Appendix A. ONLINE SUPPLEMENTAL MATERIALS FOR *How Efficient are Decentralized Auction Platforms?* by Aaron L. Bodoh-Creed, Jörn Boehnke, and Brent R. Hickman THEORY: TECHNICAL PROOFS APPENDIX

Due to the length of the technical proofs, we have divided our appendices into sections. Appendix A.1 provides some useful results from the prior literature and proves some miscellaneous results from this paper. Appendix A.2 provides the proof of the existence of an SCE of the limit model (Proposition 2.4). Appendix A.3 describes the technical details of the relationship between the large finite games and the limit model (Proposition 6.3). Each section is proceeded by a brief summary of the arguments contained therein.

A.1. **Preliminary Results.** Before beginning our main arguments, we define the Kolmogorov metric for random variables over $\mathbb{R}$, where F and G are the CDFs of the random variables.

$$d_K(F,G) = \sup_x \|F(x) - G(x)\|$$

Note that when either G or F is continuous (i.e., the underlying measure is atomless), then d_K metricizes the weak-$*$ topology.

Remark 1. *Consider a random variable $X : \Omega \to \mathbb{R}$ with CDF $F(x)$. For N i.i.d. realizations, $\{x_1, ..., x_N\}$, drawn from F, denote the N realization empirical CDF as $F_N(x)$. Then from the Glivenko-Cantelli theorem we have*

$$\sup_{x \in \mathbb{R}} \|F_N(x) - F(x)\| \to 0 \text{ almost surely as } N \to \infty$$

Now we provide a quick proof of our claim that the equilibria of static auctions can be translated directly into equilibria of the dynamic auction market that uses the same payment rule.

Proposition 6.2. Suppose that if $\delta = 0$ there exists an equilibrium $\tilde{\sigma} = \left(\tilde{\theta}, \tilde{\mathcal{O}}\right)$. Then we can define the equilibrium $\sigma = (\theta, \mathcal{O})$ when $\delta > 0$ as

$$\theta(v, C_t^N, F_{V,t}^N, G_{R,t}^N) = \tilde{\theta}(v - \delta E_t^N \left[\mathcal{V}^N(v, C_{t+1}^N, F_{V,t+1}^N, G_{R,t+1}^N | \sigma)\right], C_t^N, F_{V,t}^N, G_{R,t}^N)$$

$$\mathcal{O}(v, C_t^N, F_{V,t}^N, G_{R,t}^N) = \tilde{\mathcal{O}}(v - \delta E_t^N \left[\mathcal{V}^N(v, C_{t+1}^N, F_{V,t+1}^N, G_{R,t+1}^N | \sigma)\right], C_t^N, F_{V,t}^N, G_{R,t}^N$$

Proof. For parsimony, we have supressed all of the arguments for the functions χ^N and ρ^N other than the bid and all of the arguments of $\mathcal{V}^N$. First consider the problem facing a buyer with value $\tilde{v}$ who has entered the market in the static ($\delta = 0$) game and is choosing the optimal bid:

$$\tilde{b}(\tilde{v}) = \underset{b \geq 0}{\operatorname{argmax}}\, \chi^N(b)\tilde{v} - \rho^N(b) - \kappa \tag{48}$$

Now consider the problem of a bidder with value v with a time discount factor $\delta > 0$ facing the same auction mechanism:

$$\mathcal{O}(v) = \underset{b \geq 0}{\operatorname{argmax}}\, \chi^N(b)v - \rho^N(b) + \left(1 - \chi^N(b)\right)\delta E\left[\mathcal{V}^N\right] - \kappa.$$

A few algebraic rearrangements yields:

$$\mathcal{O}(v) = \underset{b \geq 0}{\operatorname{argmax}}\, \chi^N(b)\left[v - \delta E\left[\mathcal{V}^N\right]\right] - \rho^N(b) + \delta E\left[\mathcal{V}^N\right] - \kappa.$$

Since $E\left[\mathcal{V}^N\right]$ and κ are independent of the choice of b, an equivalent problem is:

$$\mathcal{O}(v) = \underset{b \geq 0}{\operatorname{argmax}}\, \chi^N(b)\left[v - \delta E\left[\mathcal{V}^N\right]\right] - \rho^N(b). \tag{49}$$

Clearly any solution to problem 49 for a buyer with value v is also a solution to problem 48 for a buyer with value $\tilde{v} = v - \delta E\left[\mathcal{V}^N\right]$ and vice versa.

