Bidding Behavior in Divisible Good Auctions: Theory and Evidence from the Turkish Treasury Auction Market

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Abstract

Using detailed bidder level data from the Turkish treasury auction market, we show that the behavioral assumptions underlying many models of Treasury markets eschew the richness of the strategy space available to the bidders: since the securities that are being sold are essentially divisible, bidders effectively submit entire demand functions, instead of single price bids. We show that these demand functions can be represented quite closely by a simple linear interpolation of the bids. We then develop a model of bidding in a divisible good discriminatory auction with common values and size asymmetries. We characterize the equilibrium of this game in the special case in which there are two bidders and bidders have indepedent private values. We then characterize approximate equilibria of the more general setting, and empirically test several hypotheses regarding the response of equilibrium bid strategies to changes in exogenous factors such as uncertainty in the value of the security and changes in the number of bidders.

1 Introduction

Despite the important policy questions regarding the choice of the proper auction mechanism in the sale of Treasury securities, electricity, emissions permits, etc., our present knowledge of bidder behavior in standard multi-unit auction formats is quite limited. Most attempts to answer the question theoretically have taken one of two stands in analyzing bidder behavior: while most of the early work on Treasury auctions regarded these auctions as extensions of single unit auction models, in which bidders submit a single price bid for a single unit of the multiple units of the commodity being sold (which we will call the "single-unit extension" framework), a later strand of work (originating with Wilson (1979) has taken the standpoint that bidders submit downward sloping demand functions instead of single price bids (this we will call the "divisible good auction" framework). These two modelling approaches have been shown to lead to surprisingly different conclusions: the "single-unit" framework has pointed to the desirability of a uniform price format over the discriminatory format, as the uniform price auction in this framework is essentially a second price auction.[1] In contrast, "divisible good auction" models have been less enthusiastic about the uniform price auction's advantages, as it has been shown that in certain cases, the uniform price auction possesses "collusive-seeming" equilibria that yield very low revenues for the seller.[2]

Using a detailed bidder-level data set from Turkish treasury auctions, this paper demonstrates that the divisible good framework is the better alternative in analyzing bidder behavior. We find that the observed bids in the data set can be approximated quite closely using linear demand functions. By reducing the description of the multiple bid schedule of bidders to two parameters (price-intercept and slope), we are able to characterize the variation in bids succintly.

Since there are very few existing results on the characterization of equilibria in divisible good auctions, most of which are for the uniform price auction, we then proceed to model the divisible good discriminatory auction game. We provide an analytic characterization of equilibria in a 2 bidder discriminatory auction with private values in which bidders have linear demand curves and possess exponentially distributed private signals regarding the value of the security. To our knowledge, this is the first analytical example of equilibrium strategies in the divisible good discriminatory auction. We then construct a much more general model in which we account for: 1) common values, 2) asymmetries due size/inventory considerations, and 3) an arbitrary number of bidders. We first characterize necessary conditions that the equilibrium bid functions should satisfy. We then construct an approximate solution to the problem by assuming that bidders follow a "first-order approximation" rule when computing their best-responses under uncertainty.

[1] See, for example, Bikchandani and Huang (1989) and Chari and Weber (1992).

[2] Perhaps the starkest demonstration of this type of equilibrium in Wilson's example in which N bidders all having the same constant marginal value, v, for the good being auctioned, can end up paying only $\frac{v}{2}$ for each unit they end up purchasing. Back and Zender (1993) and McAdams (1999) argue that in the discriminatory auction, the only equilibrium will be in which bidders pay v.

The approximate solution of the general model of bidding nevertheless provides us with several empirical hypotheses regarding the way exogenous factors like uncertainty and competition affect the nature of the observed bid functions. Using our data set we find empirical support for the model developed in section 3.5 and the approximate equilibrium strategies we derive.

2 A description of bidding behavior

This study uses bidder-level data from Turkish Treasury auctions. We describe the data set in much more detail in Hortaçsu (2000). To repeat some of the main features of the data: the data set contains 27 auctions of 13 week Treasury bills, spanning the period between October 1991 and October 1993 with monthly frequency. There are an average 70 unique bidders per auction, with about 15 bidders capturing 70% of market share. The bidders are mainly banks, who buy the securities primarily to fulfill their reserve requirements and for resale purposes. The auction format used is discriminatory, where bidders are allowed to submit multiple price-quantity pairs as their bids, and pay the price they indicate for bids that are above the market clearing price. For the auctions preceding February 1993, the Treasury exercised some discretion over the total supply, causing uncertainty in this dimension.

Let us recall the exact rules for bidding in the auction: bidders are allowed to submit an unlimited number of price and quantity pairs as their bids. Prices are quoted for an imaginary 100 TL face value Treasury bill, and can be specified up to 3 significant digits. Quantities are specified as the face value of Treasury bills the bidder wants to buy. The minimum quantity that can requested by a bidder is 50 million TL (about $ 6000).

As mentioned in the introduction, much of the earlier theoretical work regarding mechanism choice in Treasury auctions have modeled Treasury auctions as multi-unit auctions in which each bidder demands at most one unit of the good, and submits a single price bid for that unit. The convenience of this assumption is that it can be shown that many results from single unit auction theory, including the revenue equivalence theorem and the revenue rankings of Milgrom and Weber, can be extended to a multi-unit setting when bidders have demand for only a single unit (most notably, the work of Bikhchandani and Huang (1989) and Chari and Weber (1992) follow this vein).

In the Treasury auction context, the single-unit demand assumption makes sense if: 1) each bidder demands an identical fraction of the total supply, 2) bidders submit only a single bid. The second hypothesis is easily rejected in our sample. We find that the mean number of price-quantity pairs submitted by bidders in an auction is 6.89 with a standard deviation of 5.63. In fact, one bidder submitted 64 price-quantity pairs in one auction! We also find that 76 out of the total 101 bidders submitted more than 10 price-quantity pairs in at least one auction (this includes all of the largest 15 bidders).

As for the first hypothesis, the following histogram of the (log of) quantities demanded by the bidders shows that there is in fact wide variation in the demands of the bidders (since there were multiple bids submitted by a bidder, we take the total quantity demanded by each bidder). In fact, the variation in quantity demands can be described quite well with a log-normal distribution. We thus conclude that the "single-unit" demand assumption describes the bidding environment poorly. Bidders are quite heterogenous regarding their quantity demands, and they submit multiple bids.

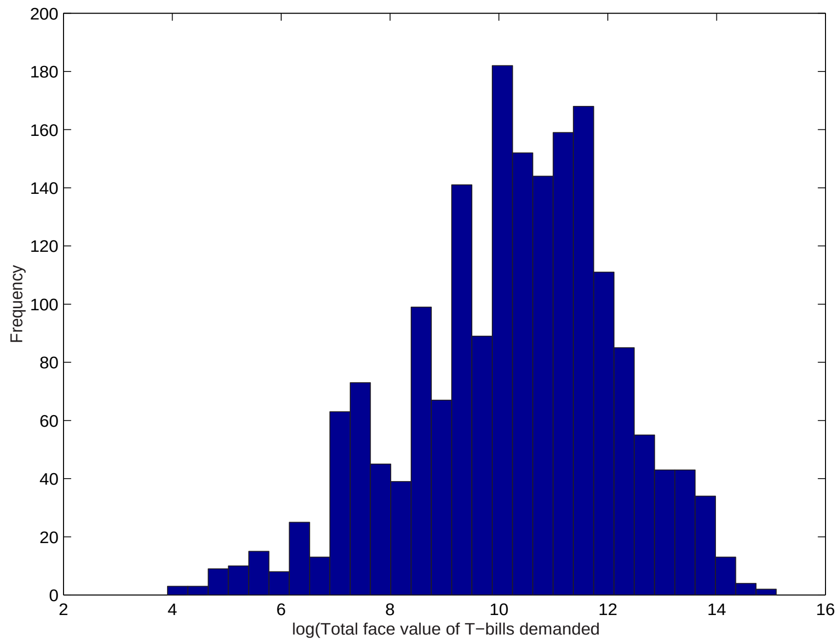

Figure 1: The distribution of quantity bids

An alternative to the "single-unit demand" assumption is that each bidder has a pre-specified "ideal" quantity in mind, and submit multiple price bids to alleviate the winner's curse associated with paying a high price for this quantity. This assumption was used by Gordy (1994) to model bidding in the Portuguese Treasury auction market by allowing bidders to submit a single price bid for an exogenously specified quantity of Treasury bills. Gordy (1999) further investigates this specification of bidding behavior by checking whether bidders tendency to submit multiple bids depends on factors that would affect the severity of the winner's curse.

