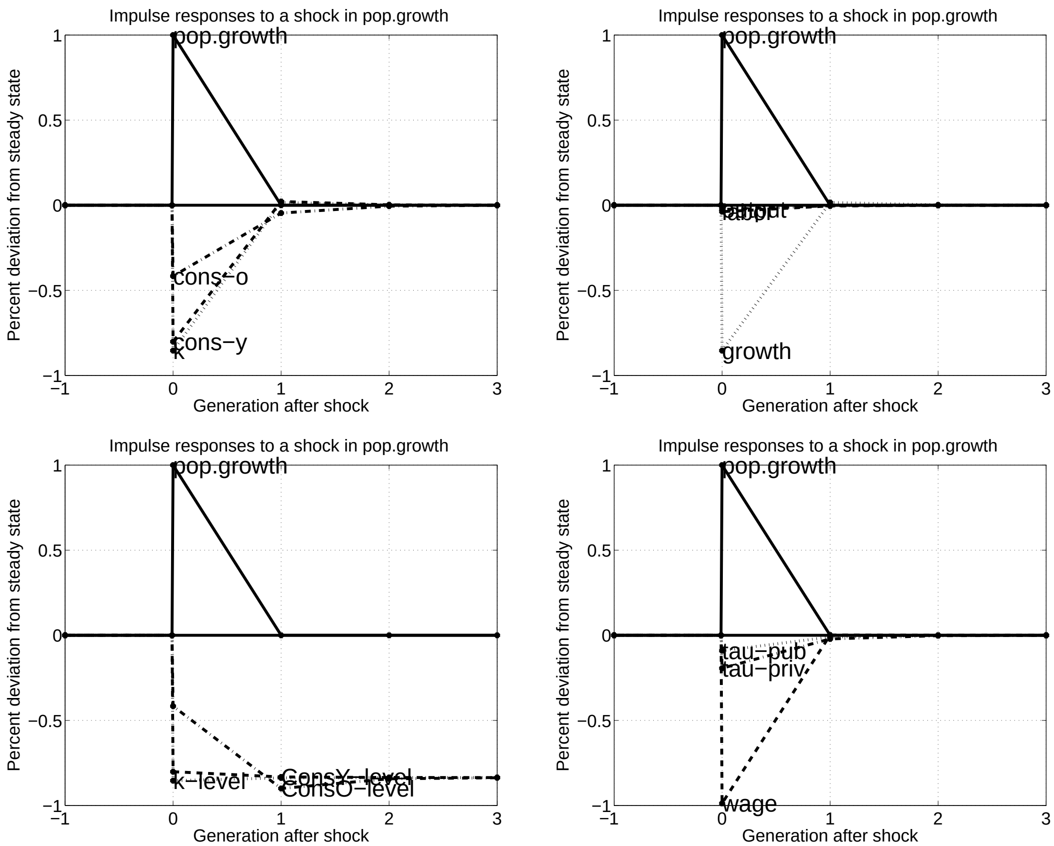


Figure 4: *Impulse responses for a one percent shock to population growth, given the benchmark parameterization*