In the static game the buyer will choose to enter the market and bid if and only if:

$$\chi^N(b)\tilde{v} - \rho^N(b) - \kappa \geq 0 \tag{50}$$

The analogous equation for the dynamic game is:

$$\chi^N(b)v - \rho^N(b) + \left(1 - \chi^N(b)\right)\delta E\left[\mathcal{V}^N\right] - \kappa \geq \delta E\left[\mathcal{V}^N\right]$$

After some algebra, we find that this is equivalent to:

$$\chi^N(b)\left[v - \delta E\left[\mathcal{V}^N\right]\right] - \rho^N(b) - \kappa \geq 0 \tag{51}$$

Again, a comparison of equations 50 and 51 implies that an agent with value v enters the dynamic market if and only if an agent with value $\tilde{v} = v - \delta E\left[\mathcal{V}^N\right]$ enters the static market in this period.

□

A.2. **Proofs from Section 2.** The highlight of this section is an equilibrium existence result using a fixed point argument. Our argument consists of several parts. First we argue that the state variables, (C, F_V, G_R), live in a compact space that we denote Γ. We must show that the best responses to the state variables and the transition operator for the state variables are continuous and closed with respect to Γ.

As part of our argument we must prove that the bid strategies we consider have, stated informally, both a lower and an upper bound on their derivatives so that the induced distributions of bids do not admit atoms. Stated formally we need to show that the best response function is closed and continuous over the space $\mathbb{C}_M[0,1|\varphi]$, $\varphi \in (0,1)$, that contains all continuous, strictly increasing mappings from $[0,1]$ into $[0,1]$ that satisfy

$$f(v) - f(v') \in \left[\varphi(v - v'), \frac{v - v'}{\varphi}\right], \ \forall v > v'.$$

The set $\mathbb{C}_M[0,1|\varphi]$ is equicontinuous and bounded over a compact domain. From the Arzelá-Ascoli theorem, $\mathbb{C}_M[0,1|\varphi]$ is compact. The strategy space we need to consider can be defined as $\sigma = (e, \beta) \in \Xi = [0,1] \times \mathbb{C}_M[0,1|\varphi]$. Once we show that our best reponse and state transition operators are closed and continuous, a straightforward fixed point argument applies.

Our proof characterizes two functions. First, we define the best response dynamics, $BR(C, F_V, G_R, \sigma) = \tilde{\sigma} = \left(\tilde{e}, \tilde{\beta}\right)$, that describes how each type of buyer responds given the belief that (C, F_V, G_R) is stationary and all other agents use the strategy σ. We show that BR is continuous in (C, F_V, G_R, σ) and closed within Ξ.

The function $T(C, F_V, G_R|\tilde{\sigma}) = (\tilde{C}, \tilde{F}_V, \tilde{G}_r)$ describes the transitions of the aggregate states. We show that T is continuous in $(C, F_V, G_R, \tilde{\sigma})$ for $(C, F_V, G_R) \in \Gamma$ and $\tilde{\sigma} \in \Xi$, and $T(\cdot|\tilde{\sigma}) : \Gamma \to \Gamma$ as long as $\tilde{\beta} \in \mathbb{C}_M[0,1|\varphi]$. Once we have proven T and BR are continuous, closed operators over compact spaces, we define the total operator $\mathcal{L} : \Gamma \times \Xi \to \Gamma \times \Xi$ where

$$\begin{aligned} \mathcal{L}(C, F_V, G_R, \sigma) &= (\tilde{C}, \tilde{F}_V, \tilde{\sigma}) \\ BR(\sigma, C, F_V, G_R, \sigma) &= \tilde{\sigma} \\ T(C, F_V, G_R|\tilde{\sigma}) &= (\tilde{C}, \tilde{F}_V) \end{aligned}$$

A straightforward application of Schauder's fixed point theorem to the operator $\mathcal{L}$ gives us existence of a stationary equilibrium. However, proving T and BR are continuous and closed operators over compact spaces requires many small steps.

