Introduction to Probability Theory for Graduate Economics: Solutions to Chapter 1 Exercises

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1 Solutions

Exercise 1 Identify the sample space, S, in the following experiments. Is S discrete or continuous?

a. Three 6-sided die are cast and the outcomes are recorded.

Answer: $S = \{(O_1, O_2, O_3) | O_i \in \{1, 2, 3, 4, 5, 6\} \ \ \forall i\}$ (discrete)

b. A researcher records how many patients enter an emergency room between November and March with respiratory problems.

Answer: $S = \mathbb{N} \cup \{0\}$ (discrete)

c. Another researcher records the amount of time it takes for an ER patient to complete triage and see a doctor.

Answer: $S = (0, \infty)$ (continuous)

d. A new coin is drawn at random from a minting machine. It is flipped once and its mass in grams is measured. The coin flip outcome and the mass are both recorded.

Answer: $S = \{H, T\} \times (0, \omega)$, where ω is some upper bound (*e.g.*, the mass of the minting machine). This sample space has a discrete component and a continuous component. ■

Exercise 2 For each of the four cases listed in Example 1, identify some elementary and compound events, and construct a collection of three mutually exclusive events. ■

Exercise 3 (*Exercise 32 from Capter 1 of "Introduction to Probability and Mathematical Statistics, 2^{nd} Ed.," Bain and Engelhardt, Duxbury, 1992*) A baseball team has 3 pitchers, A, B, and C, with winning percentages of 0.4, 0.6 and 0.8, respectively. These pitchers pitch with frequency 2, 3, and 5 out of every 10 games, respectively. In other words, for a randomly selected game, $P(A) = 0.2$, $P(B) = 0.3$, and $P(C) = 0.5$. Compute the following probabilities:

a. Pr[team wins game] $= P(W)$

Answer: $P(W) = 0.66$

b. Pr[A pitched game|team won] $= P(A|W)$

Answer: $P(A|W) = 0.1212$ ■

Exercise 4 (*Belief updating in strategic games*): A drunken sailor walks into a bar, itching to pick a fight. He sees a man sitting at the bar and he considers the choice of whether to pick a fight him. The man could either be a whimp or a tough-guy, but whimps always dress like tough guys, so the sailor can't tell them apart. However, he thinks that whimps and tough-guys hang out in bars with equal probability. When whimps are antagonized, they fight with probability w and they win with probability .4. When tough-guys are antagonized, they fight with probability t and they win with probability .7. The sailor's ship will be in port for two days, so he will have the opportunity to return the next day and fight the man again, regardless of what happens on the first day. For the next three questions, assume that the sailor decides to pick a fight on the first day.

a. What is the probability that the sailor's opponent is a tough-guy, given that he fought and won on the first day?

Answer: Let WP, TG, F and L denote, respectively, the events that a whimp is observed, a tough-guy is observed, a fight breaks out and the sailor loses a fight. We wish to compute $P(TG|F \cap L)$ and we have the following information available:

$$P(WP) = 0.5, \;\; P(F|WP) = w, \;\; P(L|WP \cap F) = 0.4$$
$$P(TG) = 0.5, \;\; P(F|TG) = t, \;\; P(L|TG \cap F) = 0.7.$$

By Bayes' rule, we have

$$P(TG|F \cap L) = \frac{P(F \cap L|TG)P(TG)}{P(F \cap L)}.$$

Since

$$P(F \cap L|TG) = P(F|TG)P(L|TG \cap F) = 0.7t$$

and

$$P(F \cap L) = P(F \cap L|TG)P(TG) + P(F \cap L|WP)P(WP) = 0.7t(0.5) + 0.4w(0.5),$$

we can re-write

$$P(TG|F \cap L) = \frac{0.7t(0.5)}{0.7t(0.5) + 0.4w(0.5)} = \frac{0.7t}{0.7t + 0.4w}.$$

b. What is the sailor's probability of losing a fight on the second day, conditional on having fought against an opponent who won on the first day?

