Introduction to Probability Theory for Graduate Economics

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CHAPTER 5 - STOCHASTIC PROCESSES

1 Stochastic Processes

A stochastic process, or sometimes a random process, is the counterpart to a deterministic process (or a deterministic system) in probability theory. Instead of dealing with only one possible 'reality' of how the process might evolve under time (as is the case, for example, for solutions of an ordinary differential equation), in a stochastic or random process there is some indeterminacy in its future evolution described by probability distributions. This means that even if the initial condition (or starting point) is known, there are many possibilities the process might go to, but some paths are more probable and others less.

In the simplest possible case ("discrete time"), a stochastic process amounts to a sequence of random variables known as a time series. Another basic type of a stochastic process is a random field, whose domain is a region of space, in other words, a random function whose arguments are drawn from a range of continuously changing values. One approach to stochastic processes treats them as functions of one or several deterministic arguments ("inputs", in most cases regarded as "time") whose values ("outputs") are random variables: non-deterministic (single) quantities which have certain probability distributions. Random variables corresponding to various times (or points, in the case of random fields) may be completely different. The main requirement is that these different random quantities all have the same "type". Although the random values of a stochastic

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process at different times may be independent random variables, in most commonly considered situations they exhibit complicated statistical correlations.

Familiar examples of processes modeled as stochastic time series include stock market and exchange rate fluctuations, and random movement such as Brownian motion or random walks.

2 The Poisson Process

2.1 Definition and Properties of a Poisson Process

Definition A stochastic process $\{N(t), t \geq 0\}$ is called ***a counting process***, if $N(t)$ represents the total number of events that have occurred up to time t. A counting process $N(t)$ must satisfy the following properties:

- $N(t) \geq 0$,
- $N(t) \in \mathbb{N},\ \forall t \geq 0$,
- If $s < t$, then $N(s) \leq N(t)$,
- For $s < t$, $N(t) - N(s)$ equals the number of events that have occurred in the interval (s, t).

Definition A counting process possesses ***independent increments*** if the number of events which occur in disjoint time intervals are independent. For example, the number of events that occurred by time 10; ie, $N(10)$, must be independent of the number of events that occurred between times 10 and 15; ie, $N(15) - N(10)$.

Definition A counting process possesses ***stationary increments*** if the distribution of the number of events that occur in any time interval depends only on the length of the time interval. In other words, if the number of events in the interval $(t_1 + s, t_2 + s)$; ie, $N(t_2 + s) - N(t_1 + s)$ has the same distribution as the number of events in the interval (t_1, t_2); ie, $N(t_2) - N(t_1)$, for all $t_1 < t_2$, and $s \geq 0$.

Definition The counting process $\{N(t), t \geq 0\}$ is called ***a Poisson process having rate*** λ, $\lambda > 0$, if

1. $N(0) = 0$,
2. The process has independent increments,
3. The number of events in any interval of length t is Poisson distributed with mean λt. That is, for all $s, t \geq 0$:

$$Pr\left(N(t+s) - N(s) = n\right) = \exp\left(-\lambda t\right) \frac{(\lambda t)^n}{n!}, \; n = 0, 1, 2, 3, \ldots$$

It is noteworthy that it follows from condition (iii) that a Poisson process has stationary increments and also that

$$\mathcal{E}\left[N(t)\right] = \lambda t$$

which explains why λ is the rate of the process.

Definition The function $f(\cdot)$ is said to be $o(\cdot)$ if

$$\lim_{h \to 0} \frac{f(h)}{h} = 0.$$

Definition The counting process $\{N(t), t \geq 0\}$ is called ***a Poisson process having rate*** λ, $\lambda > 0$, if

1. $N(0) = 0$,
2. The process has stationary and independent increments,
3. $Pr\left(N(h) = 1\right) = \lambda h + o(h)$,
4. $Pr\left(N(h) \geq 2\right) = o(h)$.

Theorem The two definitions of the Poisson process given above are equivalent.

Proof See Ross(1993). ■

2.2 Interarrival and Waiting Time Distributions

Definition Consider a Poisson process, and let T_1 denote the time of the first event. Further, for $n > 1$, let T_n denote the elapsed time between the $(n-1)^{st}$ and n^{th} events. The sequence $\{T_n, n = 1, 2, 3, \ldots\}$ is called the ***sequence of interarrival times***. For example, $T_1 = 5$ and $T_2 = 10$, then the first event of the Poisson process would have occurred at time 5, and the second at time 15.