If Gordy's specification were correct, we would expect the multiple price bids to be clustered very closely around a given quantity. Suppose q_{min} is the total quantity requested at the highest price point and q_{max} is the total quantity requested at the lowest price point. Then, under Gordy's specification of bidder behavior, $\frac{q_{max}-q_{min}}{q_{max}}$, should be very small, since the quantity bids are clustered closely around one "ideal" quantity.

However, we find that the quantity bids in the data set are spread over a wide range of quantities. The mean $\frac{q_{max}-q_{min}}{q_{max}}$ ratio for the data is 0.71, i.e. price bids are spread over 70% of the quantity request

Table 1: Linear fit to bid functions

Linear specification	Mean over i, t	Std. dev. over i, t
β_{it}	-0.0004	0.0016
α_{it}	0.8494	0.0071
R^2_{it}	0.9201	0.1004

(the standard deviation is 0.30). Therefore, we conclude that it is difficult to rationalize observed the bidding behavior as "hedging one's price bids for an ideal quantity."

As described in the introduction, an alternative modelling framework is to model a Treasury auction as a "divisible good auction, " in which bidders submit downward sloping demand functions. Since the Turkish Treasury allows each bidder to submit an unlimited number of price quantity pairs, one can think of each bidder submitting a demand function that consists of a discrete number of steps.

One might wonder whether with an average of 6.9 price-quantity pairs submitted per bidder, these step functions can be approximated by simple, continuous functions. In the next exercise, we tried a linear fit to bid functions that contained at least 3 price-quantity pairs. Specifically, we estimated the following specification for each bidder's multiple price-quantity bids:

$$p_{itj} = \beta_{it} q_{itj} + \alpha_{it} + \varepsilon_{itj} \tag{1}$$

where $\{p_{itj}, q_{itj}\}$ is the j-th price quantity submitted by bidder i in auction t. Prices are normalized by 1, and quantities are normalized by 1000 units. In Table 1, we report the averages of the coefficient estimates and the fit parameters. The overall fit, as measured by the average of R^2's over all individual bid functions, is 0.92 for the linear specification.[3]

A straight line interpolation through the multiple price-quantity pairs can explain 92% of the variation in the multiple price and quantity pairs submitted by the bidders. Hence, a divisible-good auction model that would generate linear demand functions for each bidder as equilibrium bidding strategies would provide a good description of the data.

Using the fitted slope and intercept parameters for each bidder's demand function from the preceding exercise, we now investigate the heterogeneity in bidding strategies in this reduced representation space. To partially control for exogenous factors that would affect the level of the price-intercept, we normalize it with the resale market price of the security, measured one day after each auction. Figure 2 displays the distribution of price-intercepts of the fitted linear bid functions about the resale price of the security being auctioned. We

[3] We also tried a log-linear fit, which yielded an R^2 of 0.84, so we decided to opt for the linear fit.

see that this distribution is roughly normal, with noticeable skewness, and a mode close to 1.

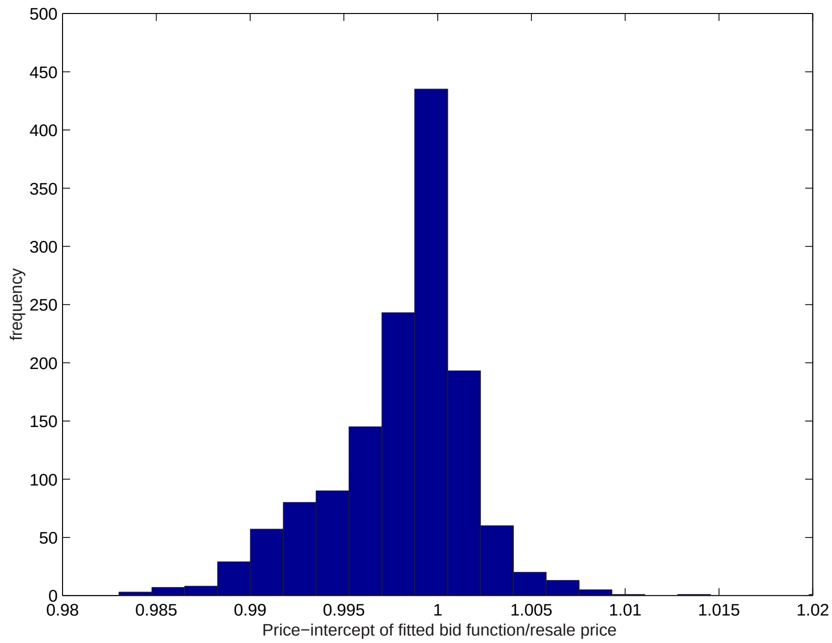

Figure 2: Distribution of normalized price intercepts

Figure 3 displays the distribution of the (log) slopes of the fitted linear bid functions. We see that the distribution of slopes has an approximate log-normal character, reflecting the distribution of quantity demands across bidders.

The wide heterogeneity in quantities demanded by the bidders, hence the heterogeneity in the slopes of their bid functions can be driven by wide variations in bidder size. Hence, a good model of bidding in the Turkish treasury auction setting should account for differences in bidders' sizes. The fact that the price-intercept of the fitted bid function is distributed almost normally centered about the resale price of the security suggests that the price-intercept of the bid function is determined by the private information of a bidder regarding the value of the security. Hence, in a realistic model of bidding that yields linear bid functions as equilibrium bid strategies, the intercept of the bid function should depend on the bidder's private information about the resale value of the security.

3 Theories of bidding in a divisible good discriminatory auctions

As we saw in the last section, an accurate model of bidding in Turkish treasury auctions should enable bidders to have demand for multiple units. One way to analyze equilibrium strategies in multi-unit auctions with

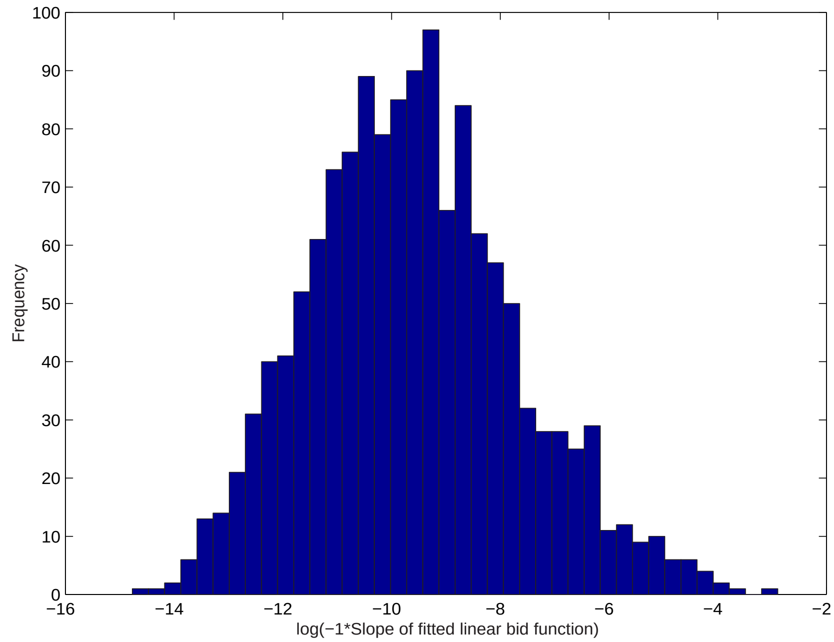

Figure 3: Distribution of slopes of bid functions

multi-unit demands is to treat a bidder bidding for $K > 1$ units as K independent bidders, where each of these bidders uses the equilibrium strategy in a "single-unit demand" equilibrium. However, a bidder who coordinates his bidding strategies across multiple units can always do (weakly) better than a bidder who makes disjoint decisions for each of the units he is bidding for – therefore, the disjoint bidding equilibrium will rarely, if ever, coincide with the equilibrium of the game in which a bidder will "spread" his bids across different units.

