Introduction to Probability Theory for Graduate Economics

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4 Transformations of Random Variables

It is often the case that on some sample space S we have a RV X with a known distribution $F_X(x)$ and density $f_X(x)$, but the object of interest is a function of X, say $Y = u(X)$. Note that Y is a RV itself, because it is the composition of two functions, X and u, on S. Thus, we can talk about the distribution $F_Y(y)$ and density $f_Y(y)$ of Y as well. Thus, the first task at hand is to derive F_Y using our knowledge of u and F_X. There are several methods for doing so, and they will be discussed in this chapter.

The first method is called the **CDF technique**, and it is based on the fact that X and Y are both functions on the same sample space. More specifically, since $Y(S) = u \circ X(S)$, it immediately follows that the sets $\{y'|y' \leq y\}$ and $A_y = \{x|u(x) \leq y\}$ are equivalent events and therefore, they occur with the same probability. Thus, we have

$$F_Y(y) = \Pr[u(X) \leq y] = \Pr[X \in A_y] = \begin{cases} \int_{A_y} f_X(x)dx & \text{if } X \text{ is continuous,} \\ \sum_{x \in A_y} f_X(x) & \text{if } X \text{ is discrete.} \end{cases}$$

Exercise 1 Let X be a RV with distribution $F_X(x) = 1 - e^{-2x},\ 0 < x < \infty$ and define Y by $Y = e^x$. Use the CDF technique to derive $F_Y(y)$ and identify the domain of F_Y.

Exercise 2 Let X be a RV with distribution $F_X(x)$ and define Y by $Y = x^2$. Use the CDF technique to derive $F_Y(y)$ and identify the domain of F_Y. Can you derive the pdf f_Y also?

4.1 One-to-One Transformations

When $Y = u(X)$ is a monotonic function so that we can find an inverse function $X = w(y) = u^{-1}(Y)$, the process of solving for the distribution and density of Y becomes simpler. I begin with the discrete case.

Theorem 1 *Suppose X is a discrete RV with pdf $f_X(x)$ and $Y = u(x)$ is a one-to-one transformation with inverse w. Then the pdf of Y is*

$$f_Y(y) = f_X(w(y)).$$

Exercise 3 Prove the previous theorem.

Now suppose that X is a continuous RV with distribution and density F_X and f_X, and let $Y = u(x)$ be a one-to-one transformation with inverse w. Monotonicity of u greatly simplifies the CDF technique, as it reduces to a simple matter of plugging in the inverse function $w(Y)$. Suppose first that u is strictly increasing. In that case, we have

$$F_Y(y) = \Pr[Y \leq y] = \Pr[X \leq w(y)] = F_X(w(y)),$$

from which it follows that

$$f_Y(y) = \frac{d}{dy} F_X(w(y)) = \frac{d}{dw(y)} F_X(w(y)) \frac{d}{dy} w(y) = f_X(w(y)) \frac{d}{dy} w(y).$$

Now suppose that u is strictly decreasing, so that $u'(x) < 0 \; \forall x$. Now we have

$$F_Y(y) = \Pr[Y \leq y] = \Pr[X \geq w(y)] = 1 - F_X(w(y)),$$

from which it follows that

$$f_Y(y) = \frac{d}{dy} \left(1 - F_X(w(y))\right) = -f_X(w(y)) \frac{d}{dy} w(y).$$

From these two examples, we get the following theorem:

Theorem 2 *If X is a continuous RV with pdf $f_X(x)$ and $Y = u(X)$ is a one-to-one transformation with inverse w, then the pdf of Y is*

$$f_Y(y) = f_X(w(y)) \left| \frac{d}{dy} w(y) \right|.$$

Exercise 4 Consider a first-price, procurement auction where there are N firms competing for a contract to build a section of highway for the government. Firms differ with respect to how costly it would be to complete the project. Each firm submits a sealed bid representing the price it will charge the government to complete the project and the lowest bidder gets the contract.