Proposition $T_n, n = 1, 2, 3, \ldots,$ are independently and identically distributed with exponential random variables having mean $1/\lambda$.

Remarks The assumption of stationary and independent increments is basically equivalent to asserting that, at any point in time, the process *probabilistically* restarts itself. In other words, the process from any point on is independent of all that has previously occurred (by independent increments), and also has the same distribution as the original process (by stationary increments). So, the process has *no memory*, and exponential arriving times are to be expected.

Definition Another quantity of interest is S_n, the arrival time of the n^{th} event, also called the ***waiting time*** until the n^{th} event. Mathematically,

$$S_n = \sum_{i=1}^{n} T_i, \ n \geq 1.$$

Finally, it is noteworthy that S_n has a gamma distribution with parameters n and λ.

2.3 Properties of the Exponential Distribution

Definition A random variable X is ***memoryless***, if

$$Pr(X \geq t) = Pr\left(X \geq t + s | X \geq s\right); \ \forall t, s \geq 0.$$

In words, the probability that the first occurrence happens at a time $X \geq t$ is equivalent to the probability that the first occurrence happens at time $X \geq t_o + t$, given that it has not yet occurred until time t_o. Whenever it is appropriate, the memoryless property of Exponential distribution is useful in economics, as one does not have to keep track of the whole history to compute the probability distribution of a variable in the current state.

Remark Exponential distribution is not only memoryless, but it is also the unique distribution possessing this property. To see this, suppose that X is memoryless, and $\bar{F}(x) = Pr\,(X > x)$. Then, it follows that:

$$\bar{F}(t+s) = \bar{F}(t)\ \bar{F}(s).$$

That is, $\bar{F}(x)$ satisfies the functional equation:

$$g(t+s) = g(t)\ g(s).$$

However, the only right continuous solution of this functional equation is

$$g(x) = \exp\,(-\lambda x)$$

and since a distribution function is always right continuous, it implies that

$$\bar{F}(x) = \exp\,(-\lambda x) \Rightarrow F(x) = Pr\,(X \leq x) = 1 - \exp\,(-\lambda x)$$

which shows that X is exponentially distributed.

Definition The memoryless property is further illustrated by the ***failure (hazard) rate function*** of the exponential distribution:

$$\mathrm{r}(t) = \frac{f(t)}{1 - F(t)}.$$

One can interpret the hazard rate as the probability that X will not survive for an additional time dt, given that X has survived until t.

Since the memoryless property is associated with the exponential distribution, the hazard rate becomes

$$\mathrm{r}(t) = \frac{f(t)}{1 - F(t)} = \frac{\lambda\ \exp\,(-\lambda t)}{\exp\,(-\lambda t)} = \lambda.$$

Thus, the hazard rate of the exponential distribution equals the reciprocal of the inverse of the mean of the exponential distribution.

3 Discrete Time Discrete Space Markov Chains

3.1 Definitions

First we consider a discrete time Markov chain. In this setup, ***a stochastic process*** $\{X_n, n = 0, 1, 2, \ldots\}$ takes on a **finite or countable number of possible values**. If $X_n = i$, then the process is said to be in state i at time n. We suppose that whenever the process is in state i, there is a fixed probability P_{ij} that it will next be in state j; ie:

$$Pr\left(X_{n+1} = j \mid X_n = i, X_{n-1} = i_{n-1}, \ldots, X_1 = i_1, X_0 = i_0\right) = P_{ij}$$

for all states $i_0, i_1, \ldots, i_{n-1}, i, j$ and all $n \geq 0$. Such a stochastic process is called ***Markov Chain***. One can interpret this equation int he following way: The conditional distribution of any future state X_{n+1} given the past states $X_0, X_1, \ldots, X_{n-1}$ and the present state X_n, is independent of the past states and depends only on the present state.