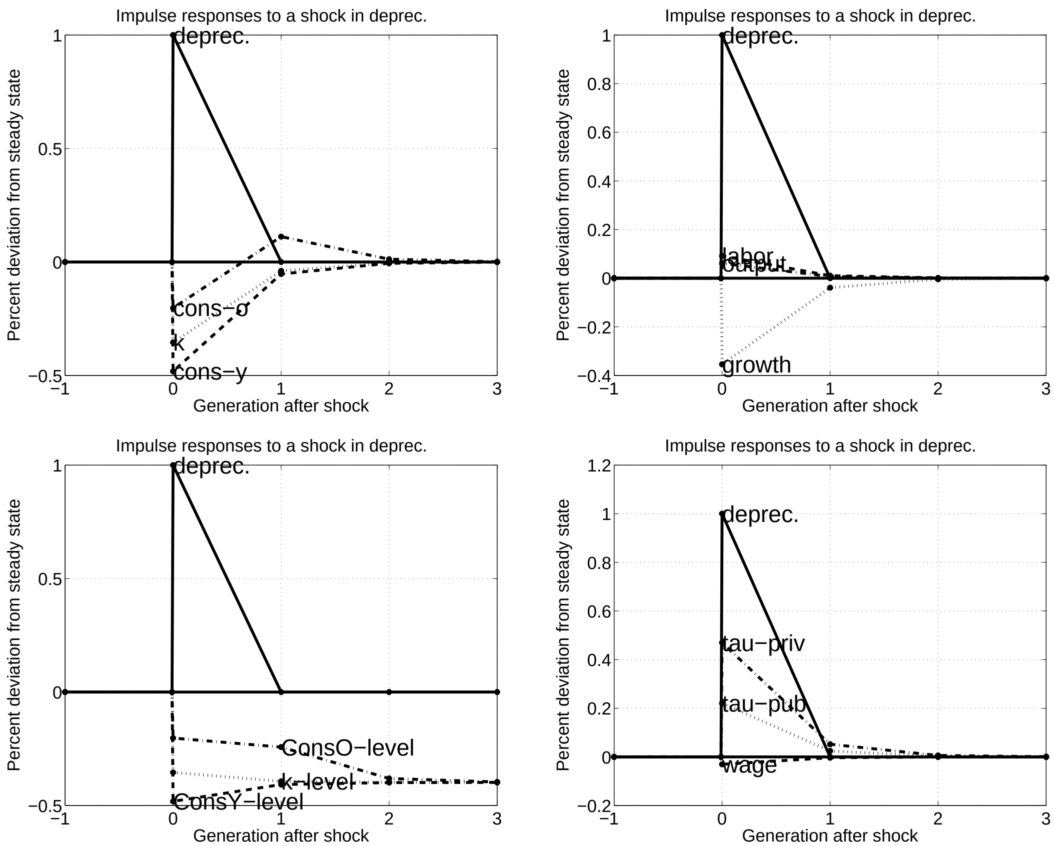


Figure 5: *Impulse responses for a one percent shock to depreciation, given the benchmark parameterization*

