Currency Substitution under Transaction Costs

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A major selling point and feature of cryptocurrencies is that they allow anonymous payments around the globe without a third party watching. For payments of certain goods, this censorship resistance feature makes cryptocurrencies more suitable or less costly a medium of exchange than traditional fiat moneys such as Dollars or Euros. On the other hand, there exist goods which are easier to acquire using traditional means of payments. The costs of employing cryptocurrencies may involve fees to miners, while traditional money might be subject to taxation. In this paper, we therefore explore how asymmetry in transaction costs as well as exchange fees drive currency substitution. We build on Schilling and Uhlig [2018]. Agents alternate in their role as buyers and sellers, necessitating currency. We assume a continuum of differentiated goods, which can be strictly ordered according to the costs agents incur when purchasing these goods with Bitcoins as opposed to Dollars. Exchanging Dollars for Bitcoins may be subject to a fee. In a market equilibrium, agents endogenously decide which goods to acquire using Dollars and which goods to purchase using Bitcoins. We characterize the nonstochastic equilibrium and the resulting exchange rate dynamics. The marginal good at which agents are indifferent between purchasing with Bitcoins or Dollars depends on and varies in the size of the value-added-tax and transaction fees to miners.

I. The Model

The model builds on Schilling and Uhlig [2018] and is related to Kareken and Wallace [1981], Obstfeld and Rogoff [1995], and [Casas et al., 2016]. Time is discrete, $t = 0, 1, 2, \ldots$. The economy is deterministic. There exists a continuum $[0, 1]$ of differentiated consumption goods $k \in [0, 1]$ which are all non-storable across time.

CURRENCIES: Trade is carried out, using two kinds of money. The first shall be called Bitcoin, and its aggregate stock is fixed at $B_t \equiv B$.[1] There exists no designated crypto central bank. The second money shall be called Dollar, and we denote its aggregate stock at time t with D_t. There is a designated Dollar central bank, which governs the aggregate stock of Dollars D_t per lump sum transfers in each period. The central bank can produce Dollars at zero cost.

AGENTS: There are two types of infinitely lived agents, called 'red' and 'green'. Each type is given by a unit interval. In odd periods, a green agent receives an exogenously given goods endowment $y_{kt,j} = y_t$ for all $k \in [0, 1]$, which he can sell to red agents, against Bitcoins or Dollars, but does not enjoy consuming herself. We assume that y_t is a deterministic function of time. In even periods t, the green agent j has zero endowment but enjoys consumption $c_{kt,j}$. To purchase goods $k \in [0, 1]$ from red agents, she employs her individual Bitcoin and Dollar stock she accumulated from previous periods when she was a seller. Additionally, she receives a lump-sum Dollar transfer τ_t from the central bank to purchase goods. She then enjoys utility $\beta^t u(c_{kt,j})$, $\beta \in (0, 1)$. Since agents are infinitely lived, they may choose to save in Bitcoins or Dollars across time. For red agents, flip even and odd peri-

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[1] This assumption corresponds to the fact that the maximum quantity of Bitcoins in circulation is bounded from above by 21 million coins. We employ the assumption of a fixed Bitcoin stock to abstract from coin mining and the associated effort in this paper. Schilling-Uhlig (2018) provide details, when allowing for mining and an increase in the Bitcoin stock.

ods. Red and green agents alternate in consuming and producing the consumption goods. Since goods are perishable, the alternation creates the absence of a double-coincidence of wants, and thereby reasons to trade using currency. Denote by $c_{t,j} = \left(\int_0^1 c_{kt,j}^{1-1/\eta}\, dk\right)^{1/(1-1/\eta)}$ the CES aggregator with $\eta > 1$ denoting the elasticity of substitution. The consumption-utility function $u(\cdot)$ is strictly increasing, concave and twice differentiable. We assume that whoever consumes first has all the money. Agents maximize discounted expected lifetime utility

$$U = E\left[\sum_{t=0}^{\infty} \beta^t \xi_{t,j} u(c_{t,j})\right] \tag{1}$$

Formally, we impose alternation of utility from consumption per $\xi_{t,j} = 1_{\{t \text{ is odd }\}}$ for $j \in [0,1)$ and $\xi_{t,j} = 1_{\{t \text{ is even }\}}$ for $j \in [1,2]$.

Prices and Transaction Costs: We assume that usage of currency comes at a cost. Let $(1-\gamma(k)) \in [0,1]$, $k \in [0,1]$ the exogenous, product-specific transaction cost for goods purchased with Dollar, reflecting perhaps a goods-specific value-added tax (VAT). Purchases with Bitcoins cannot be taxed, but are nonetheless subject to an exogenous transaction cost $(1-\alpha(k)) \in [0,1]$, perhaps reflecting transaction fees which are paid to miners.