Accordingly, attempts to model multi-unit auctions have taken one of two approaches: the first option is to discretize the quantity space into a countable number of units (say K units), and define the strategy space of the bidders to be a K-tuple, specifying a price for each unit for sale. Existing analytical characterizations of equilibria in the discriminatory and uniform price auctions focus on the case where K is small. For example Engelbrecht-Wiggans and Kahn (1995) and (1998) characterize equilibria in discriminatory and uniform price auction games in which $K = 2$; the experimental work of Kagel and Levin (2000) also utilizes equilibria derived for $K = 2$.[4]

Unfortunately, the results for the discrete-quantity models are difficult to generalize to the case where

[4]The main finding of these studies is that a bidder can do significantly better by coordinating his bids across different quantities than by bidding separately on each unit. For example, Kagel and Levin (2000) construct an example of a uniform price auction with 2 units for sale and 2 bidders who have demand for both units, in which the equilibrium is to bid zero for the second unit. This forces the market clearing price, and hence the payment of each bidder to be zero! The "disjoint equilibrium" of this game would entail bidders to reveal their true valuations for both units – resulting in positive payments.

the number of units for sale is very large – which is quite clearly the case in Treasury auctions. Hence, the second modelling option, pioneered by Wilson (1979), is to allow quantities to be perfectly divisible, and to allow bidders to submit continuous demand schedules as their bids. Such auctions are known in the literature as "share auctions" or "divisible good auctions."

In this section, we will present 3 different models of bidding in divisible good discriminatory auctions. Although the bid functions defined by these discrete price-quantity pairs are step functions, we saw in the previous section that they can be approximated closely by linear functions. Therefore, we focus on models that can generate simple functional forms for the equilibrium strategies.

The first model, originally discussed by Back and Zender (1993) and Wang and Zender (1995), encompasses the simplest imaginable case: bidders have constant marginal valuations for the good for sale and they know this value with certainty. The second model relaxes the complete information assumption by allowing the bidders to have private information. We start with the simple setting where there are 2 bidders who have linear demands for the security. We find that, with exponentially distributed private signals, the equilibrium bid functions are downward shifted versions of the demand curves of the bidders. We show that the resulting allocation is efficient – the bidder with the higher valuation for a unit of the good receives it. We also show a rather surprising result that the average revenue obtained from the discriminatory auction can exceed the revenue obtained from the revenue in a uniform price auction, in which bidders do not bid strategically, but reveal their true demands. Since it is a dominated strategy for bidders to bid above their true demands in a uniform price auction, this result shows that revenue equivalence can be violated in this simple 2 bidder example.

Unfortunately, the equilibrium bidding strategies in the case where there are more than 2 bidders is not characterizable analytically, therefore we investigate linear approximations to the equilibrium bidding strategies. While the approximation scheme is admittedly less fulfilling than a full characterization of the equilibria, the qualititative nature of the linear bidding strategies reflects the bidding strategies we see in the data.

3.1 Model setup

Let us begin with a very general setup of the divisible good or share auction of Wilson (1979). There are Q units of a divisible good for sale by the auctioneer, where, without loss of generality, we can normalize $Q = 1$. There are N risk-neutral bidders, where we assume that N, and the identity of different bidders is common knowledge. Each bidder has a downward sloping inverse demand curve, or equivalently, marginal valuation function of the form $v_i(q, s_i, s_{-i})$, where the ex-post marginal valuation of bidder i depends on his private information, s_i, and the private information of the other bidders, s_{-i}. Observe that the specification for the ex-post marginal value obtained by the bidder is so far very general, as it does not impose any distributional assumptions on the set of signals. In particular, the *independent private values* (IPV) model arises when $v_i(q, s_i)$ is a function of bidder i's signal only, and when s_i are independent. The *symmetric IPV* case arises when $v_i(.) = v(.)$, i.e. the form of the marginal valuation function is identical across bidders. The *affiliated value* case arises when the s_i's are affiliated in the sense of Milgrom and Weber (1982).

Bidders are allowed to submit bid functions $y_i(p) : \Re \to \Re$, which are constrained to be (weakly) decreasing in price. With N competing bidders, the market clearing price, p^c, will be at the point where

$$y_i(p^c) = 1 - \sum_{j \neq i}^{N} y_j(p^c) \tag{2}$$

i.e. where bidder i's demand curve intersects her "residual supply curve."

3.2 A model with no private bidder information

An important special case is one in which bidders have constant marginal value for the securities being sold and this value is known perfectly, i.e. $v_i(q, s_i, s_{-i}) = \overline{v}$. In this case, Back and Zender (1993), and Wang and Zender (1998) show that in equilibrium, bidders will make zero profit and will share the good equally among themselves. The bidding strategies are:

$$p(q) = \overline{v} \text{ if } 0 < q < \frac{1}{N} \tag{3}$$

$$= 0 \text{ if } q > \frac{1}{N} \tag{4}$$

Clearly, this equilibrium is too stylized to match the features of the data, as both the prices and the quantities requested by the bidders vary quite widely. It also predicts that bidders will submit a single price-quantity pair consituting a "single-step" demand function: $(\overline{v}, \frac{1}{N})$, which does not fit the downward sloping feature of the demand curves we found in the data. However, the significance of this result is that it is in stark contrast to the corresponding "seemingly collusive" equilibrium of the uniform price auction game found by Wilson (1979), in which a market clearing price of $\frac{\overline{v}}{2}$ can be sustained. In particular, Back and Zender (1993) used this example to demonstrate that uniform price auctions do not always perform better than discriminatory auctions.

A more realistic looking equilibrium of the discriminatory auction game is obtained by Wang and Zender (1998) by incorporating risk aversion and uncertainty into the bidding environment. Wang and Zender (1998) maintain the assumption that bidders derive the same value, $\widetilde{v}$, from the securities, but assume that the value of the securities is distributed normally, $\widetilde{v} \sim N(\overline{v}, \sigma_v^2)$. Bidders are modeled to have downward sloping demand curves for the securities, since they have CARA utilities with constant-risk aversion coefficient, ρ. The "no private information" assumption is maintained by assuming that all bidders know the distribution of $\widetilde{v} \sim N(\overline{v}, \sigma_v^2)$. Wang and Zender also assume that there is uncertainty about the total supply, i.e. $Q = 1 - \widetilde{z}$, where $\widetilde{z}$ is the "supply noise" term, assumed to be distributed uniformly over the interval $[0, Z]$.

Given this setup, the equilibrium bid functions of the bidders will be:

$$p(q) = \overline{v} - \frac{\rho\sigma_v^2}{2N-1}[1 + \frac{(N-1)(1-Z)}{N}] \quad \text{if } 0 < q < \frac{1-Z}{N} \tag{5}$$

$$= \overline{v} - \frac{\rho\sigma_v^2}{2N-1}[1 + (N-1)q] \quad \text{if } \frac{1-Z}{N} < q < \frac{1}{N} \tag{6}$$

$$= 0 \quad \text{if } q > \frac{1}{N} \tag{7}$$

Observe that this equilibrium reflects the general character of the bid curves we observe in the data: it is flat up to some quantity, and is downward sloping for the rest of the quantities. This equilibrium does not reflect the heterogeneity in the prices submitted by different bidders that we find in the data – this, admittedly, can not be take into account in a model where bidders do not have private information.

Some empirical hypothesis that can be derived from the equilibrium strategies above are:

- A1: Bid shading (defined as $\overline{v} - p(q)$) increases as the uncertainty in security value, σ_v^2 ,increases.
- A2: Bid shading increases as the supply uncertainty, Z,increases.
- A3: Bid shading decreases as the number of bidders, N, increases.
- A4: The absolute value of the slope of the downward sloping portion of the equilibrium bid functions, $\frac{\rho\sigma_v^2}{2}$, increases if the uncertainty in security value increases, i.e. bid functions become steeper.
- A5: The downward sloping portion of the bid function becomes flatter as the number of bidders, N, increases.