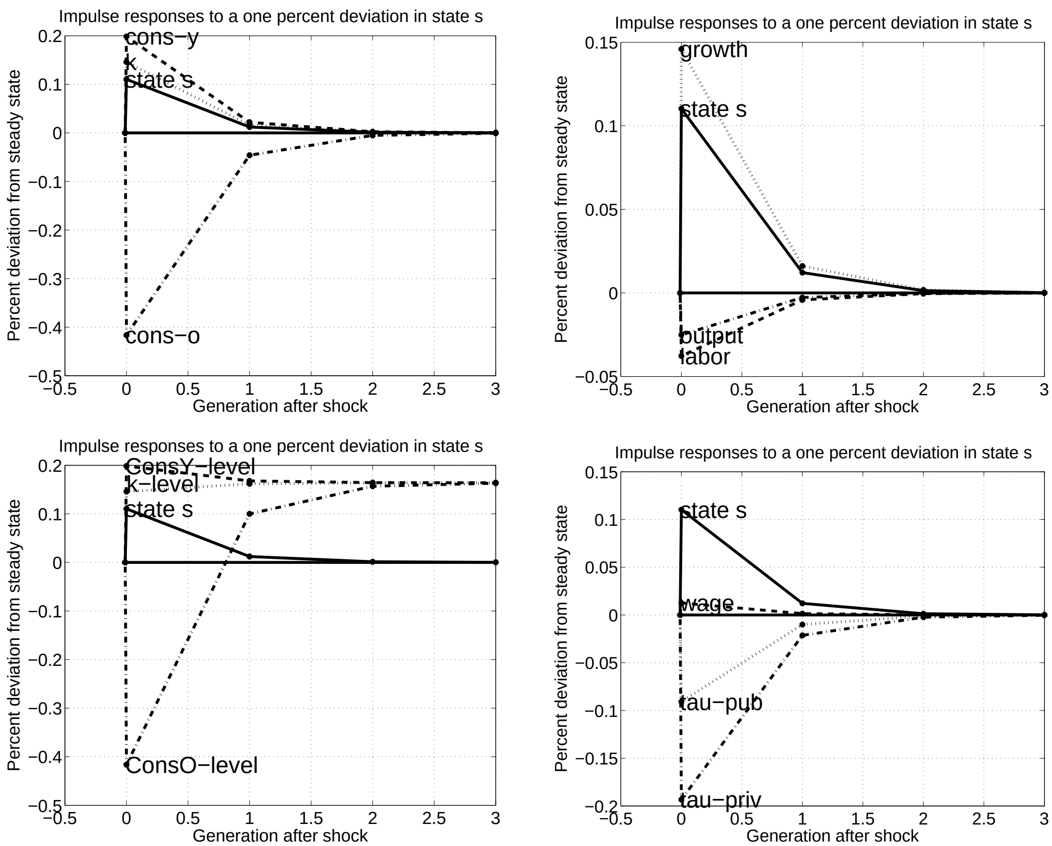


Figure 6: *Impulse responses for an initial one percent deviation of the state from its steady state.*

$\xi =$	2.00	2.00	2.00	0.50	0.50	0.50
$\bar{\delta} =$	1.00	0.00	1.00	1.00	0.00	1.00
$\omega = \beta =$	0.40	0.40	0.80	0.40	0.40	0.80
$\bar{n} =$	0.83	0.80	0.83	0.83	0.83	0.83
$\bar{k} =$	1.00	1.17	1.00	1.00	1.94	1.00
$\bar{\pi}\bar{c}_y/\bar{y} =$	0.27	0.33	0.27	0.27	0.28	0.27
$\bar{c}_o/\bar{c}_y =$	1.22	1.81	1.22	1.22	1.20	1.22
$\tau_{priv} =$	-0.65	-21.83	-0.65	-0.56	-59.68	-0.56
$\tau_{pub} =$	59.38	50.35	59.38	59.37	57.62	59.37
$\varphi_{s,s} =$	0.15	0.19	0.12	-0.00	-0.00	-0.00

Table 3: *A comparison of steady states, when varying some parameters.*

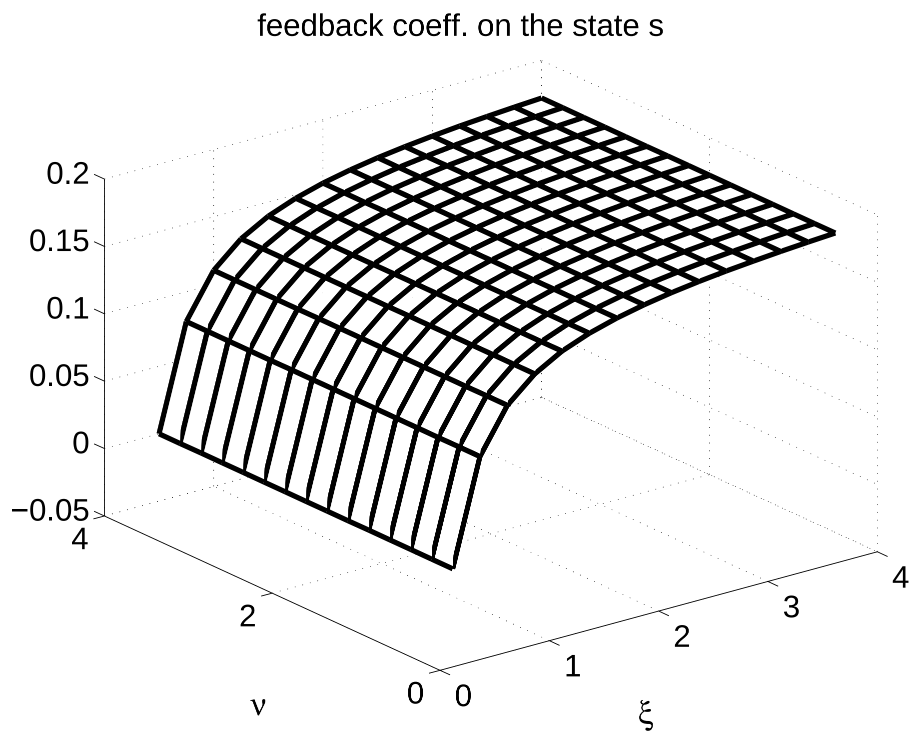


Figure 7: *Endogenous dynamics, i.e. feedback coefficient φ_{ss} on the endogenous state, as the risk aversion parameters are varied.*