Assume that $\alpha(k)/\gamma(k)$ is strictly increasing and continuous in k. In equilibrium, there will therefore exist a critical good $k_t^c \in [0,1]$, such that all goods $k \leq k_t^c$ are purchased with Dollars and all goods $k > k_t^c$ will be purchased with Bitcoins. Let $P_{k,t}$ and $\hat{P}_{k,t}$ be the period-t price of good k in Dollars and Bitcoins respectively before transaction costs, set by goods sellers at which they are indifferent between accepting either currency. A buyer, therefore, needs to pay $P_{k,t}/\gamma(k)$ Dollars or $\hat{P}_{k,t}/\alpha(k)$ Bitcoins to obtain a unit of good k.

In addition, buyers and sellers can exchange currency in either direction, at an exchange fee of $(1-\phi) \in [0,1]$. We adopt the convention that the goods buyer's shadow price of a Bitcoin in Dollar, which we shall denote with Q_t, is also the official exchange rate, if exchange takes place at all. Suppose that the goods buyer gives one Bitcoin to the goods seller in period t, and that the seller in exchange pays an amount of Dollars $\tilde{Q}_t$. We assume, that the goods seller receives the Bitcoin in full, while the goods buyer only receives $\phi\tilde{Q}_t$ Dollars, with the difference of $(1-\phi)\tilde{Q}_t$ Dollars collected as transaction cost. Per our convention, we therefore have $\phi\tilde{Q}_t = Q_t$. From the perspective of the goods seller, the exchange rate is $\tilde{Q}_t = \frac{Q_t}{\phi}$. Conversely, if the goods buyer obtains one Bitcoin in exchange for paying Q_t Dollars to the goods seller, we assume that all Q_t Dollars arrive at the seller, while the goods seller needs to give up $1/\phi$ Bitcoins for one Bitcoin to arrive at the goods buyer, and where $(1-\phi)/\phi$ Bitcoins are collected as transaction cost. The exchange rate from the perspective of the goods seller now becomes $\tilde{Q}_t = \phi Q_t$.

As usual, define the aggregate Dollar price indices $P_t = \left(\int_0^1 P_{k,t}^{1-\eta} dk\right)^{1/(1-\eta)}$. The central bank implements an exogenously given path P_t for the aggregate Dollar price index via appropriate lump sum Dollar transfers τ_t paid to the goods buyer at the beginning of the period. Let $\pi_t = \frac{P_t}{P_{t-1}}$ denote the resulting inflation. In equilibrium, the central bank achieves her price path target.

Denote by $D_{t,i}$ the Dollar stock of individual i when entering time period t. The aggregate stock equals $D_t = \int_{[0,2]} D_{t,i}\, di$ where $i \in [0,1]$ denote red agents and $i \in [1,2]$ denote green agents. Similarly, let $B_{t,i}$ be the Bitcoin stock of individual i at time t. Note that $B = \int_{[0,2]} B_{t,i}\, di$. Exchange of currency and purchases of goods happens simultaneously. Thus, the goods buyer can spend all his Bitcoins on goods, purchase Bitcoins from the goods seller against Dollar, and spend these Bitcoins again.

We shall focus on symmetric equilibria, which allows us to drop the agent-individual subscripts i and j, unless needed for clarification. Denote by

$$C_{D,t} = \int_{k \le k_t^c} \frac{P_{k,t}}{\gamma(k)} c_{k,t}\, dk \tag{2}$$

the total quantity of Dollars the representative buyer spends on consumption goods in Dollars at time t potentially after exchanging some Bitcoins to Dollars. Due to transaction costs, the representative goods seller only receives

$$R_{D,t} = \int_{k \le k_t^c} P_{k,t} c_{k,t}\, dk$$

Dollars as revenue. Similarly, let $C_{B,t} = \int_{k > k_t^c} \frac{\hat{P}_{k,t}}{\alpha(k)} c_{k,t}\, dk$ be the quantity of Bitcoins spent on consumption goods demanded in Bitcoins, and let $R_{B,t} = \int_{k > k_t^c} \hat{P}_{k,t} c_{k,t} dk$ the number of Bitcoins the seller receives through goods purchases. The goods buyer faces the time t Dollar budget constraint