The firms know their own private cost and they view their opponents' private costs as independently distributed Pareto RVs having cdf and pdf

$$F_C(c) = 1 - \left(\frac{\theta_0}{c}\right)^{\theta_1} \quad \text{and} \quad f_C(c) = \theta_1 \theta_0^{\theta_1} c^{-(\theta_1+1)}.$$

In this case, it can be shown that the equilibrium bidding function is given by $\beta(c) = c\frac{\theta_1(N-1)}{\theta_1(N-1)-1}$. Do the following:

a. Derive the equilibrium bid density $f_B(b)$ using the previous theorem.

b. Derive the equilibrium bid distribution $F_B(b)$ using the cdf technique.

c. Determine the limiting bid distribution as the number of competitors N approaches ∞.

d. Find the distribution of the winning bid for fixed N and determine the limit of this distribution as N approaches ∞.

e. What do parts (c.) and (d.) say about expected firm profits as the market becomes perfectly competitive?

It should be noted that Theorems 1 and 2 can be easily generalized to situations where $Y = u(X)$ is **piecewise monotonic**. This leads to the next theorem:

Theorem 3 *Consider a RV X with a pdf or pmf $f_X(x)$ and $Y = u(X)$. Suppose that there are k disjoint connected subsets $A_1, \ldots, A_k$ on the range of X where $u(\cdot)$ is one-to-one over A_i for each i. Moreover, let the inverse function of $u(\cdot)$ on A_i be defined by*

$$w_i(y) \equiv \begin{cases} u^{-1}(y) \cap A_i & \text{if } y \in u(A_i), \\ -\infty & \text{otherwise.} \end{cases}$$

Then the pdf of Y is given by

$$f_Y(y) = \sum_{i=1}^{k} f_X(w_i(y))$$

if X is discrete, and

$$f_Y(y) = \sum_{i=1}^{k} f_X(w_i(y)) \left| \frac{d}{dy} w_i(y) \right|$$

if X is continuous.

4.2 Transformations of High-Dimensional RVs

Thus far, we have concentrated solely on functions of univariate RVs, but what if we needed to find the distribution of a function of and n-dimensional RV $\mathbf{X} = (x_1, \ldots, x_n)$? The CDF technique can also be employed to find the distributions of real-valued functions $u : \mathbb{R}^n \to \mathbb{R}$, although it gets a bit more complicated.

Theorem 4 *Let $\mathbf{X} = (x_1, \ldots, x_n)$ be a continuous n-dimensional RV with joint pdf $f_{\mathbf{X}}(x_1, \ldots, x_n)$, and let $Y = u(\mathbf{x})$ be a real-valued function of $\mathbf{X}$. Then*

$$\begin{aligned} F_Y(y) &= \Pr[u(\mathbf{x}) \le y] \\ &= \int \cdots \int_{A_y} f_{\mathbf{X}}(x_1, \ldots, x_n) dx_1 \ldots dx_n, \\ \text{where } A_y &= \{x | u(x) \le y\}. \end{aligned}$$

Suppose now that $\mathbf{Y}$ is a vector-valued function

$$\mathbf{Y} = \mathbf{u}(\mathbf{X}) = (u_1(\mathbf{X}), \ldots, u_n(\mathbf{X})).$$

Moreover, suppose that $\mathbf{Y}$ is a one-to-one transformation in the sense that for a given $\mathbf{y}$ there is a unique $\mathbf{x}$ which solves $\mathbf{y} = \mathbf{u}(\mathbf{x})$. In this case, Theorems 1 and 2 can be generalized as follows.

Theorem 5 *If $\mathbf{X}$ is discrete, then under the above conditions the joint density of $\mathbf{Y}$ is given by*

$$f_{\mathbf{Y}}(y_1, \ldots, y_n) = f_{\mathbf{X}}(x_1, \ldots, x_n),$$

where $\mathbf{x} = (x_1, \ldots, x_n)$ is the unique solution to $(y_1, \ldots, y_n) = (u_1(\mathbf{x}), \ldots, u_n(\mathbf{x}))$.

Theorem 6 *Suppose $\mathbf{X}$ is continuous and the determinant of the* ***Jacobian***

$$\mathbf{J} = \left| \begin{pmatrix} \frac{\partial x_1}{\partial y_1} & \frac{\partial x_1}{\partial y_2} & \cdots & \frac{\partial x_1}{\partial y_n} \\ \frac{\partial x_2}{\partial y_1} & \frac{\partial x_2}{\partial y_2} & & \vdots \\ \vdots & & \ddots & \\ \frac{\partial x_n}{\partial y_1} & \cdots & & \frac{\partial x_n}{\partial y_n} \end{pmatrix} \right|$$

exists. Then under the above conditions the joint density of $\mathbf{Y}$ is given by

$$f_{\mathbf{Y}}(y_1, \ldots, y_n) = f_{\mathbf{X}}(x_1, \ldots, x_n)|\mathbf{J}|,$$

where $\mathbf{x} = (x_1, \ldots, x_n)$ is the unique solution to $(y_1, \ldots, y_n) = (u_1(\mathbf{x}), \ldots, u_n(\mathbf{x}))$.