3.3 Models with private information

One would normally expect bidders in a Treasury auction to have access to private information that would change their value for the security. Some sources of private information could be:

1. Having different expectations about the future resale value of the securities: this can be driven by different bidders' having access to differing forecasts of future inflation – since the securities are denominated in nominal terms, the projected real value will be calculated differently by different bidders
2. Having different liquidity needs due to idiosyncratic shocks in deposit flows and the corresponding reserve requirements

The equilibrium concept that we will be interested in the private information context is Bayesian Nash equilibrium. That is, we expect bidders to submit bid functions that are also functions of their information. Given that the bidding strategies of her rivals is random, each bidder has to form expectations about the market clearing price, p^c, when forming her optimal bidding strategy, since the market clearing price determines the quantity allocated to bidder i, $y_i(p^c)$, and the payment made by the bidder.

More concretely, for a given realization of the market clearing price, p^c,the surplus of bidder i in a discriminatory auction is given by:

$$E_{s_{-i}|p^c,s_i}[\int_0^{y_i(p^c)} v_i(q, s_i, s_{-i})dq] - \int_0^{y_i(p^c)} y_i^{-1}(q)dq \tag{8}$$

where, $E_{s_{-i}|p^c,s_i} \int_0^{y_i(p^c)} v(q, s_i, s_{-i})dq$ is the expected surplus that a bidder gets from winning $y_i(p^c)$ units of T-bills, *conditional* on her own signal s_i and the market clearing price, p^c (observe that for each realization of the market clearing price, the market clearing relation allows the bidder to update his prior on the distribution of her rivals' information, which, in turn, changes his expectation of the value of the security to herself). The second term is the payment of the bidder, i.e. the area under her demand curve to the point it intersects the market clearing price.

Now, following Wilson (1979), define:

$$\begin{aligned} H(p, y_i(p, s_i)) &= \Pr\{y_i(p, s_i) \leq Q - \sum_{j\neq i}^{N} y_j(p, s_j)\} && (9) \\ &= \Pr\{p^c \leq p | y_i(p, s_i)\} && (10) \end{aligned}$$

which is the probability distribution function of the market clearing price. We will assume that H as a probability distribution over the support of market clearing prices, is absolutely continuous. We will also assume that H is differentiable in both of its arguments.

Since $H(.)$ defines the probability distribution over the set of market clearing prices, the bidder's expected profit maximization problem in the discriminatory auction becomes:

$$\max_{y_i(\cdot)} \int_0^\infty \left\{ \int_0^{y_i(p^c,s_i)} E_{s_{-i}|p^c,s_i} v_i(q, s_i, s_{-i})dq - \int_0^{y_i(p^c,s_i)} y_i^{-1}(q, s_i)dq \right\} dH(p^c, y_i(p^c, s_i)) \tag{11}$$

where $dH(p, y(p))$ (denoting the total derivative of H) is the probability density function of the market clearing price.

The theoretical challenge in handling a general affiliated value model is that the expression $E_{s_{-i}|p^c,s_i} v_i(q, s_i, s_{-i})$ might be very difficult to evaluate, since the inference from the realization of the market clearing price to the distribution of s_{-i} depends on the specific functional form of the market clearing condition, which in turn depends on the equilibrium bidding strategies.[5] Therefore, we first look at the case where bidders have independent

[5]A special case in which this inference can be performed analytically is when the signals are jointly normal and the equilibrium strategies are linear in the signal and price. This "trick" has been utilized extensively in the market microstructure literature in finance, starting with the model of Kyle (1985).

private values, where the bidder does not have to update his expectation about the value of the securities given a particular realization of the market clearing price. Assuming the ex-ante symmetry of bidders, we look for symmetric bidding strategies maximizing the objective function:

$$\max_{y(.)} \int_0^\infty \left\{ \int_0^{y(p^c,s_i)} v(q,s_i)dq - \int_0^{y(p^c,s_i)} y^{-1}(q,s_i)dq \right\} dH(p^c, y(p^c,s_i)) \tag{12}$$

The difficulty in solving the above maximization problem is that that the probability distribution of the market clearing price, $H(.)$ is an endogenous quantity that depends on the equilibrium bid functions, $y(p,s_i)$. The maximization problem can be solved quite easily given a fixed $H(.)$, but the $H(.)$ used in the maximization problem to derive $y(p,s_i)$ should coincide with the $H(.)$ generated by the resulting bid functions.

The above description suggests the following analytical strategy to look for equilibrium bidding strategies:

1. Guess the form of $y(p,s_i)$.

2. Derive the probability distribution of the market clearing price,$H(.)$, generated by these bid functions

3. Verify that the $y(p,s_i)$ indeed solve the above maximization problem.

Unfortunately the cases in which the above guess-and-verify technique can yield analytical results are limited. In the next section we will analyze the solution of the problem in the simple case where there are two bidders, and bidders have independent private values. The problem loses analytical tractability when there are more than 2 bidders, so I will pursue an approximation strategy to characterize equilibria in a more realistic model with N bidders.

3.4 An IPV discriminatory auction with 2 bidders

In this section we will analyze a simple case in which the optimal bidding strategies in the discriminatory share auction can be derived. To do this we will modify the maximization problem to tranform it into a problem that can be solved using calculus of variations. Dropping the i subscript from the demand function, the objective function becomes:

$$\max_{y(.)} \int_0^\infty \left\{ \int_0^{y(p)} v(q,s_i) - y^{-1}(q)dq \right\} dH(p, y(p)) \tag{13}$$

First we have to get rid of the inner integral. Let the profit of the bidder from submitting the bid function $y(p)$ be

$$\pi(y(p)) = \int_0^{y(p)} v(q,s_i) - y^{-1}(q)dq \tag{14}$$

Now, setting $y(\infty) = 0$:

$$\pi(y(\infty)) = \pi(0) = 0 \tag{15}$$

and

$$\frac{d\pi}{dp} = (v(y(p), s_i) - y^{-1}(y(p)))y^{'}(p) \tag{16}$$

$$= (v(y(p), s_i) - p)y^{'}(p) \tag{17}$$

Substituting in the expression for $\pi(y(p))$ in the objective function and integrating by parts, we get

$$\max_{y(.)} - \int_0^{\infty} (v(y(p), s_i) - p)y^{'}(p)H(p, y(p))dp \tag{18}$$

Observe that the integrand is a function of p, y and $y^{'}$, denote it by $F(\ p, y, y^{'})$.

The Euler equation (which is a necessary condition for optimality) is given by (see Kamien and Schwartz, pg. 35):

$$F_y = \frac{d}{dp}F_{y'} \tag{19}$$

Evaluating the derivatives, we get:

$$v(y(p), s_i) = p + \frac{H(p, y(p))}{H_p(p, y(p))} \tag{20}$$

Observe that this relationship is very similar to the first-order condition relating the bid and the valuation of of a bidder in a first-price auction. The term $\frac{H(p,y(p))}{H_p(p,y(p))}$ is the inverse hazard ratio of the distribution of the market clearing price.