Since the probabilities are nonnegative, and since the process must make a transition into some state, we have that:

- Every probability must be positive and less than 1:

$$P_{ij} \geq 0; \ \forall i, j \geq 0,$$

- The sum of the probabilities must equal 1:

$$\sum_{j=0}^{\infty} P_{ij} = 1; \ \forall i = 0, 1, 2, \ldots.$$

Definition Chapman-Kolmogorov Equations We now define the n-step transition probabilities P_{ij}^n to be the probability that a process in state i will in state j after n additional transitions; ie:

$$P_{ij}^n = Pr\left(X_{n+m} = j \mid X_m = i\right),\ n \geq 0,\ i, j \geq 0$$

It is noteworthy that the matrix $\mathbf{P}$ is called ***Transition Matrix***:

$$\mathbf{P} = \begin{vmatrix} P_{00} & P_{01} & P_{02} & \dots \\ P_{10} & P_{11} & P_{10} & \dots \\ \vdots & \vdots & \vdots & \\ P_{i0} & P_{i1} & P_{i0} & \dots \\ \vdots & \vdots & \vdots & \ddots \end{vmatrix}$$

The ***Chapman-Kolmogorov equations*** provide a method for computing these n-step transition probabilities:

$$P_{ij}^{n+m} = \sum_{k=0}^{\infty} P_{ik}^n\ P_{kj}^m,\ \forall n, m \geq 0,\ \text{all } i, j$$

and are most easily understood by noting that $P_{ik}^n\ P_{kj}^m$ represents the probability that starting in i, the process will go to state j in $n + m$ transition thrugh a path which takes it int state k at the n^{th} transition. Hence:

$$\begin{aligned} P_{ij}^{n+m} &= Pr\left(X_{n+m} = j \mid X_0 = i\right) \\ &= \sum_{k=0}^{\infty} Pr\left(X_{n+m} = j, X_n = k \mid X_0 = i\right) \\ &= \sum_{k=0}^{\infty} Pr\left(X_{n+m} = j \mid X_n = k, X_0 = i\right)\ Pr\left(X_n = k \mid X_0 = i\right) \\ &= \sum_{k=0}^{\infty} P_{ik}^n\ P_{kj}^m \quad \blacksquare \end{aligned}$$

In matrix notation, above equation implies:

$$\mathbf{P}^{(n)} = \mathbf{P}^n$$

where $\mathbf{P}^{(n)}$ denotes the matrix of n-step transition probabilities P_{ij}^n.

3.2 Limiting Probabilities

Definition A Markov chain is called ***irreducible***, if for any $i \neq j$, P_{ij}^n is positive, for $n \geq 0$.

Definition If state i is ***recurrent***, then starting in state i, the process will reenter state i again and again and again— in fact, infinitely often. So, the state i is recurrent if:

$$\sum_{n=1}^{\infty} P_{ii}^n = \infty.$$

Definition If state i is recurrent, then it is said to be ***positive recurrent*** if, starting in state i, the expected time until the process returns to state i is finite. In a finite-state Markov chain, all recurrent states are positive recurrent.

Definition If state i is ***transient***, then every time process enters state i, there will be positive probability that it will never again enter that state. Thus, if state i is transient, then starting in state i, the number of time periods that the process will be in state i has a geometric distribution with finite mean. So, the state i is transient if:

$$\sum_{n=1}^{\infty} P_{ii}^n < \infty.$$

Definition State i has ***period d***, if $P_{ii}^n = 0$ whenever n is not divisible by d, which is the largest integer with this property. For example, starting in state i, it may be possible for the process to enter state i only at the times $2, 4, 6, \ldots$, in which case state i has period 2. A state with period 1 is *aperiodic*.

Definition Positive recurrent, aperiodic states are called ***ergodic***.

Theorem For an irreducible ergodic Markov chain $\lim_{n\to\infty} P_{ij}^n$ exists and is independent of i. Furthermore, letting

$$\pi_j = \lim_{n\to\infty} P_{ij}^n, \ j \geq 0,$$

then π_j is the nonnegative solution of

$$\pi_j = \sum_{i=0}^{\infty} \pi_i P_{ij}, \ j \geq 0,$$
$$\sum_{j=1}^{\infty} \pi_j = 1.$$

Remarks It can be shown that π_j, the limiting probability that the process will be in state j at time n, also equals the *long-run proportion of time that the process will be in state* j. These long-run proportions are often called ***stationary*** probabilities. Also, the vector $\mathbf{\Pi} = (\pi_0, \pi_1, \pi_2, \ldots)$ is also referred to as ***invariant*** distribution. This is because if the initial state is chosen according to probabilities $\pi_j,\ j \geq 0$, then the probability of being in state j at any time n is also equal to π_j:

$$Pr\left(X_0 = j\right) = \pi_j \Rightarrow Pr\left(X_n = j\right) = \pi_j;\ \forall n, j \geq 0.$$

Moreover, the proportion of time in state j equals the inverse of the mean time between visits to j. This result is a special case of "The Strong Law of Renewal Process".