Several of our results require that F_V and G_R admit bounded PDFs, f_V and g_r, except at $r = 0$.[45] For notational purposes, we let $\mathcal{Q}[0, \bar{q}]$ denote the space of measures over $[0,1]$ that admit pdfs bounded from above by $\bar{q} > 0$.

A.2.1. *Continuity of Best Responses.* In this section we prove that under certain conditions on the aggregate state, the best responses are continuous as required and lie in $\mathbb{C}_M[0,1|\varphi]$ for some choice of $\varphi \in (0,1)$. Our first result is that the measure of entering agents and

[45]This may fail to be true if low value bidders accumulate in the market by entering even though they have arbitrarily low probabilities of consummating a trade.

the distribution of bids are continuous in the underlying economy. To prove this, we need the following intermediate result. We add the "open neighborhood" qualifier to the statement of Lemma A.1 so that our result applies to empirical measures that are close (in the weak-* sense) to a nonatomic measure with a bounded PDF. We exclusively work with measures with bounded PDFs in Section A.2, but Section A.3 requires us to work with empirical measures near the SCE steady-state distributions.

Lemma A.1. *Consider any increasing function $f \in \mathbb{C}_M[0,1|\varphi], \varphi \in (0,1)$, and let Z be an atomless CDF over $\mathbb{R}$ that admits a pdf that is bounded from above by M. Then $Y(s) = Z(f^{-1}(s))$ is uniformly continuous in $f \in \mathbb{C}_M[0,1|\varphi], \varphi \in (0,1)$ and an open neighborhood of Z.*

Proof. Consider $f, \widetilde{f} \in \mathbb{C}_M[0,1|\varphi]$. For any x and $\gamma, v \in \mathbb{R}_{++}$ we can write

$$f(x+\gamma/v) > \widetilde{f}(x) > f(x-\gamma/v)$$

so for any y we have $\widetilde{f}^{-1}(y) \in [f^{-1}(y)-\gamma/v, f^{-1}(y)+\gamma/v]$, which implies

$$Z\left(\widetilde{f}^{-1}(y)\right) \in \left[Z\left(f^{-1}(y)-\gamma/v\right), Z\left(f^{-1}(y)+\gamma/v\right)\right]$$

Therefore

$$\left\|Z\left(\widetilde{f}^{-1}(y)\right) - Z\left(f^{-1}(y)\right)\right\| < \frac{2M\gamma}{v}$$

Since this bound holds uniformly over x, our result regarding continuity with respect to f is proven. Continuity with respect to Z is immediate. □

Lemma A.1 yields the following result.

Lemma A.2. *The distribution of bids, G_B and the measure of entering buyers $\mathcal{C}$ are continuous in C, F_V, and $\sigma = (e, \beta)$ provided that the entry cutoff point $e < 1$, F_V admits a bounded pdf, and $\beta \in \mathbb{C}_M[0,1|\varphi], \varphi \in (0,1)$.*

Proof. The distribution of entering buyers is

$$F_V^E(x) = \frac{F_V(x) - F_V(e)}{1 - F_V(e)} \text{ for } x \geq e, 0 \text{ otherwise}$$

and the measure of entering buyers is $\mathcal{C} = C\left[1 - F_V(e)\right]$. Note that F_V^E and $\mathcal{C}$ are continuous in (C, F_V). The distributions of bids can be described as

$$G_B(b) = F_V^E(x)(\beta^{-1}(b))$$

Using Lemma A.1, we find that G_B is continuous in β. Finally, if F_V is atomless, then $F_V(e)$ varies continuously in e, so $(\mathcal{C}, G_B)$ is also continuous in e. □