Now, suppose there are only 2 bidders, and that their signals are distributed exponentially, with $F(s) = e^{\lambda s}, s \leq 0$. Also assume that their true demand for T-bills is linear in price and their signal:

$$D(p, s_i) = \alpha + \beta p + \gamma s_i \tag{21}$$

where $\alpha, \gamma > 0, \beta < 0$. The inverse demand (marginal valuation) function is:

$$v(q, s_i) = \frac{1}{\beta}(q - \alpha - \gamma s_i) \tag{22}$$

Let us first guess that a bid function that is also linear in price and signal:

$$y(p, s_i) = a + bp + cs_i \tag{23}$$

where a, b, c will be determined as functions of $\alpha, \beta, \gamma, \lambda$. Taking the linear bid function as given, the cdf of the market clearing price will be:

$$H(p, y) = e^{\lambda[\frac{1-y-a-bp}{c}]} \tag{24}$$

and

$$H_p(p, y) = -\frac{\lambda b}{c}e^{\lambda[\frac{1-y-a-bp}{c}]} \tag{25}$$

Observe that we did not take into account the indirect effect of changes in p through $y(p, s_i)$. Substituting into the optimality condition:

$$a + bp + cs_i = \alpha + \beta(p - \frac{c}{\lambda b}) + \gamma s_i \tag{26}$$

Equating coefficients, we get:

$$y(p, s_i) = \alpha - \frac{\gamma}{\lambda} + \beta p + \gamma s_i \tag{27}$$

or

$$p(q, s_i) = \frac{1}{\beta}(q - \alpha + \frac{\gamma}{\lambda} - \gamma s_i) \tag{28}$$

Immediately we see that both bidders shade their bids by the constant amount $\frac{\gamma}{\beta\lambda}$ in the equilibrium of the discriminatory auction. The fact that bidders shade their bids by a constant amount implies that the allocation in this example will be efficient (the bid function of the bidder with the higher signal will always be above the bid function of the other bidder). We might wonder whether this implies a revenue equivalence result with the uniform price auction and/or the Vickrey auction. The following result implies that there exists circumstances in which revenue equivalence is violated in favor of the discriminatory auction:

Proposition 1. *If $\frac{\gamma}{\lambda} < 2 - \sqrt{2}$ or $\frac{\gamma}{\lambda} > 2 + \sqrt{2}$, the expected revenue from the discriminatory auction analyzed above exceeds the expected revenue of the auctioneer in a "best-case" uniform price auction in which bidders reveal their true valuations.*

(Sketch of proof) Given two bidders, i and j, compute the market clearing price as a function of s_i and s_j, for both auction formats. Compute the payment made by the bidders for the particular realization of s_i and s_j.Take expectations over s_i and s_j, which is easy to do since the moments of the exponential distribution are well-known.

The above revenue inequivalence result provides the first example in the divisible good auction literature in which the discriminatory auction outperforms the uniform price auction in a setting with private bidder information. The comparison case of the "best-case" uniform price auction was chosen to avoid having to characterize the equilibria in the uniform price auction.[6] However, the above revenue inequivalence result does not lose bite when we concentrate on the hypothetical best-case scenario. The auctioneer's expected revenue in a uniform price auction where the bidders are acting strategically is necessarily higher than the expected revenue in the uniform price auction where the bidders are revealing their true valuations, since bidding above one's true valuation is a dominated strategy.[7]

Unfortunately this 2 bidder example is not very rich in terms of providing testable empirical hypotheses, aside from:

[6] It turns out that there are no linear equilibria of the uniform price auction. In particular, it can be shown that there are no equilibria of the uniform price auction in this case in which $y(p, s_i) = f(p) + g(s_i)$, an additively separable function in price and signal.

[7] Similarly, the expected revenue in a Vickrey auction will always be less than the expected revenue in the "best-case" uniform price auction. The Vickrey auction leads to truthful revelation in this case (since this an IPV auction), since a bidder's payment in the Vickrey auction is defined as the area under the residual supply function she faces (up to the market clearing price), making it a dominant strategy to reveal one's true demand curve. However, with an upward sloping residual supply function, this payment is always less than the payment in a uniform price auction where the bidders reveal their true demand functions.

- B1: Bid shading increases as the uncertainty in security value, $\frac{1}{\lambda}$, increases.

Also, the assumption of exponentially distributed private information is not a very realistic one, since, in this example, this would cause the price-intercepts of the bid functions to be distributed exponentially. However, in section 2, we found the price intercepts to be distributed normally.

What this example illustrates is the difficulty of finding suitable assumptions to make the solution of a general model tractable. Here, what aided us was the fact the inverse-hazard ratio of the exponential distribution is a constant. For most distributions, this will not be true. Hence, it will be difficult to pursue the above "guess and verify" strategy when solving for the equilibrium strategies.

3.5 A more general model

The next model we will discuss attempts to incorporate several aspects of the Turkish treasury auction market that were not contained in the previous two models. First of all, we do allow for private information about the value of the security. However, unlike the 2 bidder symmetric IPV example discussed previously, we allow a more general value specification for the bidders, in which *the marginal valuation of a bidder from a given quantity of the security depends on the realization of the market clearing price.* Observe that the addition of this feature allows us to model the dependence of a bidder's marginal valuation on other bidders' signals through a reduced form relationship, since in a Bayesian Nash equilibrium of the game, the market clearing condition suggests that the realized value of the market clearing price is a statistic aggregating the private information of the bidders.

We also allow two dimensions of heterogeneity in bidders' demands for the security: the first dimension of heterogeneity is a private signal each bidder receives about the marginal value of the security, which the bidder weights with his expectation of the market clearing price. Hence, depending on the magnitude of this signal, the prices submitted by different bidders will vary.

The second dimension of heterogeneity, designed to create variation in bidder sizes, is a bidder specific "optimal inventory position." Since one of the main reasons why bidders participate in Turkish treasury auctions is to satisfy liquid asset reserve requirements that are generated by idiosyncratic deposit flows, specifying an optimal inventory requirement for the security being auctioned, and allowing the ex-post utility of the bidder to depend on deviations from this optimal inventory position introduces a realistic feature into the model.

The information structure of the game is as follows: Bidders receive independent identically distributed private signals about the (constant) marginal value of the Treasury bills, $s_i \sim N(\bar{s}, \sigma_s^2)$. The total supply of T-bills is normalized to 1, with a normally distributed noise term, $\widetilde{z} \sim N(0, \sigma_z^2)$, since there was considerable supply uncertainty in the auctions in the data set.

To model the "optimal inventory position" feature, assume that the treasury department of the bank comes up with a purchase target, q_i, based on the liquid asset reserve requirements and other portfolio/liquidity constraints. Let $y_i(p^c)$ be the amount of T-bills that the bank ends up buying as the result of a particular realization of the market clearing price, p^c, and the strategic demand function, $y_i(p^c)$, submitted by the bidder. We will assume that the bank (or the bidding desk manager) incurs a quadratic cost of $\frac{\theta}{2}\left(y_i(p^c)-q_i\right)^2$ when there is a deviation from the purchase target.[8]

To reduce the complexity of incorporating the strategic effects of two separate components of private information, we will assume that the optimal inventory positions of the banks are common knowledge, in particular, the total optimal inventory need by the banking system, $Q=\sum_i q_i$, is known. It is perhaps too demanding to assume that banks know this quantity with certainty; however, bank managers have indicated that keeping a close eye on the interbank money markets and telephone conversations with other bidders give them a good indication about the overall purchase need of the market.

Assume that bidders are constrained to submit continuous, decreasing demand functions as their bids. Let $y_i(p)=y(p,s_i,q_i)$ be the demand function submitted by bidder i and p^c be the realized market clearing price under the market clearing condition $\sum_{i=1}^N y_i(p^c)=1-\widetilde{z}$.

As described above, instead of specifying a full affiliated values model, we will assume a reduced form specification of a "common value" model in which the ex-post value of the market-clearing price enters into the utility of a bidder directly. Specifically, we will assume that for each unit of T-bill they win, bidders get a value of $(1-\alpha)s_i+\alpha p^c$, i.e. a bidder's utility depends on a weighted average of her private information of the value of the security and the ex-post realization of the market clearing price. Observe that the case where $\alpha=0$ is the pure private values case.

The ex-post utility that a risk neutral bidder gets from winning $y_i(p^c)$ units of T-bills, with the market clearing price p^c revealed, is:

$$U(y_i(p^c),s_i,p^c)=\left[(1-\alpha)s_i+\alpha p^c\right]y_i(p^c)-\left[p^c y_i(p^c)+\int_{p^c}^{\overline{p}} y_i(p)dp\right]-\frac{\theta}{2}\left(y_i(p^c)-q_i\right)^2 \tag{29}$$

where the first term is the utility from winning $y_i(p^c)$ units, the second term is the payment made by the bidder[9], and the third term is the inventory adjustment cost. Observe that this utility specification implies a linear marginal valuation schedule, with intercept $(1-\alpha)s_i+\alpha p^c+\theta q_i$ and slope $-\theta$. Since this implies the demand curve $D\left(p,s_i\right)=\frac{1}{\theta}\left((1-\alpha)(s_i-p)\right)+q_i$, we get the asymmetry in bidder sizes due to differing liquidity needs.