Definition ***Time Reversibility*** The idea of a reversible Markov chain comes from the ability to "invert" a conditional probability using Bayes' Rule:

$$\begin{aligned} Pr\left(X_n = i \mid X_{n+1} = j\right) &= \frac{Pr\left(X_n = i, X_{n+1} = j\right)}{Pr\left(X_{n+1} = j\right)} \\ &= \frac{Pr\left(X_n = i\right)\ Pr\left(X_{n+1} = j \mid X_n = i\right)}{Pr\left(X_{n+1} = j\right)}. \end{aligned}$$

It now appears that time has been reversed. Thus, a Markov chain is said to be ***reversible*** if there is a $\mathbf{\Pi}$ such that:

$$\pi_i P_{ij} = \pi_j P_{ji},\ \forall\ i, j \geq 0.$$

This condition is also known as the "detailed balance" condition. Summing over i gives

$$\sum_{i=0}^{\infty} \pi_i P_{ij} = \pi_j,\ \forall\ j \geq 0.$$

So, for reversible Markov chains, $\mathbf{\Pi}$ is always a stationary distribution.

3.3 Computing an Invariant Distribution and Simulating Markov Chains

Below is a MATLAB code to compute the invariant distribution given a transition matrix for a discrete time Markov chain:

Input 1: MATLAB Code to Find Invariant Distribution:

```
function[INVD] = invdist(TM)
% invdist.m - function
%% CODE
N = size(TM,2);
B = (TM-eye(N));
B(1:N,N) = ones(N,1);
%
o = zeros(1,N);
o(N) = 1;
INVD = o*inv(B); % Invariant Distribution
%
```

Below is a MATLAB code to simulate a stochastic process which follows a discrete time Markov chain:

Input 2: MATLAB Code to Simulate Markov Chains:

```
function [theta,ug] = simulationMC(G,TM,T)
% simulation - function
%% Arguments:
% G = grid points for the Markov chain;
% TM = transition matrix for the Markov chain;
% T = number of periods to simulate;
%% CODE:
CDF = cumsum(TM,2); % cumulative distribution function.
i = 1; % index of the initial shock.
ind = zeros(1,T);
ind(1) = i;
for t = 1:T; % simulation for T periods.
ug(t) = rand(1); % create pseudo-random numbers from the uniform dist.
j = find(CDF(i,:)>=ug(t),1,'First');
ind(t) = j;
i = j;
end
theta = G(ind); % pseudo-random numbers from the Markov chain for simulation.
%
```

3.4 Simple Example

Grid Points Suppose there are uncertainty is caused by a stochastic shock which follows a known discrete space Markov Chain. First, let G denote the grid points for the shock.

```
>> G = 1:5;
```

Thus, the grid points are the integers from 1 to 5.

Transition Matrix One can also create a transition matrix for the shock in MATLAB in the following way:

```
>> rand('state',12356); % Seed the uniform, pseudo-random number generator.
TM = rand(length(G));
for i=1:length(G);
TM(i,:)  = TM(i,:)./sum(TM(i,:));
end
```

$$\mathbf{TM} = \begin{vmatrix} 0.0496 & 0.2110 & 0.4210 & 0.1936 & 0.1249 \\ 0.1906 & 0.0637 & 0.0141 & 0.2412 & 0.4905 \\ 0.2165 & 0.2522 & 0.2904 & 0.1877 & 0.0532 \\ 0.0854 & 0.3903 & 0.0401 & 0.4400 & 0.0442 \\ 0.2385 & 0.1702 & 0.0283 & 0.4751 & 0.0880 \end{vmatrix}$$

One can interpret the transition matrix in the following way: The number 0.0496 is the probability to go stay in state 1, given that the current state is already 1. Similarly, the number 0.0401 is the probability to go to state 3, given that the current state is 4.