$\xi =$	2.00	2.00	2.00	0.50	0.50	0.50
$\bar{\delta} =$	1.00	0.00	1.00	1.00	0.00	1.00
$\omega = \beta =$	0.40	0.40	0.80	0.40	0.40	0.80
$\varphi_{cy,Z}$	1.00	0.71	1.00	1.38	0.83	1.38
$\varphi_{co,Z}$	1.00	0.72	1.00	0.30	0.23	0.30
$\varphi_{n,Z}$	0.00	0.07	-0.00	-0.07	0.03	-0.07
$\varphi_{y,Z}$	1.00	1.04	1.00	0.95	1.02	0.95
$\varphi_{growth,Z}$	1.00	0.91	1.00	1.20	1.77	1.20
$\varphi_{\tau,priv,Z}$	0.00	0.36	0.00	-0.32	0.21	-0.32
$\tau_{priv,tot}$ for Z	-0.01	0.13	-0.01	-0.33	-0.40	-0.33
$\varphi_{\tau,pub,Z}$	0.00	0.17	-0.00	-0.17	0.08	-0.17
$\tau_{pub,tot}$ for Z	0.59	0.69	0.59	0.39	0.67	0.39
c_o/c_{tot} for Z	0.00	0.00	-0.00	-0.49	-0.27	-0.49
$\varphi_{cy,\pi}$	-0.69	-0.58	-0.67	-0.87	-0.90	-0.87
$\varphi_{co,\pi}$	-0.69	-0.60	-0.71	-0.23	-0.24	-0.23
$\varphi_{n,\pi}$	-0.06	-0.10	-0.06	-0.02	-0.02	-0.02
$\varphi_{y,\pi}$	-0.04	-0.06	-0.04	-0.02	-0.01	-0.02
$\varphi_{growth,\pi}$	-0.74	-0.80	-0.75	-0.93	-1.84	-0.93
$\varphi_{\tau,priv,\pi}$	-0.32	-0.52	-0.33	-0.11	-0.12	-0.11
$\tau_{priv,tot}$ for π	-0.32	-0.51	-0.33	-0.11	-0.11	-0.11
$\varphi_{\tau,pub,\pi}$	-0.14	-0.24	-0.15	-0.06	-0.05	-0.06
$\tau_{pub,tot}$ for π	-0.17	-0.27	-0.18	-0.07	-0.05	-0.07
c_o/c_{tot} for π	-0.45	-0.36	-0.47	-0.16	-0.15	-0.16
$\varphi_{cy,\varpi}$	-0.93	-0.85	-1.02	-3.49	-7.95	-3.49
$\varphi_{co,\varpi}$	-0.39	-0.24	-0.20	-0.96	-1.56	-0.96
$\varphi_{n,\varpi}$	0.18	0.19	0.19	0.66	1.57	0.66
$\varphi_{y,\varpi}$	0.12	0.13	0.13	0.44	1.05	0.44
$\varphi_{growth,\varpi}$	1.25	1.34	1.18	4.26	9.50	4.26
$\varphi_{\tau,priv,\varpi}$	-0.25	-0.25	-0.16	-0.70	-0.69	-0.70
$\tau_{priv,tot}$ for ϖ	-0.25	-0.28	-0.17	-0.70	-1.31	-0.70
$\varphi_{\tau,pub,\varpi}$	0.43	0.48	0.47	1.60	3.81	1.60
$\tau_{pub,tot}$ for ϖ	0.50	0.55	0.54	1.86	4.42	1.86
c_o/c_{tot} for ϖ	0.25	0.22	0.37	1.14	2.91	1.14