Finally, it should also be noted that Theorems 5 and 6 can be generalized to deal with transformations $\mathbf{Y} = \mathbf{u}(\mathbf{X})$ that are one-to-one on partition $A_1, \ldots, A_k$ of the range of $\mathbf{X}$. The generalization is similar to Theorem 3.

4.3 Sums of RVs

There are many situations in which the sum of two or more random variables is of interest.

Theorem 7 *Let X_1 and X_2 be two RVs with joint density $f(x_1, x_2)$ and define $z = x_1$ and $y = x_1 + x_2$ for a given (x_1, x_2) pair. Then the density of the sum Y is given by*

$$f_Y(y) = \int_{-\infty}^{\infty} f(z, y - z)dz.$$

Moreover, if X_1 and X_2 are independent, then the formula is

$$f_Y(y) = \int_{-\infty}^{\infty} f_{X_1}(z) f_{X_2}(y - z)dz.$$

When X_1 and X_2 are independent, the above is called the **convolution formula**. There is also another useful technique for characterizing the distribution of a sum of independent RVs; it is called the **MGF method**.

Theorem 8 *If $X_1, \ldots, X_n$ are independent RVs with MGFs $M_{X_i}(t)$, then the MGF of their sum $Y = \sum_{i=1}^n X_i$ is*

$$M_Y(t) = \prod_{i=1}^{n} M_{X_i}(t).$$

Moreover, if the X_is are identically distributed with MGF $M_X(t)$, then the MGF of their sum is

$$M_Y(t) = [M_X(t)]^n.$$

4.4 Moments of Transformations of RVs

In many situations it is sufficient to simply compute certain moments of $Y = u(X)$, rather than characterizing the entire distribution of Y. This can often be accomplished simply using only the distribution of X, as shown in the following theorem.

Theorem 9 *If X is a RV with pdf $f_X(x)$ and $u(x)$ is a real-valued function on the range of X, then the* *expectation of $u(x)$ is given by*

$$\mathrm{E}[u(X)] = \sum_x u(x) f_X(x)$$

if X is discrete and

$$\mathrm{E}[u(X)] = \int_{-\infty}^{\infty} u(x) f_X(x) dx$$

if X is continuous.

Note that the theorem is general enough to provide a way to compute any of the moments of $Y = u(X)$. However, in cases where exactness is not crucial, there is another a simpler method of computing the approximate mean and variance of Y, using only the mean and variance of X and the derivatives of $u(\cdot)$.

Assuming that X has mean μ and variance σ^2 and $u(\cdot)$ is twice continuously differentiable, then its expectation can be approximated by

$$\mathrm{E}[u(X)] \approx u(\mu) + \frac{u''(\mu)\sigma^2}{2} \tag{1}$$

and it's variance can be approximated by

$$\mathrm{Var}[u(X)] \approx [u'(\mu)]^2 \sigma^2. \tag{2}$$

The accuracy of these approximations depends both on the curvature of $u(\cdot)$ around the mean and on the variability of X. A higher degree of curvature or variability lessens the accuracy of the approximation.

Exercise 5 Do the following:

a. Show that equation 1 provides a second-order approximation of the expectation of $Y = u(X)$. (*HINT:* Take a second-order Taylor expansion of $u(\cdot)$ centered at μ and compute its expectation.)

b. Show that equation 2 provides a first-order approximation of the variance of $Y = u(X)$. (*HINT:* Take a first-order Taylor expansion of $u(\cdot)$ centered at μ and compute the variance.)

c. Let X be a normal RV with mean $\mu = 0$ and variance $\sigma^2 = 4$. Define $Y = e^X$ and recall that Y is said to be a lognormal RV with exact mean of $e^{\mu + \frac{\sigma^2}{2}}$ and variance $e^{\sigma^2 + 2\mu}\left(e^{\sigma^2} - 1\right)$. Compute the approximate mean and variance of Y using equations (1) and (2). What are the approximation errors? How does this error change if $\mu = 4$? How does it change if $\sigma^2 = 9$?