Throughout this paper we maintain an assumption on the bidder arrival process $\pi(k;\boldsymbol{\lambda})$ that $\mathrm{E}[K^2] < \infty$. Recall that a bidder's beliefs about the number of other agents assigned to her auction are

$$\pi_M(k;\boldsymbol{\lambda}) = \pi(k;\boldsymbol{\lambda})\frac{(k+1)}{\mathrm{E}[K]} \ \text{ with } \ \mathrm{E}[K] = \mathcal{C}$$

Since $\mathcal{C}$ is continuous in C, F_V, and σ, $\pi_M(k;\boldsymbol{\lambda})$ is continuous with respect to these variables as well. The probability of winning the good (from a bidder's perspective) is

$$\chi(b) = \sum_{k=0}^{\infty} \pi_M(k;\boldsymbol{\lambda}) * G_R(b) * G_B(b)^k$$

Lemma A.3. *Assume G_B is atomless. Then χ is continuous in $\boldsymbol{\lambda}$, G_R, G_B, and b.*

Proof. $\chi(b)$ is clearly continuous in $\boldsymbol{\lambda}$, G_R, and G_B. $\chi(\cdot)$ is continuous in b if G_B is atomless. □

This result on the continuity of the bids combined with Assumption 2.3 yields the following result.

Lemma A.4. *$BR\left(C, F_V, G_R, \sigma\right) = \left(\widetilde{e}, \widetilde{\beta}\right)$ is continuous in (C, F_V, G_R) and σ if F_V admits a bounded PDF.*

Proof. Given the continuity of G_B and $\boldsymbol{\lambda}$ with respect to (C, F_V, G_R) and σ, Assumption 2.3 immediately yields continuity of $\tilde{\beta}$ with respect to (C, F_V, G_R).

Let $\mathcal{V}(v_i, C, G_B, G_R|\tilde{\sigma})$ denote the value function generated by best responding to (C, G_B, G_R) given a value of v_i. Theorem 1 of Pavan et al. [2014] implies that $\mathcal{V}$ is almost everywhere differentiable and the derivative, where it exists, takes the value

$$\frac{\partial \mathcal{V}(v, C, G_B, G_R|\tilde{\sigma})}{\partial v} = \sum_{\tau=t}^{\infty} \delta^{\tau-t}(1-\chi(\beta(v)))^{\tau-t}\chi(\beta(v)) \tag{52}$$

Since the strategies are best responses, the probability of sale must be positive for all types that enter. This implies the sum that equation 52 is strictly bounded between 0 and 1. From the continuity of χ, we know that $\mathcal{V}$ is continuous in $(C, G_B, G_R, \tilde{\sigma})$

$\tilde{e}$ can be defined implictly using

$$\mathcal{V}(\tilde{e}, C, G_B, G_R|\tilde{\sigma}) = 0 \tag{53}$$

Since $\mathcal{V}$ is continuous in v, has a strictly positive lower bound on its derivative, and is continuous in (C, G_B, G_R) and σ, the solution to equations 53 (i.e., $\tilde{e}$) are also continuous in these variables. □

A.2.2. *Continuity of Aggregate States.* Now we turn to proving that the evolution of the aggregate states is continuous as required. Before doing so, let us describe the operator $T(C, F_V, G_R|\tilde{\sigma})$ in more depth. We iterate the states by using steady-state relations that, in equilibrium, are consistent with the aggregate state. We will need to use the following function, which the reader is encouraged to think of as an unnormalized version of $\tilde{f}_V$

$$\tilde{j}(v) = \frac{\mu t_V(v)}{\chi\left[\tilde{\beta}(v)\right]} \text{ if } \tilde{e} \leq v$$

and $\tilde{j}(v) = 0$ if $\tilde{e} > v$. Having defined $\tilde{j}$, we can define $\tilde{C}$ as

$$\tilde{C} = \int_0^1 \tilde{j}(s)ds$$

The distributions of buyer types and reservation prices are

$$\tilde{f}_V(v) = \frac{\tilde{j}(v)}{\tilde{C}}$$

$$\tilde{G}_R = G_R$$

There is, at this point, no assurance that $\tilde{f}_V$ is bounded (or even well-defined) since χ could have arbitrarily low values for some entrants. We rule this problem out with the following lemma.

Lemma A.5. $\chi(b) \geq \kappa > 0$.