Let $H(p,y_i(p))$ denote the distribution of the market clearing price, which is equivalent to:

$$H(p,y_i(p))=\Pr\{\sum_{i=1}^N y_i(p)\le 1-\widetilde{z}\}$$

[8]The quadratic inventory adjustment cost specification is prevalent in the market microstructure literature in finance.

[9]$\overline{p}$ is the upper boundary of the support of the probability distribution of the market clearing price

Given this probability distribution, the expected utility maximization problem of a risk-neutral bidder becomes:

$$\max_{y_i} \int_{\underline{p}}^{\overline{p}} \left\{ (1-\alpha)(s_i - p) y_i(p) + \int_{p^c}^{\overline{p}} y_i(p')dp' - \frac{\theta}{2} (y_i(p) - q_i)^2 \right\} dH(p, y_i(p)) \tag{30}$$

Integrating by parts:

$$\max_{y_i} - \int_{\underline{p}}^{\overline{p}} \left\{ (1-\alpha)(s_i - p) y_i^{'}(p) - \alpha y_i(p) - \theta (y_i(p) - q_i) \right\} H(p, y_i(p)) dp \tag{31}$$

Assuming the differentiability of $H(p, y_i(p))$, we can use calculus of variations to analyze this problem, as in the previous section. This yields the Euler equation:

$$(1-\alpha)(s_i - p) = \frac{H}{H_p} + \alpha \frac{H_y}{H_p} y(p, s_i) + \theta(y(p, s_i, q_i) - q_i) \tag{32}$$

Observe that if $\alpha = 0$ and $\theta = 0$, we get the first order condition for the private values case:

$$s_i - p = \frac{H}{H_p} \tag{33}$$

i.e. the amount of bid-shading (the difference between the price bid and the private information of the bidder) is determined by the inverse hazard ratio of the market clearing price distribution. We also see that the bid-shading is less for quantities on the bid curve that are below the optimal reserve level, q_i, and higher for quantities that exceed q_i.

The Euler equation can be solved to get an expression for $y(p, s_i, q_i)$:

$$y(p, s_i, q_i) = (1-\alpha)(s_i - p) \frac{H_p}{\alpha H_y + \theta H_p} + \theta q_i \frac{H_p}{\alpha H_y + \theta H_p} - \frac{H}{\alpha H_y + \theta H_p} \tag{34}$$

Following the "guess-and-verify" technique of the previous section, I will first guess a functional form for $y(p, s_i, q_i)$ and then check the conditions under which this specification is valid. As a tractable first guess, we will seek the validity of the following specification:

$$y(p, s_i, q_i) = \mu + \beta s_i + \delta q_i - \gamma p \tag{35}$$

Assuming that all other bidders follow such strategies, the market clearing condition becomes:

$$1 - \widetilde{z} = y_i + (N-1)\mu + \beta \sum_{j \neq i} s_j + \delta(Q - q_i) - (N-1)\gamma p \tag{36}$$

where I have taken into account that $\sum_i q_i = Q$ is common knowledge.[10] Since $\widetilde{z}$ and s_j are normal random variables, where $\beta \sum_{j \neq i} s_j + \widetilde{z} = \phi \sim N(\beta(N-1)\overline{s}, (N-1)\beta\sigma_s^2 + \sigma_z^2)$, $H(p, y)$ becomes:

$$\begin{aligned} H(p, y) &= \Pr\{1 - \widetilde{z} \geq y + (N-1)\mu + \beta \sum_{j \neq i} s_j + \delta(Q - q_i) - (N-1)\gamma p \\ &= \Phi\left(\frac{1 - y - (N-1)\mu - \delta(Q - q_i) + (N-1)\gamma p - \beta(N-1)\overline{s}}{\sqrt{(N-1)\beta\sigma_s^2 + \sigma_z^2}}\right) \end{aligned} \tag{37}$$

[10] Hence bidder i does not have to be concerned with the distribution of his competitor's private information regarding their reserve needs.

Then

$$\frac{\partial H}{\partial p} = \frac{(N-1)\gamma}{\sqrt{(N-1)\beta\sigma_s^2 + \sigma_z^2}} \phi(\frac{1 - y - (N-1)\mu - \delta(Q - q_i) + (N-1)\gamma p - \beta(N-1)\overline{s}}{\sqrt{(N-1)\beta\sigma_s^2 + \sigma_z^2}}) \tag{38}$$

and

$$\frac{\partial H}{\partial p} = -(N-1)\gamma \frac{\partial H}{\partial y} \tag{39}$$

In order for the "guess-and verify" technique to work for the conjectured linear strategy equilibrium, the terms $\frac{H_p}{\alpha H_y + \theta H_p}$ and $\frac{H}{\alpha H_y + \theta H_p}$ have to reduce to constants, or linear function in y and p. Fortunately, by condition (39), $\frac{H_p}{\alpha H_y + \theta H_p} = \frac{1}{\theta - \frac{\alpha}{(N-1)\gamma}}$, so we get the simplification we need. However, the "inverse-hazard ratio" term $\frac{H}{H_p}$ will typically be a non-linear function for a normally distributed random variable.

The non-linearity of the "inverse-hazard ratio" term suggests that the equilibrium bidding strategy is not going to be linear. However, an inspection of the inverse hazard ratio of a normal random variable around its mean suggests that it can be approximated quite closely with a linear function in the range $[\mu - \frac{\sigma}{2}, \mu + \frac{\sigma}{2}]$. [11]

By making a linear approximation of the inverse-hazard ratio, we depart from a full equilibrium analysis in which all bidders make the correct conjecture about the shape of the inverse-hazard ratio when forming their best-responses. The behavioral restriction introduced by the approximation is to have all bidders form their best-responses based on a first-order approximation of the inverse-hazard ratio. Hence, the equilibrium we solve for can be interpreted as one in which the bidders' computational or analytical capabilities are limited.

Given this caveat, we will make a first-order approximation of $\frac{H}{H_p}$ about the value $\overline{p}$ that sets $H(\overline{p}, y(\overline{p}, \overline{s})) = \frac{1}{2}$, i.e. the market clearing price of the auction if all bidders received the mean value of their private signals:

$$\overline{p} = \frac{1}{N\gamma}[\beta N\overline{s} + N\mu + \delta Q - 1] \tag{40}$$

Then, letting $\Theta(p, s_i) = \frac{H(p, y(p, s_i))}{H_p(p, y(p, s_i))}$

$$\Theta(p, s_i) \simeq \Theta(\overline{p}, \overline{s}) + \Theta_p(\overline{p}, \overline{s})(p - \overline{p}) + \Theta_y(\overline{p}, \overline{s})(s_i - \overline{s})\frac{dy}{ds_i} \tag{41}$$

Given this first-order approximation for the "inverse hazard ratio" term, the linear strategy bid function that

[11] If x is distributed $N(\mu, \sigma^2)$, the linear approximation to its inverse hazard ratio around $x = \mu$ is: $\frac{F(x)}{f(x)} \simeq \sqrt{\frac{\pi}{2}\sigma^2} + (x - \mu)$.

satisfies Euler equation becomes:

$$y(p, s_i) = \mu + \beta s_i + \delta q_i - \gamma p \tag{42}$$

$$\beta = \frac{(1-\alpha)}{\theta} \frac{2N - 2 - \alpha N + 2\alpha}{2N - 2 - \alpha N + \alpha} \tag{43}$$

$$\delta = \frac{2N - 2 - \alpha N + 2\alpha}{2N - 2 - \alpha N + \alpha} \tag{44}$$

$$\gamma = \frac{1}{\theta}\left[2 - \frac{\alpha(N-2)}{N-1}\right] \tag{45}$$

$$\mu = \frac{1}{1-\alpha}\left[\beta\bar{s}\frac{N-2}{N-1} + \frac{\delta}{N}Q - \frac{1-\bar{z}}{N} - \frac{M}{N}\right] \tag{46}$$

$$M = \sqrt{\frac{\pi}{2}\left[(N-1)\beta^2\sigma_s^2 + \sigma_z^2\right]} \tag{47}$$

Recall that $\alpha = 0$ corresponds to the case of private values. In this case, the bid function the bidder submit will be:

$$p(y, s_i) = \frac{\theta}{2}(\mu + q_i) + \frac{s_i}{2} - \frac{\theta}{2}y \tag{48}$$

Comparing this to the true marginal valuation function of the bidder when $\alpha = 0$:

$$v(y, s_i) = \theta q_i + s_i - \theta y \tag{49}$$

and noting that, for large N, $\mu = \frac{\bar{s}}{\theta}$, we see that the strategic bid function is flatter than the true marginal valuation function, and will be "shaded" by $\frac{\theta q_i}{2} + \frac{s_i - \bar{s}}{2}$.