Finally, for a RV X and an associated transformation $Y = u(X)$, it is possible to put bounds on probabilities associated with outcomes of $u(X)$ based on the moments of $u(\cdot)$ as in the next theorem.

Theorem 10 *If X is a (discrete or continuous) RV and $u(x)$ is a non-negative real-valued function, then for any positive constant $c > 0$, we have*

$$\Pr[u(X) \geq c] \leq \frac{\mathrm{E}[u(X)]}{c}.$$

PROOF FOR CONTINUOUS RVs: Let $A = \{x | u(x) \geq c\}$. Then

$$\begin{aligned}
\mathrm{E}[u(X)] &= \int_{-\infty}^{\infty} u(x) f_X(x) dx \\
&= \int_A u(x) f_X(x) dx + \int_{A^c} u(x) f_X(x) dx \\
&\geq \int_A u(x) f_X(x) dx \\
&\geq \int_A c f_X(x) dx \\
&= c\Pr[\mathrm{X} \in A] \\
&= c\Pr[u(\mathrm{X}) \geq c]. \quad \blacksquare
\end{aligned}$$

A special case of Theorem 10 is known as **Markov's inequality**:

$$\Pr[|X| \geq c] \leq \frac{\mathrm{E}[|X|^r]}{c^r}, \; r > 0.$$

Another important special case of Theorem 10 is called **Chebyshev's inequality:**

Theorem 11 *If X is a RV with mean μ and variance σ^2, then for any $k > 0$,*

$$\Pr[|X - \mu| \geq k\sigma] \leq \frac{1}{k^2}.$$

Exercise 6 Prove Chebyshev's inequality. (*HINT*: use Theorem 10 and let $u(X) = (X - \mu)^2$.)

Although Chebyshev's inequality may not seem striking at first glance, it is a significant and powerful result. First, if $k = 2$ then it immediately follows from Chebyshev's inequality that a realization of any RV, whether continuous, discrete or mixed, will be within two standard deviations of it's mean with at least 75% probability. The fact that such a general statement can be made about any RV having a finite mean is remarkable. Second, the **Law of Large Numbers** (LLN), which states that the sample mean of an iid random process **converges in probability** to the true mean, comes from a straightforward application of Chebyshev's inequality.

Theorem 12 *If X_i is a RV representing the result of an independent trial of a random experiment with mean μ and variance σ^2, $i = 1, \ldots, n$, and if one defines the* ***sample mean*** *as $\overline{X}_n \equiv \frac{\sum_{i=1}^n X_i}{n}$, the arithmetic mean of the n observations, then for all $\varepsilon > 0$ we have*

$$\lim_{n\to\infty} \Pr\left[|\overline{X}_n - \mu| \geq \varepsilon\right] = 0.^1$$

PROOF: It is easy to show that $\mathrm{E}\left[\overline{X}_n\right] = \mu$ and $\mathrm{Var}\left[\overline{X}_n\right] = \frac{\sigma^2}{n}$. Then by Chebyshev's inequality, we have

$$\begin{aligned}\lim_{n\to\infty} \Pr\left[|\overline{X}_n - \mu| \geq \varepsilon\right] &\leq \lim_{n\to\infty} \frac{\mathrm{Var}\left[\overline{X}_n\right]}{\varepsilon^2} \\ &= \lim_{n\to\infty} \frac{\sigma^2}{n\varepsilon^2} = 0. \blacksquare\end{aligned}$$

4.5 Probability Integral Transformation

No discussion on transformations of RVs would be complete without giving special attention to one transformation in particular: the cumulative distribution function, also known as the **probability integral transformation**. For the remainder of this section, consider a RV X with cdf $F_X(x)$ and define another RV by $Y = F_X(X)$. If X is continuous, what might the distribution and density of Y be?

Theorem 13 *If X is a continuous RV with cdf $F_X(x)$ and if $Y = F_X(X)$, then Y is a uniform RV on the interval* $[0, 1]$.

PROOF: Using the cdf technique, we can derive the distribution of Y as follows:

$$\begin{aligned}F_Y(y) &= \Pr[Y \leq y] \\ &= \Pr[X \leq F_X^{-1}(y)] \\ &= F_X\left[F_X^{-1}(y)\right] = y. \blacksquare\end{aligned}$$

Once again, it is remarkable that such a broad statement can be made about any continuous RV, regardless of its range or distribution. Even more remarkable is how useful this theorem is in practice.