Cumulative Distribution Function Now, one can easily compute the cumulative distribution function (CDF) in the following way:

```
>> CDF = cumsum(TM,2);
```

$$\mathbf{CDF} = \begin{vmatrix} 0.0496 & 0.2605 & 0.6815 & 0.8751 & 1.0000 \\ 0.1906 & 0.2543 & 0.2684 & 0.5095 & 1.0000 \\ 0.2165 & 0.4687 & 0.7591 & 0.9468 & 1.0000 \\ 0.0854 & 0.4756 & 0.5158 & 0.9558 & 1.0000 \\ 0.2385 & 0.4087 & 0.4369 & 0.9120 & 1.0000 \end{vmatrix}$$

One can interpret the cumulative distribution function in the following way: The number 0.2605 is the probability that tomorrow state is less than or equal to 2, given that the current state is 1. Similarly, the number 0.4369 is the probability that tomorrow's state is less than or equal to 3, given that the current state is 5.

Invariant Distribution Now, one can easily compute the invariant distribution (INVD) using the transition matrix in the following way:

```
>> INVD = invdist(TM);
```

$$\mathbf{INVD} = \begin{vmatrix} 0.1460 & 0.2343 & 0.1169 & 0.3339 & 0.1690 \end{vmatrix}$$

One can interpret the invariant distribution in the following way: The number 0.1460 is the probability to go to state 1, regardless of the current state. Similarly, the number 0.1690 is the probability to go to state 3, regardless of the current state.

Simulating from the Transition Matrix One can simulate the shock from the Markov chain using the transition matrix for a fixed number of periods (T) in the following way:

```
>> T = 10;
>> [theta,ug] = simulationMC(G,TM,T);
```

$$\textbf{theta} = \begin{vmatrix} 5 & 4 & 2 & 5 & 1 & 3 & 2 & 4 & 2 & 5 \end{vmatrix}$$

$$\textbf{ug} = \begin{vmatrix} 0.9939 & 0.6425 & 0.4353 & 0.5441 & 0.0064 & 0.5779 & 0.2280 & 0.5083 & 0.4612 & 0.7689 \end{vmatrix}$$

In this setup, let theta and ug denote the shock value and the respective probability draw in a given period. One can interpret the simulation results in the following way:

- Starting from state 1 (fixed inside the code), the state in period 1 is $theta = 5$. This is because looking at the first row of the CDF, the probability draw $ug = 0.9939$ corresponds to the range $(0.8751, 1]$. Thus, the shock value equals 5 in the first period.
- In the second period, the shock value is $theta = 4$, as the probability draw $ug = 0.6425$ corresponds to the range $(0.4369, 0.9120]$, the fifth row of the CDF.

It is noteworthy that the simulation results *theta* may change from one trial to another. This is because the pseudo-random number generation is not fixed with a given state in the code "simulationMC.m".

Example "Example.m" in the zip file for Chapter 5 contains the steps carried out above.

4 Continuous Time Continuous Space Stochastic Processes

Since most of the definitions and theorems in discrete time Markov chains can be extended to continuous time Markov chains, we will rather focus on a particular application of the continuous time Markov chains in economics.

4.1 Approximating A First-Order Autoregressive AR(1) Process With Finite Space Markov Chains

Procedure Suppose that a stochastic shock follows the following AR(1) process:

$$s' \quad = \mu + \lambda s + \epsilon, \text{ with } \text{var}(\epsilon) = \sigma^2, \text{ where } |\lambda| < 1. \tag{1}$$

Here ϵ is white noise; ie, ϵ is distributed with mean zero and variance σ^2.

To discretize the AR(1) process, one must assume the process stays within a bounded interval to be able to solve the problem. To use the approach laid out in Tauchen (1986). Specifically, Tauchen (1986) considers an AR(1) process like the one in (1), where $\mu = 0$. Denote the distribution function of ϵ as $F(\epsilon)$. We are going to approximate the continuous process in (1) by a discrete process with N values where the shock s can take on the values $\tilde{s}_1 < ... < \tilde{s}_N$. To determine the values $\tilde{s}_i$ take on, Tauchen recommends the following:

$$\tilde{s}_N \equiv m\sigma_s = m\left(\frac{\sigma_\epsilon^2}{1-\lambda^2}\right)^{\frac{1}{2}},$$

$$\tilde{s}_1 = -\tilde{s}_N, \text{ and}$$

$$\tilde{s}_i = \tilde{s}_1 + \frac{(i-1)(\tilde{s}_N - \tilde{s}_1)}{N-1} \quad \text{(ie, use equidistant spacing).}$$