Table 4: *Parameter variations and feedback coefficients.*

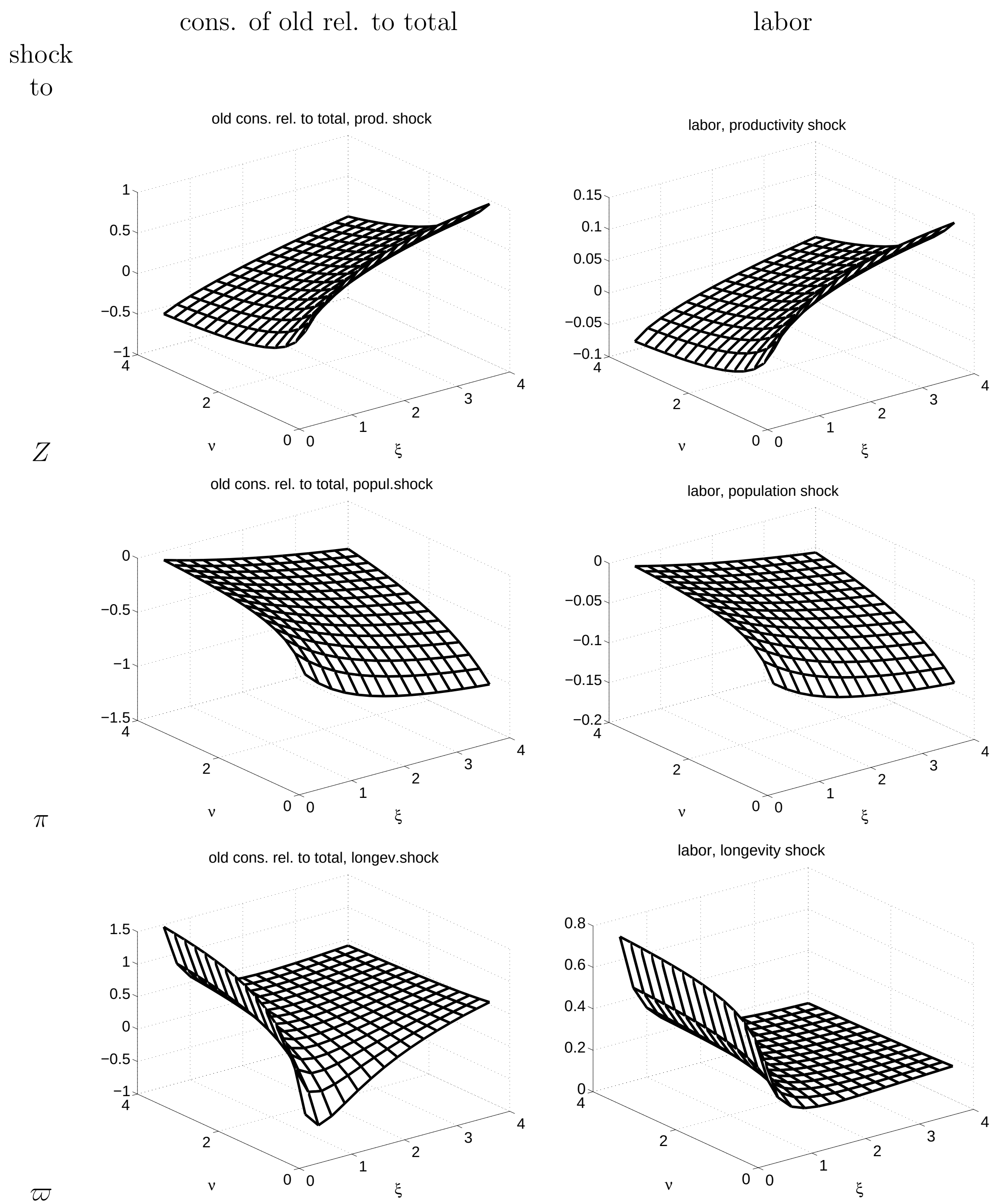
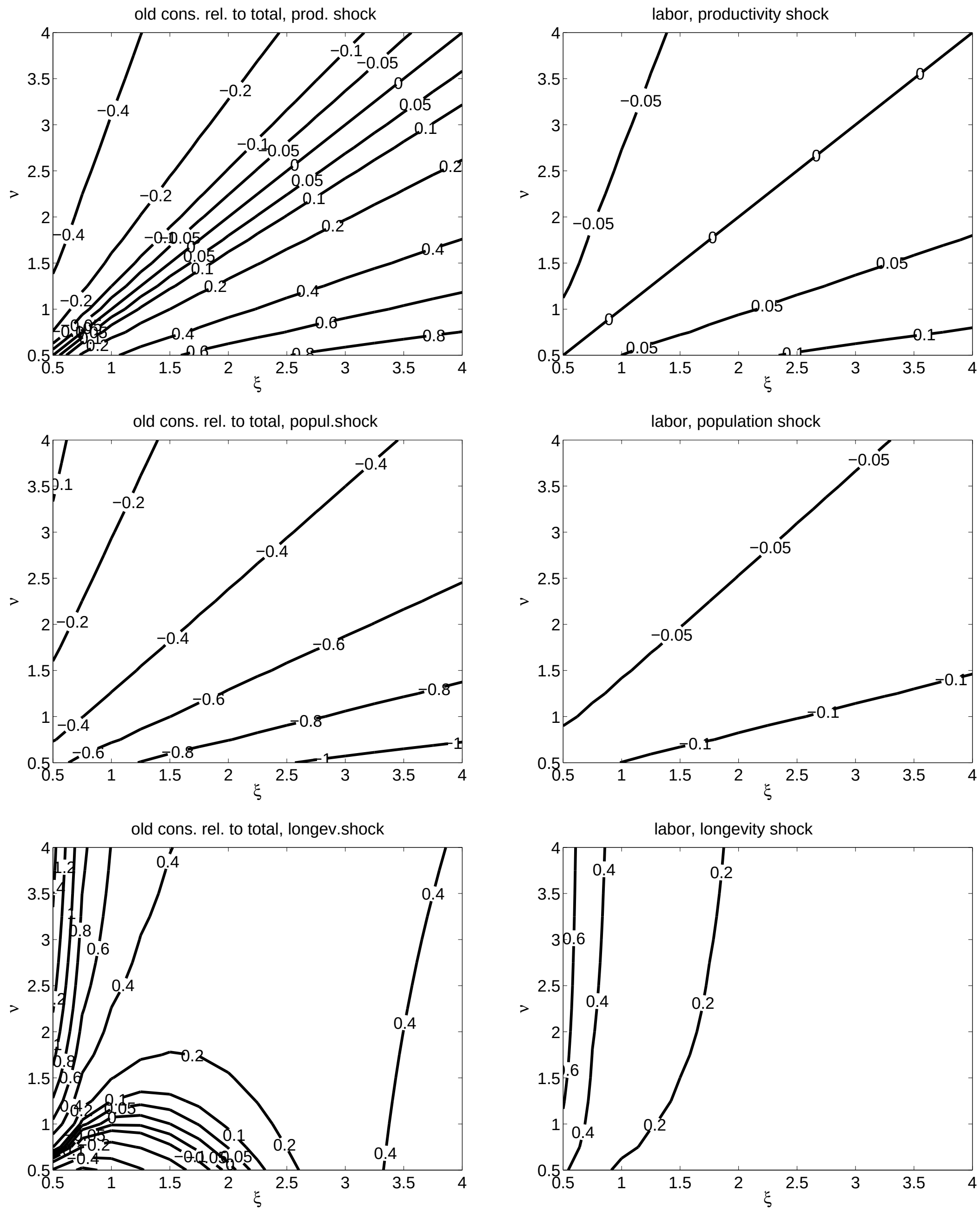