[1] This theorem is sometimes known as the **Weak Law of Large Numbers** and its counterpart, the **Strong Law of Large Numbers**, states that the sample average **converges almost surely** to the true mean, or

$$\Pr\left[\lim_{n\to\infty} \overline{X}_n = \mu\right] = 1.$$

They are so named because the latter implies the former, but not vice versa.

It implies that if one can generate uniformly distributed random numbers on a computer, then one can easily generate random numbers from ANY continuous distribution by simply inverting the uniform observations via the inverse cdf function. This eliminates the need to find a different strategy for generating random numbers from each of the infinitely many possible continuous distributions one might encounter.

Example 1 Suppose that a researcher wishes to obtain a sample $\{x_i\}_{i=1}^n$ of exponentially distributed random numbers, but he only has access to a sample $\{y_i\}_{i=1}^n$ of uniformly-distributed random numbers between 0 and 1. Recall that the exponential distribution is

$$F_X(x) = 1 - e^{-\lambda x}.$$

A little algebra reveals that

$$F_X^{-1}(y) = -\frac{\ln(1-y)}{\lambda},$$

from which it follows that

$$\{x_i\}_{i=1}^n = \left\{-\frac{\ln(1-y_i)}{\lambda}\right\}_{i=1}^n$$

is equivalent to a sample of random numbers drawn from the exponential distribution.

Although Theorem 13 is stated in terms of continuous RVs, this technique can also be used to generate discrete random numbers as well.

Example 2 A researcher wishes to obtain a sample $\{x_i\}_{i=1}^n$ of discrete random numbers drawn from the following distribution:

$$\Pr[X = x_1] = p_1, \ \ldots, \ \Pr[X = x_k] = p_k, \ \sum_{j=1}^{k} p_j = 1,$$

but he only has access to a sample $\{y_i\}_{i=1}^n$ of uniformly-distributed random numbers between 0 and 1. By performing the following transformation, he obtains the equivalent of the random sample of interest:

$$x_i = \sum_{j=1}^{k} \mathbb{I}_j x_j,$$

where

$$\mathbb{I}_j = \begin{cases} 1 & \text{if } \sum_{l=1}^{j-1} p_l < y_j \leq \sum_{l=1}^{j} p_l, \\ 0 & \text{otherwise.} \end{cases}$$

Now, how does one go about generating uniform random numbers? The short answer is that it's impossible. This should not surprising as truly random uniform numbers from $[0, 1]$ will take on irrational values with probability one and it is impossible to precisely represent irrational numbers.[2] However, one can approximate random numbers, or in other words, one can generate sets of rational numbers which in many ways appear to behave as if they were random uniform draws. These are commonly referred to as **pseudo-random numbers** and one of the most common algorithms for constructing them is called a **congruential pseudo-random number generation rule** or congruential rule for short.

A congruential rule is a deterministic algorithm which generates sets of numbers which appear to be random samples from the uniform distribution on the interval $[0, 1]$. The intuition behind the congruential rule is that when one divides strings of very large numbers and then discards the integer part of the quotient, the result is a set of uniform-like numbers. More specifically, the rule generates a set of numbers

$$x_i = (\alpha + \beta x_{i-1}) \bmod l, \; i = 1, 2, \ldots,$$

where x_0, l, α and $\beta < l$ are known beforehand, and $\bmod$ denotes the modulus operator.[3]

One drawback of the congruential rule is that it has a period of at most l, or in other words, it can generate at most l numbers before arriving back at x_0. Choosing l prime helps to maximize the period of the algorithm.[4] A larger prime l also lengthens the period and generates a better approximation to true randomness. Typically, l is set to $(2^{31} - 1)$, the largest prime number that can be represented on a 32-bit computer. Also, $\alpha = 0$ usually, and $\beta = 397, 204, 094$ in order to deliver a good mix of computational speed and randomness.

I generated a sample of 1,000,000 Weibull pseudo-random numbers using MATLAB, and Figure 1 displays a comparison of the histogram and the actual density of the Weibull distribution. The two are obviously very close, indicating that the pseudo-random number generator in MATLAB performs its function well.

Example 3 APPLICATIONS OF THEOREM 13 IN ECONOMICS

1. Theory: "Rank-Based Methods for the Analysis of Auctions," by Ed Hopkins (2007, typescript)
2. Theory: model simulation
3. Econometrics: Monte-Carlo experiments
4. Bayesian Econometrics: simulating draws from prior distributions

[2]Recall from Chapter 2 that countable sets have Lebesgue measure zero, and since the uniform distribution is absolutely continuous with respect to Lebesgue measure, it assigns zero probability to the rationals.