Proof. For a buyer to find it optimal to enter given a positive entry cost, his expected payoff from entering must at least cover the cost of entry. This implies:

$$\chi v - \rho \geq \kappa$$

Therefore

$$\chi = \Pr\{\text{Transaction}\} \geq \frac{\kappa}{v} \geq \kappa$$

where the last inequality follows from the fact that $v \in [0, 1]$. □

Lemma A.5 implies that

$$\tilde{j}(v) \leq \frac{\mu t_V(v)}{\kappa}$$

for entrants. These relations give us

$$\tilde{C} \in [\mu, \mu/\kappa]$$

$$\tilde{f}_V(v) \in \left[0, \frac{t_V(v)}{\kappa}\right]$$

Therefore we can restrict attention to $\tilde{f}_V \in \mathcal{Q}\,[0, \overline{q}]$ where $\overline{q}$ is the maximum of $t_V(\cdot)/\kappa$. In other words, f_V admits a bounded PDF. Furthermore, $\tilde{f}_V$ is continuous in χ as required.

Finally, $\tilde{C}$, $\tilde{F}_V$ and $\tilde{G}_R$ inherit continuity with respect to $\boldsymbol{\lambda}$, G_R, G_B, and σ from the continuity of χ.

We can now prove the existence of a stationary equilibrium of the continuum model.

Proposition 2.4. A stationary competitive equilibrium exists, and a positive mass of buyers choosing to enter the market if κ is not too large.

Proof. Lemmas A.1 through A.5 in conjunction with the continuity provided by Assumption 2.3 prove that the dynamics of our limit model imply $\mathcal{L}$ is a continuous mapping from $\Xi \times \Gamma$ into itself. Given the continuity of the mapping and the compactness of the spaces, Schauder's fixed point theorem implies that there exists a fixed point that defines a stationary equilibrium of our model.

To see that it is optimal for some buyers to enter for κ sufficiently small, first assume there is an equilibrium where none of the buyers enter (*i.e.*, $e = 1$). Suppose a buyer with a value of $v = 1$ deviated to choosing *Enter* and bid $b = 0$. Given there is an atom of reservation prices at $r = 0$, there is a positive probability that the buyer is matched into an auction with a reservation price of 0. Since there are no competing bidders, the buyer wins the good at a price of 0. Since the expected payoff of this deviation is $G_R(0)v$, if κ is less than this value it cannot be an equilibrium for no buyers to choose *Enter*. □

A.3. **Proofs from Section 6.2.** Our approximation result requires two steps. First, we must show that the limit game has a utility structure that is close to the utility structure of a sufficiently large finite game. In particular, we must show that the correlation in sale price and winning probability across auctions vanishes as $N \to \infty$. Second, we must show that these facts imply that with high probability there are no deviations that yield a significant improvement in the utility of any agent. We conduct each task in separate sections. Throughout the sections we focus on an SCE strategy of the limit game (σ, C, F_V).

A.3.1. *Convergence of Utility.* First note that if an agent chooses *Out*, then his utility is 0 regardless of the number of other agents. For the remainder of the section we will assume the agent in question chooses *Enter.* The utility of a bidder in the current period of the N-agent game given the bidder enters and bids b is

$$\chi^N(b)v_i - \rho^N(b) - \kappa$$

If we can show that

$$\begin{aligned} \chi^N(b) &\to \chi(b) \\ \rho^N(b) &\to \rho(b) \end{aligned} \tag{54}$$

uniformly over b when we hold b, C, F_V, and G_R fixed, then we will have shown that the utility function in the N agent game converges to the utility functions of the limit game.

We first show that the probability of buyers being unmatched vanishes as $\mathcal{C}^N$ increases.