Answer: Similarly as in part (a.), We can say that

$$P(WP|F \cap L) = \frac{0.4w}{0.7t + 0.4w}.$$

This gives the sailor a set of updated beliefs on who the man in the bar is for the second day. For notational ease, we will simply redefine all events in terms of the second day, including

$$P(WP) = \frac{0.4w}{0.7t + 0.4w}$$

and

$$P(TG) = \frac{0.7t}{0.7t + 0.4w}.$$

Now, the probability of the sailor's opponent winning on the second day is just the total probability of his opponent fighting and winning, with the updated beliefs of his type:

$$\begin{aligned} P(L) =& P(F|WP)P(L|WP \cap F)P(WP) + P(F|TG)P(L|TG \cap F)P(TG) \\ =& w(.4)\left(\frac{0.4w}{0.7t + 0.4w}\right) + t(.7)\left(\frac{0.7t}{0.7t + 0.4w}\right). \end{aligned}$$

c. Assume that the sailor antagonized the man at the bar on the first day, and that the man fought and won. On the second day, the sailor's payoff is 0 if there is no fight, 10 if he fights and wins, and -10 if he fights and looses. What choice gives a higher expected payoff on the second day, challenging the man at the bar to a re-match or backing down?

Answer: Left to the student. ■

Exercise 5 (*Exercise 1.2.1 from "Contemporary Bayesian Econometrics and Statistics," by John Geweke, Wiley, 2005*) Let D denote the event that a particular disease is present in an individual. A test for the disease is administered by a nurse and the result comes up as either $+$ or $-$, meaning, respectively, that the patient either has the disease or not. Like all tests, this one is not 100% accurate. The *sensitivity* of the test, denoted by q, is the probability of a "positive" result conditional on the disease being present. For this test, $q = P(+|D) = 0.98$. The *specificity* of the test, denoted by p, is the probability of a "negative" result conditional on the disease being absent; it is $p = P(-|D') = 0.9$. The *incidence* of the disease is the probability that the disease is present in a randomly selected individual; it is $\pi = 0.005$. Compute the probabilities of the following events:

a. The disease is present given a "negative" outcome

Answer:

$$\begin{aligned} P(D|-) &= \frac{P(-|D)P(D)}{P(-|D')P(D') + P(-|D)P(D)} \\ &= \frac{(1-q)\pi}{p(1-\pi) + (1-q)\pi} \\ &= \frac{0.02(0.005)}{0.9(.995) + 0.02(0.005)} \approx 0.000112. \end{aligned}$$

b. The disease is present given a "positive" outcome

Answer:

$$\begin{aligned} P(D|+) &= \frac{P(+|D)P(D)}{P(+|D')P(D') + P(+|D)P(D)} \\ &= \frac{q\pi}{(1-p)(1-\pi) + q\pi} \\ &= \frac{0.98(0.005)}{0.1(.995) + 0.98(0.005)} \approx 0.0469. \ \blacksquare \end{aligned}$$

Note that the test is much better at telling you when you are not sick than at telling you when you are sick.

Exercise 6 (*Exercise 6 from the August 1, 2005 qualifying exam*) Suppose that a decision maker maximizes subjective expected utility and revises beliefs according to Bayes' rule. Assume that her utility function is $u(x) = x, \quad x \in \mathbb{R}$; *i.e.*, outcomes are monetary prizes. She has the opportunity to gamble on the toss of a coin: She can not participate, or she can say either "H" or "T" and she will win \$30 if she is correct and she loses \$50 otherwise. Without additional information, she thinks that it is equally likely that the coin is two-headed, fair or two-tailed. How much will she be willing to pay for the chance to observe one toss before playing? ■

Theorem 1 *Two events A and B are independent if and only if the following pairs of events are also independent:*

1. *A and B′*
2. *A′ and B*
3. *A′ and B′*

Exercise 7 Prove the above theorem. ■

Exercise 8 Consider a test with 5 true-false questions and 3 multiple choice questions with four choices each. How many total ways are there in which the test could be answered?

Answer: $(2^5)(4^3) = 2048$.

■

Exercise 9 There are 5 indivisible units of a good to be allocated among 9 consumers. Assume that no one gets more than one unit.

a. How many ways are there for consumer 1 to receive one?

Answer: If one unit is given to consumer 1, then that leaves 4 units left and 8 consumers, so there are $\binom{8}{4} = \frac{8!}{4!(8-4)!} = 70$ ways.

b. How many ways are there for both consumers 1 and 2 to receive a unit of the good?