[3]The modulus operator divides the left number by the right number and discards the integer part of the quotient.

[4]A necessary condition form the congruential rule to have full period is that l and β must be relatively prime, meaning that they share no common factors other than 1. Primality of l guarantees that this condition is met.

Figure 1: Weibull Peudo-Random Variables

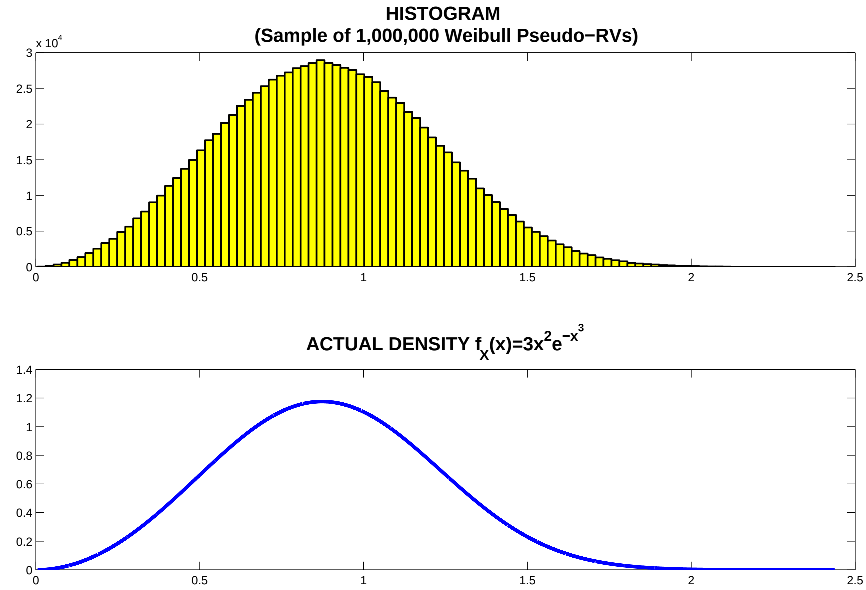
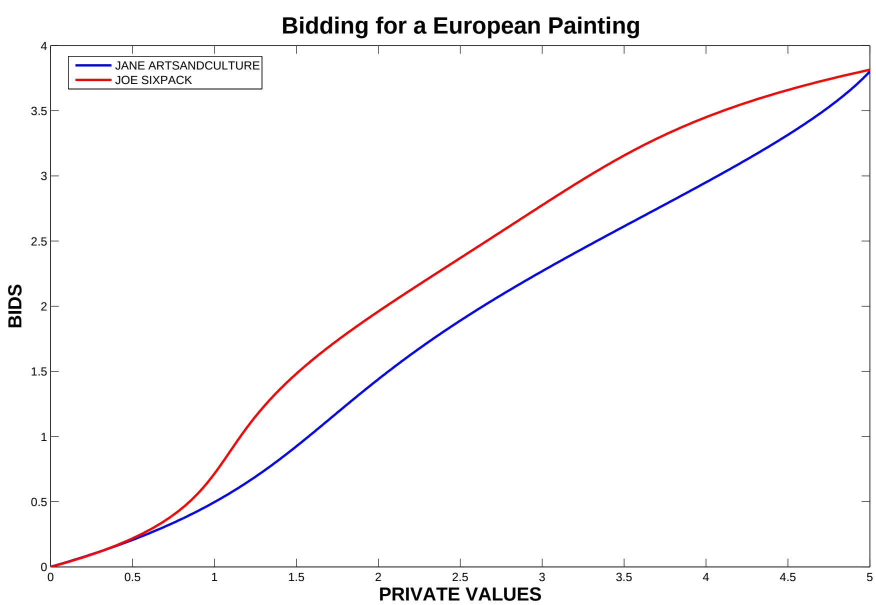


Figure 2:



Exercise 7 SIMULATION EXERCISE Consider a first-price, winner-pay auction for a European painting. Joe Sixpack, a toilet bowl dealer, wishes to use the painting as a decoration in his showroom, whereas his competitor, Jane Artsandculture, wishes to add the painting to her extensive collection of fine art. Both bidders are risk-neutral, but because of the differences in tastes among toilet bowl dealers and art connoisseurs, private values in this auction are asymmetric.