Lemma A.6. *We have*

$$\Pr\left(\textit{A particular buyer is unmatched}\right) = O\left(\frac{1}{\mathcal{C}^N}\right)$$

Proof. Let D_l denote the number of bidders matched to auction l. For any buyers to be unmatched, the total "demand" for bidders from sellers must fall short of the supply, $\mathcal{C}^N$, which means that i bidders are not matched if and only if

$$\sum_{l=1}^{\mathcal{S}^N} D_l = \mathcal{C}^N - i$$

Since any bidder is equally likely to be amongst the unmatched buyers, conditional on i bidders being unmatched, the probability that a particular bidder is unmatched is $i/\mathcal{C}^N$. The total probability a particular bidder is unmatched is

$$\Pr\left(\text{A particular buyer is unmatched}\right) = \sum_{i=1}^{\mathcal{C}^N} \frac{i}{\mathcal{C}^N} \Pr\left(\sum_{l=1}^{\mathcal{S}^N} D_l = \mathcal{C}^N - i\right)$$

Using assumption 2.1, we can approximate the probaility mass function of the sum of the D_l using a normal distribution probability density function. This lets us write:

$$\begin{aligned} \sum_{i=1}^{\mathcal{C}^N} \frac{i}{\mathcal{C}^N} \Pr\left[\sum_{l=1}^{\mathcal{S}_N} D_l = \mathcal{C}^N - i\right] &\simeq \frac{1}{\mathcal{C}^N} \sum_{i=1}^{\mathcal{C}^N} \frac{i}{\sqrt{\mathcal{S}^N Var[K]}} \psi\left[\frac{-i}{\sqrt{\mathcal{S}_N Var[K]}}\right] \\ &\leq \frac{1}{\mathcal{C}^N} \int_0^\infty x\psi(x)dx = O\left(\frac{1}{\mathcal{C}^N}\right) \end{aligned}$$

where ψ is the standard normal PDF. □

The asymptotics results required to prove equation 54 holds is complicated by the fact that in the finite game the buyers are sampled without replacement when assigned to auctions. Denote the number of auctions generated in the N agent game as A_N, and $A_N \to \infty$ almost surely as $N \to \infty$. Intuition suggests that as $N \to \infty$ the auctions become asymptotically independent, and we prove this in the following lemma.

Lemma A.7. *Supose that all buyers follow some SCE of the limit game, $\sigma = (e, \beta)$. For any $\varepsilon, \gamma > 0$ we can choose N^* such that for any $N > N^*$ and any (C^N, F_V^N, G_R) we have*

$$P\left(\frac{1}{\mathcal{C}^N}\left\|\sum_{i=1}^{\mathcal{C}^N}\left[x_i^N(b) - \chi^N(b)\right]\right\| > \varepsilon\right) \;<\; 1-\gamma \tag{55}$$

$$P\left(\frac{1}{\mathcal{C}^N}\left\|\sum_{i=1}^{\mathcal{C}^N}\left[p_i^N(b) - \rho^N(b)\right]\right\| > \varepsilon\right) \;<\; 1-\gamma$$

$$\sup_{b\in[0,1]}\left\|\chi^N(b) - \chi(b)\right\| = O\left(\frac{1}{\sqrt{\mathcal{C}^N}}\right) \tag{56}$$

$$\sup_{b\in[0,1]}\left\|\rho^N(b) - \rho(b)\right\| = O\left(\frac{1}{\sqrt{\mathcal{C}^N}}\right)$$

The choice of ε and ρ can be chosen uniformly over (C^N, F_V^N, G_R).

Proof. We provide a proof for Equations 55 and 56, but essentially identical arguments suffice for the other results. For notational cleanliness, we will provide a proof of the probability of large positive deviations, but the analogous result for large negative deviations is essentially identical.

From Chebyshev's inequality we have

$$P\Big(\frac{1}{\mathcal{C}^N}\sum_{i=1}^{\mathcal{C}^N}\left(x_i^N(b) - \chi^N(b)\right) \;>\; \varepsilon\Big) \le \frac{1}{\varepsilon^2} Var\left(\frac{1}{\mathcal{C}^N}\sum_{i=1}^{\mathcal{C}^N}\left(x_i^N(b) - \chi^N(b)\right)\right) \tag{57}$$

$$= \frac{1}{\varepsilon^2}\frac{1}{(\mathcal{C}^N)^2}\Big[\sum_{i=1}^{\mathcal{C}^N} Var\left(x_i^N(b) - \chi^N(b)\right) + \tag{58}$$

$$2\sum_{i=2}^{\mathcal{C}_N}\sum_{j=1}^{\mathcal{C}_N} cov\left(x_i^N(b) - \chi^N(b), x_j^N(b) - \chi^N(b)\right)\Big]$$