Answer: If one unit each is given to consumers 1 and 2, then that leaves 3 units left and 7 consumers, so there are $\binom{7}{3} = \frac{7!}{3!(7-3)!} = 35$ ways.

c. How many ways are there for the combined allocation to consumers 1 and 2 to be at least one unit?

Answer: We wish to count the occurrences of the event A = consumers 1 and 2 together receive at least one unit. Alternatively, one could count the number of ways that A' could occur and subtract it from the total number of outcomes of all types. With consumers 1 and 2 out of the pool, there are $\binom{7}{5} = \frac{7!}{5!(7-5)!} = 21$ ways to allocate the 5 goods among the other consumers. Since there are $\binom{9}{5} = \frac{9!}{5!(9-5)!} = 126$ ways to allocate the object among everyone in the pool, it turns out that 105 of those possible allocations involve consumers 1 and 2 receiving at least one unit together.

d. Assuming that a unit of the good is allocated to any given consumer with equal probability, what are the probabilities of the events in the previous three questions?

Answer: Since there are $\binom{9}{5} = \frac{9!}{5!(9-5)!} = 126$ ways to allocate the object among everyone in the pool, the probability of the events described in (a.), (b.) and (c.) are, respectively, $\frac{70}{126} = 0.5556$, $\frac{35}{126} = 0.2778$, and $\frac{105}{126} = 0.8333$.

■

Exercise 10 *Order Statistics*: Consider a set of outcomes $\{X_i\}_{i=1}^n$ from n independent trials of the same experiment. In technical terms, we say that the X_is are **identically and independently distributed**, or **iid**. Suppose that they are real numbers and that we order them in increasing fashion in terms of their values. The k^{th} value in the ordered sample, known as the k^{th} **order statistic**, will be denoted by V_k, so that $V_1 \leq V_2 \leq \ldots \leq V_n$. Finally, let $F(v)$ represent the probability that any one observed outcome from the experiment is less than or equal to some benchmark, v. Derive the following probabilities using your knowledge of probability theory and counting techniques:

a. $\Pr[V_n \leq v] = G_n(v)$

Answer: Note that the event $(V_n \leq v)$ is the same as the event $(X_1 \leq v) \cap (X_2 \leq v) \ldots \cap (X_n \leq v)$. Since the X_is are all independent, we know that $P((X_1 \leq v) \cap (X_2 \leq v) \ldots \cap (X_n \leq v)) = \prod_{i=1}^{n} P(X_i \leq v)$, and since they are identically distributed we know that $\prod_{i=1}^{n} P(X_i \leq v) = F(v)^n = G_n(v)$.

b. $\Pr[V_{n-2} \leq v] = G_{n-2}(v)$

Answer: The event $(V_{n-2} \leq v)$ can be viewed as the union of mutually exclusive events $A_0 = (V_n \leq v)$, $A_1 = (V_{n-1} \leq v) \cap (V_n > v)$ and $A_2 = (V_{n-2} \leq v) \cap (V_{n-1} > v)$. We already know the probability of A_0 from the previous exercise. In order for A_1 to occur, there must be $n-1$ of the observations below v and exactly one above. There are n ways for this to occur, and the probability of any one occurrence is $F(v)^{n-1}(1-F(v))$. Thus, $P(A_1) = nF(v)^{n-1}(1-F(v))$. In order for A_2 to occur, there must be $n-2$ observations below v and the rest above, which can happen in $\binom{n}{n-2}$ different ways (note that ordering doesn't matter, as long as $n-2$ are below v), so $P(A_2) = \binom{n}{n-2}F(v)^{n-2}(1-F(v))^2$. Since the A_is are mutually exclusive, it follows then that

$$G_{n-2}(v) = F(v)^n + nF(v)^{n-1}(1-F(v)) + \binom{n}{n-2}F(v)^{n-2}(1-F(v))^2.$$

c. $\Pr[V_{n-k} \leq v] = G_{n-k}(v)$, for arbitrary k.