Figure 8: *Reaction of consumption of old relative to total consumption as well as reaction of labor, as the risk aversion parameters are varied.*

cons. of old rel. to total　　　　labor

shock to



Z

π

ϖ

Figure 9: *Contour plots: Reaction of consumption of old relative to total consumption as well as reaction of labor, as the risk aversion parameters are varied.*

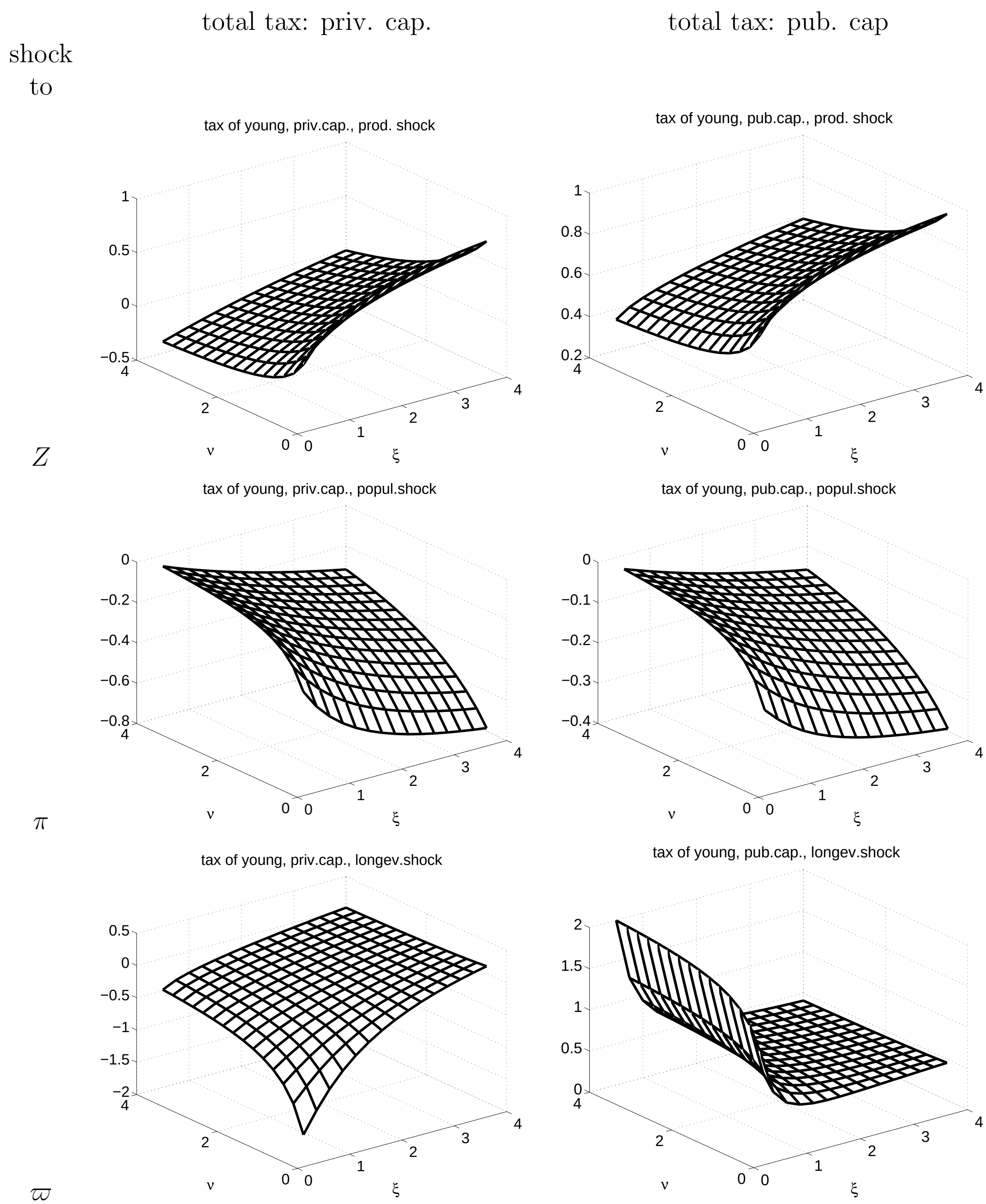


Figure 10: *Reaction of total tax payments by young relative to unchanged labor income, as the risk aversion parameters are varied.*