From this, one can calculate a transition matrix. Define $p_{ij} \equiv \text{Prob}(s(t) = \tilde{s}_j | s(t-1) = \tilde{s}_i)$, this would be the element in the transition matrix row i and column j - such that $\sum_j p_{ij} = 1$. For $j \in [2, N-1]$, p_{ij} can be computed as

$$p_{ij} = F\left(\frac{\tilde{s}_j - \lambda\tilde{s}_i + w/2}{\sigma_\epsilon}\right) - F\left(\frac{\tilde{s}_j - \lambda\tilde{s}_i - w/2}{\sigma_\epsilon}\right)$$

where $w = \tilde{s}_k - \tilde{s}_{k-1}$ (note that given we are using equidistant points this value is the same for all k). These expressions can be thought of as the probability that $\lambda\tilde{s}_i + \epsilon \in [\tilde{s}_j - w/2, \tilde{s}_j + w/2]$.

Otherwise the probability of transitioning from state i into state 1 is

$$p_{i1} = F\left(\frac{\tilde{s}_1 - \lambda\tilde{s}_i + w/2}{\sigma_\epsilon}\right)$$

and the probability of going from state i to state N is

$$p_{iN} = 1 - F\left(\frac{\tilde{s}_N - \lambda\tilde{s}_i - w/2}{\sigma_\epsilon}\right).$$

This discrete approximation to the conditional distribution of $s(t)$ given $s(t-1)$ will converge in the sense of probability to the true conditional distribution for the stochastic process articulated in (1).

One practical concern for the above approach is how to deal with negative values for the shock. Specifically this means the firm's technology produces negative output, which does not make much economic sense. To prevent this situation we transform the shocks by letting $s = \exp(s)$ which ensures all values of the shock are positive. Another benefit of this transformation is that the grid becomes finer at the lower end and more coarse for high shock values.

Example "Example2.m" in the zip file for Chapter 5 provides a simple implementation of the algorithm. Please refer to the "markovprob.m" file on the next page for a way to implement the algorithm.

5 References

Ross, Sheldon M. *Introduction to Probability Models.* Fifth Edition. San Diego: Academic Press, 1993.

Silos, Pedro. "Assessing Markov-Chain Approximations: A Minimal Econometric Approach." *Journal of Economic Dynamics and Control*, 30 (2006), 1063–1079.

Tauchen, George. "Finite State Markov-Chain Approximations to Univariate and Vector Autoregressions." *Economic Letters*, 20 (1986), 177–181.

Implementation Tauchen Algorithm.

Input 3: MATLAB Code for Tauchen Algorithm:

```
function [Gl, TM, CDF, INVD, sz] = markovprob(mue, p, s, N, m)
% markovprob - function
%% Arguments:
% mue = intercept of AR(1) process;
% p = slope coeff. of AR(1) process;
% s = std. dev. of residuals in AR(1) process;
% N = # of grid points for the 'z' variable;
% m = Density of the grid for 'z' variable;
%% CODE:
sz = s / ((1-p^2)^(1/2)); % Std. Dev. of z.
zmin = -m * sz + mue/(1-p);
zmax = m * sz + mue/(1-p);
z = linspace(zmin,zmax,N); % Grid Points
%% Transition Matrix:
TM = zeros(N,N); % Transition Matrix
w = z(N) - z(N-1);
for j = 1:N;
TM(j,1) = cdf('norm',(z(1)+w/2-mue-p*z(j))/s,0,1);
TM(j,N) = 1 - cdf('norm',(z(N)-w/2-mue-p*z(j))/s,0,1);
for k = 2:N-1;
TM(j,k) = cdf('norm',(z(k)+w/2-mue-p*z(j))/s,0,1)- ...
cdf('norm',(z(k)-w/2-mue-p*z(j))/s,0,1);
end
end
%% Cumulative Distribution Function:
CDF = cumsum(TM,2);
%% Invariant Distribution:
INVD = invdist(TM)'; % Invariant Distribution
%% Grids:
Gl = exp(z');
fprintf('If we have a lognormal var. (log(z)) in AR(1) process, \n');
fprintf('To make the interval finer at the lower end and coarser at the upper end.\n');
%
```