Since $x_i^N(b)$ is bounded, we know

$$\frac{1}{(\mathcal{C}^N)^2}\sum_{i=1}^{\mathcal{C}_N} Var\left(x_i^N(b) - \chi^N(b)\right) = O(\mathcal{C}_N^{-1}) \tag{59}$$

We now bound the covariance term. If the agents assigned to the auctions were generated through a process of sampling with replacement (SWR), then

$$cov\left(x_i^N(b) - \chi^N(b), x_j^N(b) - \chi^N(b)\right) = 0$$

The only difference between sampling with and without replacement is an event wherein one of the following occurs:

(1) The bidder is unmatched
(2) The two auctions share a common bidder

Lemma A.6 provides a uniform upper bound on the probability of event (1). Let $\mathcal{E}_S$ denote the event the auctions share a bidder in a SWR regime, which allows us to write

$$cov\left(x_i^N(b) - \chi^N(b), x_j^N(b) - \chi^N(b)\right) < \Pr\{\mathcal{E}_S\} + \frac{1}{\mathcal{C}^N}$$

where the second term captures the probability of a buyer being unmatched.

Fix two auctions with m and n bidders. The probability the auctions do not share a bidder is[46]

$$\left(1 - \frac{1}{\mathcal{C}^N}\right)\left(1 - \frac{2}{\mathcal{C}^N}\right)...\left(1 - \frac{m+n-1}{\mathcal{C}^N}\right) > \left(1 - \frac{m+n}{\mathcal{C}^N}\right)^{m+n}$$

The probability the auctions share an agent is no more than

$$\sum_{\{m,n:m+n\leq\mathcal{C}_N\}} \pi(m;\boldsymbol{\lambda})\pi(n;\boldsymbol{\lambda})\left[1 - \left(1 - \frac{m+n}{\mathcal{C}^N}\right)^{m+n}\right]$$

A binomial expansion yields:

$$\left(1 - \frac{m+n}{\mathcal{C}^N}\right)^{m+n} = 1 - \frac{(m+n)^2}{\mathcal{C}^N} + o\left(\mathcal{C}_N^{-1}\right)$$

Therefore

$$\begin{aligned}
&\sum_{\{m,n:m+n\leq\mathcal{C}^N\}} \pi(m;\boldsymbol{\lambda})\pi(n;\boldsymbol{\lambda})\left[1 - \left(1 - \frac{m+n}{\mathcal{C}^N}\right)^{m+n}\right] \\
&\qquad = \sum_{\{m,n:m+n\leq\mathcal{C}^N\}} \pi(m;\boldsymbol{\lambda})\pi(n;\boldsymbol{\lambda})\frac{(m+n)^2}{\mathcal{C}^N} + o\left(\frac{1}{\mathcal{C}_N}\right) \\
&\qquad = \sum_{\{m,n:m+n\leq\mathcal{C}^N\}} \pi(m;\boldsymbol{\lambda})\pi(n;\boldsymbol{\lambda})\frac{m^2+2mn+n^2}{\mathcal{C}^N} + o\left(\frac{1}{\mathcal{C}^N}\right) \\
&\qquad \leq \frac{2E[K^2] + 2E[K]^2}{\mathcal{C}^N} + o\left(\frac{1}{\mathcal{C}^N}\right) \\
&\qquad = O\left(\frac{1}{\mathcal{C}^N}\right)
\end{aligned}$$

[46] If we think of drawing the $m+n$ buyers in order, the second buyer chosen cannot have the same identity as the first (first term), the the third buyer chosen must not have the same identity as either of the first two (second term), etc.