$$0 \le C_{D,t} \le D_{t,i} + \tau_t + Q_t \Delta_{t,i} \tag{3}$$

where $\Delta_{t,i}$ is the quantity of Bitcoins exchanged for Dollars by the goods buyer. She then carries the difference $D_{t+1,i}$ between the right-hand side of (3) and $C_{D,t}$ over as Dollars into period $t+1$. If $\Delta_{t,i} > 0$ ($\Delta_{t,i} < 0$), the goods buyer is selling (acquiring) Bitcoins for Dollars to the goods seller. Likewise, the goods-buying agent faces the time t Bitcoin budget constraint

$$0 \le C_{B,t} \le B_{t,i} - \Delta_{t,i} \tag{4}$$

and carries the difference between the right-hand side and $C_{B,t}$ as Bitcoin balance $B_{t+1,i}$ into the next period. In the symmetric equilibrium, all choose the same $\Delta_t = \Delta_{t,i}$. Next, consider the goods sellers in period t. We assume that they receive all transaction costs and Bitcoin-Dollar exchange fees, $\chi_{D,t}$ and $\chi_{B,t}$, as lump sum payments.

$$\begin{aligned} \chi_{D,t} &= C_{D,t} - R_{D,t} + \frac{1-\phi}{\phi} Q_t \max\{\Delta_t, 0\} \\ \chi_{B,t} &= C_{B,t} - R_{B,t} - \frac{1-\phi}{\phi} \min\{\Delta_t, 0\} \end{aligned}$$

As she sells goods to green agents for Dollars $C_{D,t}$ and Bitcoins $C_{B,t}$, her currency stocks increase to

$$\begin{aligned} 0 \le D_{t+1,j} &= D_{t,j} + R_{D,t} - \tilde{Q}_t \tilde{\Delta}_{t,j} + \chi_{D,t} \\ 0 \le B_{t+1,j} &= B_{t,j} + R_{B,t} + \tilde{\Delta}_{t,j} + \chi_{B,t} \end{aligned}$$

where $\tilde{\Delta}_{t,j}$ is the quantity of Bitcoins obtained ("sold", if negative) by the goods seller j and $\tilde{Q}_t$ is the price from the perspective of that goods seller. In equilibrium, all goods sellers choose the same currency trade $\tilde{\Delta}_t = \tilde{\Delta}_{t,j}$ and the market clears via corresponding currency trades by goods buyers and, possibly, transaction fees. If goods buyers sell Bitcoins, the market clearing condition is $\Delta_t = \tilde{\Delta}_t > 0$ and the price from the perspective of goods sellers is $\tilde{Q}_t = Q_t/\phi$. If goods buyers buy Bitcoins, the market clearing condition is $\Delta_t = \phi \tilde{\Delta}_t < 0$ and the price from the perspective of goods sellers is $\tilde{Q}_t = \phi Q_t$. Market clearing for all goods markets requires $y_{k,t} = y_t = c_t = c_{k,t}$ for all $k \in [0,1]$ and t, since buyers cannot produce and goods are not storable.

II. Analysis

Consider the goods seller at time t, who becomes a goods buyer in $t+1$. Given her shadow price Q_{t+1} for a Bitcoin tomorrow, she sets goods prices in terms of Bitcoins and Dollars according to

$$Q_{t+1} = \frac{P_{k,t}}{\hat{P}_{k,t}}, \quad \text{for all } k \in [0,1] \tag{5}$$

in order to be indifferent between receiving either currency. Now consider the buyer. With Q_t Dollars, a buyer can purchase $Q_t \gamma(k)/P_{k,t}$ units of good k. On the other hand, one Bitcoin buys $\alpha(k)/\hat{P}_{k,t}$ units of good k. The buyer is indifferent at a critical good $k = k_t^c$, $k_t^c \in (0,1)$, if $\frac{\alpha(k_t^c)}{\gamma(k_t^c)} = Q_t \frac{\hat{P}_{k_t^c,t}}{P_{k_t^c,t}}$. Equation (5) then implies

$$\frac{\alpha(k_t^c)}{\gamma(k_t^c)} = \frac{Q_t}{Q_{t+1}} \tag{6}$$

The left hand side is a strictly increasing function in k. Thus, the good k_t^c

is unique for every t if it exists. Otherwise, if $\frac{\alpha(k)}{\gamma(k)} > \frac{Q_t}{Q_{t+1}}$ for all $k \in [0,1]$, set $k_t^c = 0$, and if $\frac{\alpha(k)}{\gamma(k)} < \frac{Q_t}{Q_{t+1}}$ for all k, set $k_t^c = 1$. All goods $k \leq k_t^c$ are payed in Dollars, and it holds $\frac{\alpha(k)}{\gamma(k)} \leq \frac{Q_t}{Q_{t+1}}$ while goods $k > k_t^c$ are paid for in Bitcoins with $\frac{\alpha(k)}{\gamma(k)} > \frac{Q_t}{Q_{t+1}}$. To determine prices for goods purchased in Dollars and Bitcoins, consider two goods $k, k' < k_t^c$ which a buyer pays for in Dollars. Both goods are provided at the same supply and are thus consumed in the same quantity in equilibrium. After transaction costs, they must therefore have the same Dollar price. The same remark holds for the after-transaction Bitcoin prices for goods $k, k' > k_t^c$. This implies