Answer: Following the same reasoning in the previous question, it is easy to see that $G_{n-k}(v) = \sum_{i=0}^{k} P(A_i)$, where A_i is the event that exactly i observations exceed v. This can be written as

$$G_{n-k}(v) = \sum_{i=0}^{k} \binom{n}{n-i} F(v)^{n-i}(1-F(v))^i. \blacksquare$$

Exercise 11 *Endogenous State Transition in Dynamic Oligopoly* (Erickson & Pakes, *Review of Economic Studies*, 1995): Consider an industry in which there are N firms competing in each of infinitely many periods indexed by t. For simplicity, assume that there is no entry or exit. In each period, firms base their production decisions on an individual state and an aggregate state. Individual firm states are denoted by $\omega \in \{1, 2, \ldots, K\}$ and the aggregate state of the industry is $\mathbf{s} = (s_1, s_2, \ldots, s_K)$, where s_j is the number of firms with individual state $\omega = j$ in the current period. In each period, firms chose an input $x(\omega, s)$ which affects their transition to tomorrow's state through the stochastic function $\pi(i|j, x(\omega_t, \mathbf{s}^t)) = \Pr[\omega_{t+1} = i | \omega_t = j, x(\omega_t, \mathbf{s}^t)]$. Let $\mathbf{y}_j = (y_{1j}, y_{2j}, \ldots, y_{Kj})$ denote a profile of tomorrow's firms who started out in state j today, where y_{ij} is the number of firms who find themselves in state $\omega = i$ tomorrow. Finally, let $\mathbf{Y} = (\mathbf{y}_0, \ldots, \mathbf{y}_K)$ denote combined profile of all transitions from today's states to tomorrow's. Derive the following probabilities using your knowledge of probability theory and counting techniques:

a. $m(\mathbf{y}_j|\mathbf{s}) = \Pr[\mathbf{y}_j|\mathbf{s}]$ (*i.e.*, the probability that s_j firms will transition to profile $\mathbf{y}_j$ tomorrow).

Answer: The task of assigning the state-j firms to tomorrow's states is one of partitioning. Recall that there are s_j firms in state j today, and for a given profile $\mathbf{y}_j$, there will be y_{ij} firms assigned to state i tomorrow. Thus, there are $\left(\frac{s_j!}{y_{0j}!\cdots y_{Kj}!}\right)$ ways in which $\mathbf{y}_j$ can be realized. By independence, the probability of any given realization is $\prod_{i=0}^{K} \pi\left(i|j,x(j,s)\right)^{y_{ij}}$. Therefore, we have

$$m(\mathbf{y}_j|\mathbf{s}) = \left(\frac{s_j!}{y_{0j}!\cdots y_{Kj}!}\right)\prod_{i=0}^{K} \pi\left(i|j,x(j,s)\right)^{y_{ij}}.$$

b. $\mathcal{Q}(\mathbf{s}^{t+1}|\mathbf{s}^t) = \Pr[\mathbf{s}^{t+1}|\mathbf{s}^t]$ (*i.e.*, the probability that the industry will transition from aggregate state $\mathbf{s}^t$ today to $\mathbf{s}^{t+1}$ tomorrow)

Answer: Note that only certain configurations of $\mathbf{Y} = (\mathbf{y}_0, \ldots, \mathbf{y}_K)$ will be consistent with a particular $\mathbf{s}'$ resulting from $\mathbf{s}$. For example, if $s_1' = 5$, then only $\mathbf{Y}$s where $\sum_{j=1}^{K} y_{1j} = 5$ are possible. Note also, that each consistent $\mathbf{Y}$ represents a unique (*i.e.* mutually exclusive) transition from $\mathbf{s}$ to $\mathbf{s}'$. Moreover, for a given $\mathbf{Y} = (\mathbf{y}_0, \ldots, \mathbf{y}_K)$ to occur, the intersection of the independent events $\mathbf{y}_j$ must occur. This leads us to the following

$$\mathcal{Q}(\mathbf{s}'|\mathbf{s}) = \sum_{\{(\mathbf{y}_0,\ldots,\mathbf{y}_K)|\sum_{k=0}^{K}\mathbf{y}_{jk}=s_j'\}} \prod_{i=0}^{K} m(\mathbf{y}_i|\mathbf{s}),$$

where the summation follows from mutual exclusivity of the different configurations of $\mathbf{Y}$ and the product follows from the independence of the $\mathbf{y}_i$s. ■