Using the derived expression for the parameters of the bid function, we can arrive at the following empirical predictions of the model.

- C1: Bid shading (defined as $v(0, s_i) - p(0, s_i)$) increases as the uncertainty in security value, σ_s^2 ,increases.
- C2: Bid shading increases as the supply uncertainty, σ_Z^2,increases.
- C3: If $Q > 1 - \bar{z}$, i.e. the aggregate optimal inventory position of the banks exceeds the (expected) supply of securities, then the effect of the number of bidders, N, on the amount of bid-shading is ambiguous. If, however, $Q < 1 - \bar{z}$, i.e. the supply of the security exceeds the aggregate optimal inventory position of the market, then bid shading increases as the number of bidders, N, increases.
- C4: The slope of the bid function is independent of the uncertainty in security value.
- C5: The downward sloping portion of the bid function becomes flatter as the number of bidders, N, increases.

Perhaps the most interesting finding above is C3. Several papers in the literature (Nyborg, Rydqvist and Sundaresan (1997), Gordy (1999) and Cammack (1991)) have treated the finding of an increase in bid-shading with the number of bidders to be a sign of the winner's curse. We should first point out that the finding in C3 holds true even in the case when $\alpha = 0$, when bidders have pure private values. The case where inventory requirements do not play a role ($\delta = 0$), we have the result that bid-shading increases with the number of bidders. However, when inventory requirements are important, and there is an expected shortage of securities, one would expect bidders to bid more aggresively, especially if they are facing a large number of competitors.

Observe that hypotheses C1, C2 and C5 of this more general model are in line with the complete information model discussed in section 3.2. Of course, this model makes the much more realistic prediction that bidders who have different private information and reserve needs will submit different price bids. Also, this model matches the stylized the fact that the price-intercepts of the bid functions are distributed normally.

The fact that this model predicts that the slope of the bid function will be independent of the uncertainty in security value as opposed to the prediction by the complete information model that bid functions will become steeper with uncertainty can be attributed to the lack of risk-aversion in this model. Observe that, in the complete information model, risk-neutrality would mean perfectly flat bid functions – since risk-neutral bidders' marginal valuations are flat. In this model, bidders' marginal valuations are downward sloping due to the inventory adjustment cost – this drives the downward sloping nature of the strategic bid functions, not the uncertainty in security value. Therefore, a test of risk-aversion can be performed by checking whether the slopes of the bid functions depend on σ_s^2 .

4 Testing the models

In this section, we investigate several empirical specifications implied by the models in the previous section. We first construct the variables to be used in the empirical tests. Recall that in section 2, we computed the slope and intercept terms for each bidder's "linearized" bid function. We will use these estiamtes to form our dependent variables to test the hypotheses regarding bid-shading and the response of the slopes of the bid functions.

To construct a measure of the uncertainty regarding the value of the security, we adopt 3 alternative methods: first we use the variance of trading prices of matching securities in the secondary market. Specifically, we take a 5-day window preceding the auction, and calculate the variance of the trading prices in this window. All bidders have access to this data before the auction, hence it is reasonable to expect that they make the same calculation when entering the auction.

Since secondary market trading is somewhat thin for the early part of my sample, we also use the 5-day volatility of the interbank money market rates as an alternative proxy. Since securities traded in this market are of much shorter maturity (1 day to 1 week), the connection between the volatilities of the two market rates is more tenuous. However, since banks trade heavily in this market to cover their reserve requirements, this market can be thought of a close substitute for the auction and secondary bills market. Hence, once adjusted for the maturity difference, volatilities in this market should be comparable to the T-bill market.

A third measure of the uncertainty in security value is contained in the dispersion of different bidders' linearized demand schedules. According to the model of section 3.5, the variance of price-intercepts of the bidders' demand functions should be:

$$STDEV(intcpt) = \frac{\theta}{2} STDEV(q_i) + \frac{1}{2}\sigma_s$$

So, assuming the distribution of the size and thus the inventory requirements of the banks remain constant over time, the variance of the price intercepts yields a constant plus the variance of the value of the security. Hence, we also use this as a measure of the uncertainty in security value.

The extent of the supply uncertainty also plays an important role in the shape of the bid functions. Although we do not have data on bidders' expectation of this uncertainty, since we know that the Turkish Treasury stopped exercising discretion on the amount of Treasury bills to be sold after February 1993, we can construct a dummy variable $POSTFEB93$ to capture the decline in supply uncertainty.

We also found that the size of the aggregate optimal inventory position of the banks relative to the issue size plays an important role in changing the sign of the hypotheses regarding bid shading. Unfortunately, we do not have data on the banks' optimal inventory positions. However, we can measure the ratio of total demand to total supply. Since, during the period studied, banks were required to hold 35% of their holdings in Treasury securities, we assume that there is an supply shortfall ($SHORTFALL = 1$) if total supply is more than 35% of total demand.

We now construct the empirical specification to test hypotheses A1-A3 for the complete information model. Recall that these hypotheses relate the amount of bid shading to uncertainty in security value, total supply, and the number of bidders in the auction. Bid shading in the auction is given by the formula:

$$\overline{v} - p(0) = \frac{\rho\sigma_v^2}{2N-1}\left[1 + \frac{(N-1)(1-Z)}{N}\right] \tag{50}$$

where $p(0)$ is the price-intercept of the bid function, $\overline{v}$ is the mean value of the security, ρ the risk aversion coefficient, and Z is the "spread" of supply uncertainty.

The challenge of this exercise is to construct a measure for $\overline{v}$, the average value of the security. Following the convention in the literature, we will assume that in a perfect resale market, the price of the security will not be manipulated – so, on average, the resale price will equal $\overline{v}$. Hence, we construct the measure of bid-shading as:

$$SHADE_t = RESALE_t - E_i[INTCPT]_t \tag{51}$$

the difference between the resale price of the security and the average price-intercept of the bid functions.

Therefore, we run the following specification to test hypotheses A1-A3:

$$SHADE_t = \beta_0 + \beta_1\sigma_{vt} + \beta_2 POSTFEB93_t + \beta_3 N_t \tag{52}$$

where the hypotheses are $\beta_1 > 0, \beta_2 < 0$ and $\beta_3 < 0$.

In the private information model, average bid shading in an auction is given by formula:

$$E\left[v(0,s_i) - p(0,s_i)\right] \approx \frac{\theta}{2}Eq_i + \frac{\theta}{2}\overline{s} - \frac{\theta}{2}\frac{Q-(1-\overline{z})}{N} + \frac{\theta}{2}\frac{\sqrt{\frac{\pi}{2}\left[(N-1)\beta^2\sigma_s^2+\sigma_z^2\right]}}{N} \tag{53}$$

Once again, assuming we can use the resale market price as a measure of $E[v(0,s_i)]$, we can estimate the specification:

$$\begin{aligned} SHADE_t &= \beta_0 + \beta_1\sigma_{st} + \beta_2 POSTFEB93_t \\ &\quad +\beta_3(1-SHORTFALL_t)N_t + \beta_4 SHORTFALL_t N_t \end{aligned} \tag{54}$$

where hypotheses C1-C3 translate to: $\beta_1 > 0, \beta_2 < 0, \beta_3 > 0$, and the sign of β_4 is ambiguous.