$$P_k = P_{k_t^c} \frac{\gamma(k)}{\gamma(k_t^c)}, \quad \text{for all } k \leq k_t^c \tag{7}$$

$$\hat{P}_k = \hat{P}_{k_t^c} \frac{\alpha(k)}{\alpha(k_t^c)}, \quad \text{for all } k > k_t^c \tag{8}$$

ALLOCATION The purchase of y_t units of a good $k > k_t^c$ requires $y_t \hat{P}_k / \alpha(k) = y_t \hat{P}_{k_t^c} / \alpha(k_t^c)$ Bitcoins. Spending is constant across k. Analogous for Dollar purchases. Therefore, $C_{D,t} = k_t^c y_t P_{k_t^c} / \gamma(k_t^c)$ and $C_{B,t} = (1 - k_t^c) y_t \hat{P}_{k_t^c} / \alpha(k_t^c))$. What if the buyer enters the period with amounts of Dollars and Bitcoins which differ from the intended spending with each currency? Suppose, say, $D_t < C_{D,t}$ and $B_t > C_{B,t}$. To acquire the additional $C_{D,t} - D_t$ Dollars, the goods buyer transfers $\Delta_t = \frac{C_{D,t} - D_t}{Q_t}$ Bitcoins to the goods seller, with the exchange subject to the Dollar-Bitcoin exchange fee described above, implying $Q_{t+1} = Q_t / \phi$. Equivalently, if $D_t > C_{D,t}$ and $B_t < C_{B,t}$, the goods buyer acquires Bitcoins in exchange for Dollars and $Q_{t+1} = \phi Q_t$.

COLLECTING EQUATIONS The following set of equations characterize the aggregate evolution of the economy, assuming that buyers never hold on to Dollars or Bitcoins to the next period.

$$C_{D,t} - Q_t \Delta_t = D_t \tag{9}$$

$$C_{B,t} + \Delta_t = B \tag{10}$$

The justification for this latter assumption is the No-Speculation-Theorem in Schilling and Uhlig [2018], assuming the strong impatience condition there holds.

$$\begin{aligned}
\text{seller indiff:} \quad Q_{t+1} &= \frac{P_{k_t^c, t}}{\hat{P}_{k_t^c, t}} \\
\text{buyer indiff:} \quad Q_t &= \frac{\alpha(k_t^c)}{\gamma(k_t^c)} \frac{P_{k_t^c, t}}{\hat{P}_{k_t^c, t}}
\end{aligned}$$

$$\begin{aligned}
\text{combined:} \quad \frac{Q_t}{Q_{t+1}} &= \frac{\alpha(k_t^c)}{\gamma(k_t^c)} \\
\text{Dollars spent:} \quad C_{D,t} &= k_t^c \frac{P_{k_t^c,t}}{\gamma(k_t^c)} y_t \\
\text{Bitcoins spent:} \quad C_{B,t} &= (1 - k_t^c) \frac{\hat{P}_{k_t^c,t}}{\alpha(k_t^c)} y_t \\
\text{ratio:} \quad \frac{C_{D,t}}{C_{B,t}} &= \frac{k_t^c}{1 - k_t^c} Q_t \\
\text{exch. arbitr.:} \quad \phi Q_{t+1} &\leq Q_t \leq \frac{1}{\phi} Q_{t+1} \\
\text{seller Dollars:} \quad 0 &\leq C_{D,t} - Q_t \Delta_t \\
\text{seller Bitcoins:} \quad 0 &\leq C_{B,t} + \Delta_t
\end{aligned}$$

Note that we only need to keep track of the prices for the critical good, as the pricing for all other goods follows from the indifference conditions stated above. The goods sellers receive the entire monetary amount resulting from goods spending by the goods buyers because they receive the direct payments *and* the transaction fees and exchange fees as a lump sum payment. The latter two equations say that the goods seller cannot transfer more Dollars (Bitcoins) in the exchange rate market to the goods buyer, than she receives in total (positive money balances). In both of the following examples, assume (9) and (10), i.e. buyers never hold on to Dollars or Bitcoins at any t.