Jane believes that Joe's private value comes from the interval $[0, 5]$ (measured in \$USD thousands) according to a truncated Weibull distribution:

$$F_S(v) = \left[1 - e^{-\left(\frac{v}{\eta}\right)^\gamma}\right] \frac{1}{1 - e^{-\left(\frac{5}{\eta}\right)^\gamma}}, \qquad \eta, \gamma > 0.$$

Joe, on the other hand, expects that Jane is likely to value the painting more highly than he, as she gets utility not only from owning the painting, but also from rescuing it from the fate of hanging in a toilet bowl showroom. Therefore, Joe's belief is that Jane's private value follows the distribution

$$F_A(v) = [F_S(v)]^\theta,\ \theta > 1$$

on the same interval. Both competitors believe that $\eta = 4$, $\gamma = 3$, and $\theta = 6$.

The equilibrium in this game is a set of functions, β_s and β_a, such that Mr. Sixpack maximizes his expected payoff by choosing a bid of $b_s = \beta_s(v)$, given that Ms. Artsandculture chooses a bid of $b_a = \beta_a(v)$, and vice versa. As it turns out, there is no closed-form solution for the equilibrium, but I have solved it on a computer and plotted in Figure 2. Notice that Joe's bid function lies strictly above Jane's. This is a well known consequence of the stochastic dominance relationship between their private value distributions.

The difference in the bidding functions also gives rise to inefficiency. An allocation is said to be efficient only if the bidder with the higher private value wins the object. However, since the bid functions are continuous and since Joe bids more than Jane at a given private value, it is possible for Jane to have a slightly higher private value and still be outbid by Joe. For this exercise, you will simulate 100,000 auction matchups between Joe and Jane in order to find out how prevalent the inefficiency is. Using MATLAB and the accompanying data files, `grid.dat` (a grid of private values on $[0, 5]$), `beta_S.dat` (Joe's bids at each point on the grid) and `beta_A.dat` (Jane's bids at each point on the grid), you will do the following:

a. Write down expressions for the inverse private value distributions F_S^{-1} and F_A^{-1}.

b. Generate a $100,000 \times 2$ matrix U of uniform random numbers on $[0, 1]$.

c. Invert the first column using your expression for F_S^{-1} to get a sample of private values for Joe, and do likewise using the second column and your expression for F_A^{-1} to get a sample of private values for Jane. Call the resulting matrix V.

d. Generate a sample of bids for Joe by computing the equilibrium bids which correspond to each of Joe's random private values stored in V. You can use MATLAB's built-in interpolation utility `interp1` to do this (see footnote).[5] Do the same for Jane and call the resulting matrix B.

e. Note that each row of V and each corresponding row of B represents the outcome of one simulated auction. Compute the average bid submitted by both players.

f. Compute the percentage of the auctions in which Jane has a higher private value but Joe wins with a higher bid. This is an estimate of the probability of inefficient allocations in this auction.

References

[1] BAIN, LEE J. and MAX ENGELHARDT (1992): *Introduction to Probability Theory and Mathematical Statistics*, Duxbury.

[2] HOPKINS, ED (2007): *Rank-Based Methods for the Analysis of Auctions*, Typescript, University of Edinburgh Department of Economics.

[3] PAARSCH, HARRY J. and HAN HONG (2006): *An Introduction to the Structural Econometrics of Auction Data*, MIT Press.

[4] SHANMUGHAVEL, S. "Random Number Generation." From Networks Research Group web site, http://www.cse.msu.edu/ nandakum/nrg/others.html

[5] WEISSTEIN, ERIC W. "Strong Law of Large Numbers." From MathWorld–A Wolfram Web Resource. http://mathworld.wolfram.com/StrongLawofLargeNumbers.html

[6] WEISSTEIN, ERIC W. "Weak Law of Large Numbers." From MathWorld–A Wolfram Web Resource. http://mathworld.wolfram.com/WeakLawofLargeNumbers.html

[5] For example, suppose you have a vector of domain points `dom` and a vector of functional values `f` at the domain points. Then if there is another vector, say `points`, whose entries are contained on the interval [min(`dom`), max(`dom`)], the functional values corresponding to `points`, call them `fpoints`, can be approximated via linear interpolation by the following command: `fpoints = interp1(dom,f,points,'linear')`.