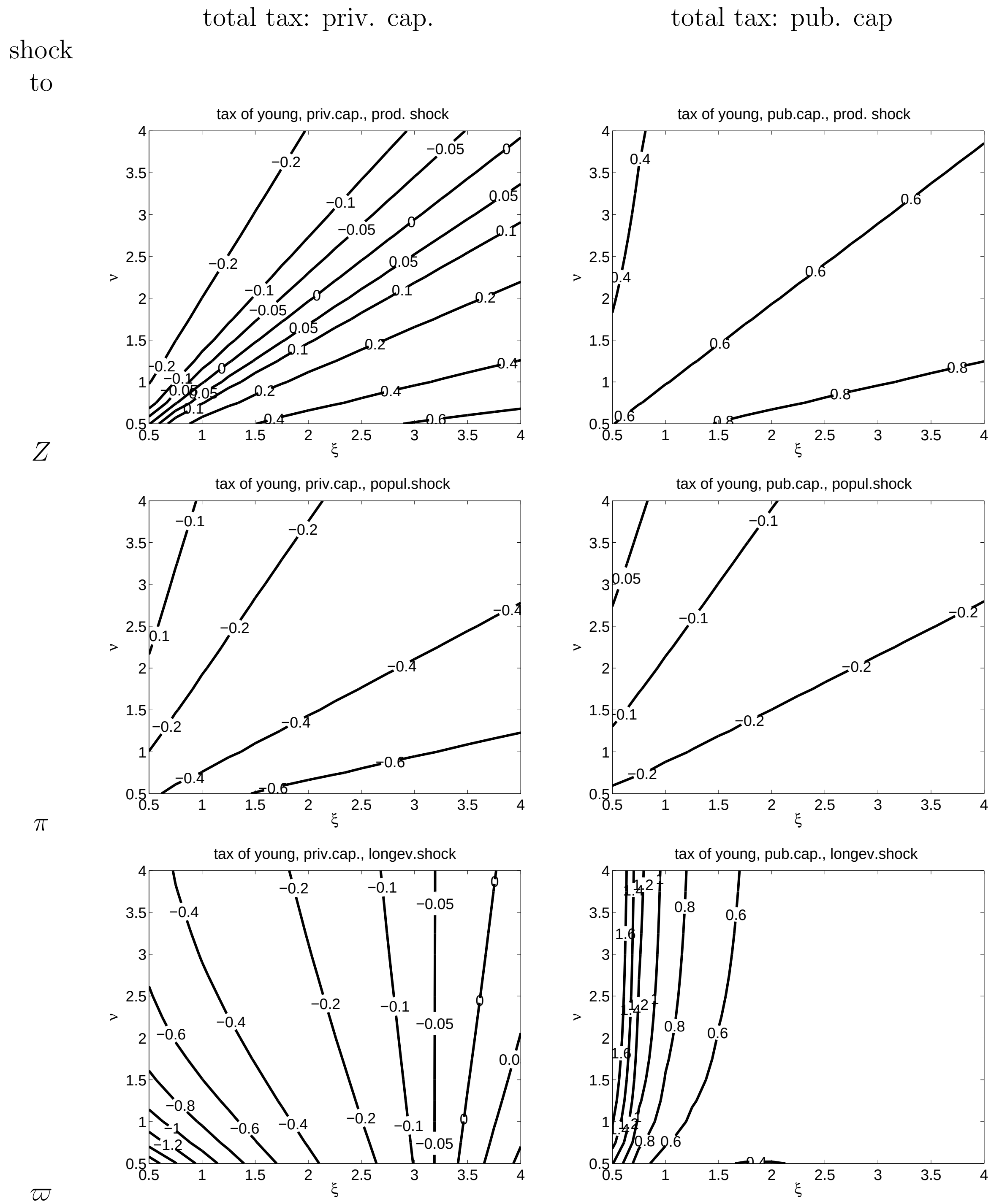


Figure 11: *Contour plots: Reaction of total tax payments by young relative to unchanged labor income, as the risk aversion parameters are varied.*