III. Example 1: no currency exchange

Suppose that Dollar inflation and output are constant $\pi_t \equiv \pi \geq 1$, $y_t \equiv y$, with π and y parameters. We wish to feature an example, where no currency exchange takes place. Thus, $\Delta_t = 0$, $C_{B,t} = B$, $C_{D,t} =$

$D_t := D_0\pi^t$ for all t. Further, assume that k_t^c is constant in t. The Bitcoins-spent equation then implies that $\hat{P}_{k^c,t}$ is constant in t, while the Dollars-spent equation implies that $P_{k_t^c,t}$ and, therefore, the Dollar price of a Bitcoin must rise at the rate of inflation, $Q_t = \pi^t Q_0$, $P_{k^c,t} = P_{k^c,t}\pi$. The combined equation allows to solve for k^c per $\gamma(k^c)/\alpha(k^c) = \pi$. Assume that there is an interior solution. Given the initial Dollar quantity D_0 and exploiting $D_0 = C_{D,0}$ as well as $B = C_{B,0}$, the ratio equation delivers Q_0 and thus $\{Q_t\}_{t\geq 0}$. The Bitcoins-spent equation can be solved for $\hat{P}_{k_t^c}$, while $P_{k_t^c,0}$ can be calculated, using the Dollars-spent equation. Two things remain to be checked. First, absence of Dollar-Bitcoin exchanges requires that the exchange fee $1-\phi$ is sufficiently high. Via the exchange-arbitrage condition, this requires $\phi Q_{t+1} < Q_t < \frac{1}{\phi} Q_{t+1}$, i.e. $\pi \in (\phi, 1/\phi)$. Second, the no-speculation theorem logic in Schilling-Uhlig (2018) indeed implies that all Dollars and Bitcoins are spent, provided the strong impatience assumption there holds.

IV. Example 2: currency exchange

Suppose again that Dollar inflation and output are constant $\pi_t \equiv \pi \geq 1$, $y_t \equiv y$, with π and y parameters. We wish to feature an example, where Bitcoins and Dollars are traded and where goods buyers always sell some Bitcoins to goods sellers against Dollars, $\Delta_t > 0$, thus trading more goods with Dollars than they own when entering the period, $D_t < C_{D,t}$. The exchange arbitrage equation implies that we are at the bound $Q_t = \phi Q_{t+1}$. This allows us to calculate the critical good per the combined equation, $\phi = \alpha(k_t^c)/\gamma(k_t^c)$, where we further note that $k^c \equiv k_t^c$ must be constant in t. Consider first the case, that k_t^c is interior. At time zero, the goods buyers have all the Dollars and Bitcoins. Since the buyer is acquiring Dollars for spending purposes, the ratio equation shows that Q_0 and D_0 must satisfy $D_0/B < (k^c/(1-k^c))Q_0$. Fix some initial Bitcoin price Q_0 and Dollar quantity D_0, satisfying this inequality. Let the Dollar quantity and Dollar prices rise at the rate of inflation, $D_t = \pi^t$ and $P_{k^c,t} = \pi^t P_{k^c,0}$. From the seller indifference equation, it follows that Bitcoin prices evolve according to $\hat{P}_{k^c,t} = \pi^t \cdot \phi^t \hat{P}_{k^c,0}$. In order for the goods buyer to keep selling Bitcoins to the goods seller, requires $\Delta_t = B - C_{B,t} > 0$. Sufficient for this is that $\hat{P}_{k^c,t}$ is not rising in t, i.e. that $\pi\phi \leq 1$. Consider next the case, that $k^c = 1$ and that no transactions ever take place with Bitcoin, $\Delta_t = B$, because $\alpha(k)/\gamma(k) < \phi$ for all k. It follows that $C_{D,t} = P_{1,0}\pi^t y/\gamma(1)$. The Bitcoins-spent equation does not impose a restriction on the rise in prices $\hat{P}_{k^c,t}$ and therefore no restriction on ϕ. A particularly interesting special case is the case of a frictionless exchange, i.e. $\phi = 1$. Then $Q_{t+1} = Q_t \equiv Q_0$, which can take any initial value, as long as the seller can afford to purchase all Bitcoins in all periods without encountering a negative dollar balance. Per the seller Dollar equation, this means we need $Q_t B \leq C_{D,t}$ or $Q_0 \leq P_{1,0}\, y/(B\gamma(1))$ at date $t = 0$, which suffices. Each period unfolds per the buyer repeatedly spending all his Dollars and then selling as many Bitcoins to the seller against Dollars as possible, until all Bitcoins are ultimately sold to the seller. The seller is indifferent between keeping Dollars or Bitcoins to the next period.

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