Table 2 reports the results of estimating specification (54). The first 3 regressions correspond to the estimations using the 3 different measures of the uncertainty of the asset value. We see that there is a weak positive relationship between the amount of bid-shading and the uncertainty in asset value, consistent with both the complete information and private information models. We also see a negative correlation between the amount bid-shading and uncertainty in supply, as predicted by both models. However, we see that the amount of shading increases with the number of bidders, contrary to what the complete information model predicts. Although we would expect the effect of the number of bidders to on the amount of shading to be less when there is no supply shortage in the auction, we find that in periods of supply shortage, bid-shading is more strongly correlated with the number of bidders in the auction.

To investigate the last result further, we reran specification (54) using the lagged values of the $SHORTFALL$ variable instead of the contemporaneous shortfall in supply. The result is reported in the 4th column of Table 2. Since bidders do not observe the shortfall in supply until after the auction ends, this specification is motivated by a model in which bidders form adaptive expectations about the supply shortfall. We see that with this modified specification, the effect of the number of bidders on bid shading is in fact less than when there is a shortfall as opposed when there is not – satisfying the prediction of the model.

Next, we investigate the hypotheses regarding effect of value uncertainty and the number of bidders on the slope of the bid functions. Recall that both the complete information and the private information model predict that the absolute value of the slope of the bid functions should decrease with N. The complete information model predicts an increase in the slopes of the bid functions given an increase in the uncertainty of the security value – whereas the private information model, in which bidders are risk neutral, predicts no effect.

Table 3 reports the results of the regression of the (log) absolute value of the slopes of the bid functions on the number of bidders, and the the uncertainty of the security value.[12]

[12]We take the log of the slopes, since, by figure 2, this follows a near-spherical distribution.

Observe that when we regress the average bid function slope in the auction on the variance of the security value and the number of bidders, we get results that are opposite to what the theory predicts: bid functions seem to get steeper with increases in N and flatter with increases in uncertainty.

However, when we run the regression using individual bid function slopes, rather than averages within auctions, we get the signs that we expect: bid functions get flatter as N increases – consistent with both (A5) and (C5). Risk aversion also seems to play a role: bid functions become steeper as the uncertainty in security value increases.

5 Conclusion

Using a detailed bidder-level data set from Turkish treasury auctions, we have tried to describe the main features of the bidding strategies that we observe in the data. Motivated by the stylized fact that bid functions are accurately represented by linear functions, we developed a model of bidding in a divisible good discriminatory auction which incorporated several realistic features like the existence of resale incentives, reserve requirements/inventory needs, and private information. A special case of the model was solved explicitly for the 2 bidder private case, and an approximate solution was found for the more general N bidder case. Several empirical hypotheses regarding the response of bid-shading and slopes of bid functions to changes in value uncertainty and number of competitors was investigated, and we found empirical support for the validity of these hypotheses.

A shortcoming of the empirical analysis conducted in section 4 and the model developed in section 3 is that the number of bidders in the auction has been treated as an exogenous variable. However, we believe that the wide variation in the number of bidders who attend these auctions is driven by expectations of profit – hence a complete model of bidding should take into account the endogenous participation decisions of the bidders. The endogeneity of entry would cause the parameter estimates on N in Table 2 to be biased in the presence of auction level unobservables. We have not attempted to control for this endogeneity bias, lacking a natural instrumental variable; however, the study of settings where entry is exogenously determined (such as primary dealer markets that exist in many countries) would be helpful in getting around this problem.

An important empirical weakness of the model that has been developed in this paper is that it can not account for the heterogeneity in the slopes of the bid functions. An inspection of the slope parameter found in equation (45) will reveal that the model predicts that different bidders, even those with different inventory needs, are going to submit demand functions with the same slope.

The variation in the slopes can be incorporated into the model predictions by allowing the inventory adjustment cost parameter, θ, to vary across bidders. However, we do not think that this addition is realistic: although it is perfectly reasonable for bidder sizes/inventory needs to be widely distributed, it is difficult to fathom that a cost parameter would be very different across bidders. We believe that a resolution to this puzzle is an important avenue of research.

We believe that the method of analysis in the theoretical section of this paper can be extended to a more general computational algorithm to solve for the equilibrium bidding strategies in divisible good auctions. The "guess-and -verify" method can be implemented using higher-order approximation techniques. Given that understanding bidder behavior is a necessary prerequisite for the proper design of Treasury auction markets, we believe that analytical hurdles should not deter researchers from seeking approximate or computational techniques to characterize the equilibria of standard multi-unit auction formats.

References

Back, Kerry and Jaime F. Zender, "Auctions of Divisible Goods: On The Rationale for the Treasury Experiment," *Review of Financial Studies*, 1993, *6*, 733–664.

Bikchandani, Sushil and C. Huang, "Auctions with Resale Markets: An Exploratory Model of Treasury Bill Markets," *Review of Financial Studies*, 1989, *2*, 311–340.

Cammack, Elizabeth B., "Evidence on Bidding Strategies and Information in Treasury Bill Auctions," *Journal of Political Economy*, 1991, *99*, 100–130.

Chari, V. V. and Robert Weber, "How The U.S. Treasury Should Auction Its Debt," *Quarterly Review of the Federal Reserve Bank of Minneapolis*, Fall 1992, pp. 3–12.

Gordy, Michael B., "Structural Analysis of Reservation Price Policy in a Treasury Bill Auction," September 1994. Working paper.

——, "Hedging winner's curse with multiple bids: evidence from the Portuguese Treasury bill auction," *Review of Economics and Statistics*, 1999, *81*, 448–465.

Hortaçsu, Ali, "Mechanism Choice and Strategic Bidding in Divisible Good Auctions: An Empirical Analysis of the Turkish Treasury Auction Market," November 2000. Stanford Institute for Economic Policy Research Discussion Paper.

McAdams, David, "Increasable Supply and "Collusive-Seeming Equilibria" in the Uniform Price Auction," September 1999. Working paper.

Nyborg, Kjell, Kristian Rydqvist, and Suresh Sundaresan, "Bidder Behavior in Multiple Unit Auctions," June 1997. mimeo.

Wang, James and Jaime Zender, "Auctioning Divisible Goods," 1995. Duke Finance Research Working paper.

Wilson, Robert, "Auctions of Shares," *Quarterly Journal Of Economics*, 1979, pp. 675–689.

Table 2: Test of shading hypotheses

Dependent Variable	*SHADE*			
Specification	(1)	(2)	(3)	(4)
Secondary market volatility	1.83e-06			
	(0.49)			
IMM volatility		.0011		
		(1.26)		
Std. dev. of intercepts			.173	.383*
			(1.39)	(2.21)
Low supply uncertainty	-.0032*	-.0044*	-.0032*	-0.0002
($POSTFEB$)	(-3.04)	(-3.05)	(-3.22)	(-0.16)
$N * (1 - SHORTFALL)$	.00003*	.00004	.00003	
	(2.31)	(1.77)	(1.90)	
$N * (SHORTFALL)$	.00011*	.00012*	.00010*	
	(5.27)	(5.64)	(4.59)	
$N * (1 - SHORTFALL_{t-1})$				.00004
				(1.53)
$N * (SHORTFALL_{t-1})$				.00003
				(0.99)
Constant	-.003*	-.003*	-.003*	-.003
	(-2.07)	(-2.50)	(-2.17)	(-1.32)
No. of observations	25	25	25	24
R^2	0.65	0.68	0.68	0.35

T-statistics in parentheses. Variables significant at the 95% level are marked with an asterisk.

Table 3: Test of slope hypotheses

Dependent Variable	$log(\|SLOPE\|)$	
Specification	Auction level	Bidder level
Std. dev. of intercepts	-18.01	36.87*
	(-0.22)	(2.88)
N	.044*	-.022*
	(3.44)	(-9.17)
Constant	-4.08*	-7.95*
	(-4.45)	(-41.68)
No. of observations	27	1580
R^2	0.33	0.052

T-statistics in parentheses. Variables significant at the 95% level are marked with an asterisk.