Putting all of these results together, we have that

$$\Pr\{\mathcal{E}_S\} = O\left(\frac{1}{\mathcal{C}^N}\right)$$

Pulling our argument together we have

$$\frac{1}{\varepsilon^2}\frac{1}{(\mathcal{C}^N)^2}\sum_{i=2}^{\mathcal{C}^N}\sum_{j=1}^{i-1} cov\left(x_i^N(b)-\chi^N(b), x_j^N(b)-\chi^N(b)\right) = O\left(\frac{1}{\mathcal{C}^N}\right)$$

Referring back to equation 57, we then have

$$P\Big(\frac{1}{\mathcal{C}^N}\sum_{i=1}^{\mathcal{C}^N}\left(x_i^N(b)-\chi^N(b)\right) > \varepsilon\Big) = O\left(\frac{1}{\mathcal{C}^N}\right)$$

The final step is proving that χ^N and χ are close to one another. Note that the difference between these two functions is generated by the fact that χ^N is generated by constructing the set of auctions with a sampling without replacement process and that buyers can be unmatched, whereas the process generating χ insures that all buyers are matched into auctions with opponent types generated by a sampling with replacement process. Our argument regarding the correction required to account for these differences implies

$$\left\|\chi^N(b)-\chi(b)\right\| = O\left(\frac{1}{\mathcal{C}^N}\right)$$

The uniformity with respect to b follows from standard arguments based on the separability of the reals and both functions being monotone in b. The uniformity with respect to (C^N, F_V^N, G_R) follows from the fact that the proof is completely independent of these variables. □

The following lemma implies that the within-period utility of the game with a finite number of agents converges to the within-period utility function of the continuum limit game.

Lemma A.8. *Supose that all agents follow some SCE of the limit game, (e, β). For any $\varepsilon > 0$ we can choose N^* such that for any $N > N^*$ and any (C^N, F_V^N, G_R^N) we have*

$$\sup_{b,v\in[0,1]}\left\|\chi^N(b)v_i-\rho^N(b)-(\chi(b)v_i-\rho(b))\right\| < \varepsilon \tag{60}$$

Proof. The result follows from Lemma A.7. □

A.3.2. *Mean Field Lemma.* The goal of this subsection is to prove that the evolution of the finite game approaches the deterministic evolution of the limit game as $N \to \infty$. We start by showing the initial distribution of agent types in the finite and limit game converges in this limit.

Lemma A.9. *The empirical distribution of types and reserve prices in period* 0 *of the* N*-agent game converges weakly to* F_V *and* G_R *with a convergence rate of* $O(N^{-0.5})$.

Proof. Follows from Remark 1. □

We use time indices for the variables in the next proposition to make the evolution of the aggregate variables clear. Let $Q_N(C_t^N, F_{V,t}^N, F_{C,t}^N|\sigma) = (C_{t+1}^N, F_{V,t+1}^N, F_{C,t+1}^N)$ denote the aggregate state iterator in the N-agent game, where $(C_{t+1}^N, F_{V,t+1}^N, F_{C,t+1}^N)$ is a random variable.

Lemma A.10. *Consider a stationary SCE strategy* $\sigma = (e, \beta)$ *and aggregate state* (C, F_V, G_R)*. For any* $\eta, \gamma > 0$*, we can choose* N^* *such that for all* $N > N^*$ *and* $\left(\widehat{C}_t^N, \widehat{F}_{V,t}^N, \widehat{G}_{R,t}^N\right)$ *such that*

$$\left\|\widehat{C}_t^N - C\right\| + \left\|\widehat{F}_{V,t}^N - F_V\right\| + \left\|\widehat{G}_{R,t}^N - G_R\right\| < \frac{\eta}{2}$$

we have the following with probability at least $1 - \gamma$

$$\left\|\widehat{C}_{t+1}^N - C\right\| + \left\|\widehat{F}_{V,t+1}^N - F_V\right\| + \left\|\widehat{G}_{R,t+1}^N - G_R\right\| < \eta \tag{61}$$

where $Q_N\left(\widehat{C}_t^N, \widehat{F}_{V,t}^N, \widehat{G}_{R,t}^N|\sigma_{-i}, \sigma_i'\right) = \left(\widehat{C}_{t+1}^N, \widehat{F}_{V,t+1}^N, \widehat{G}_{R,t+1}^N\right)$.

Proof. As an initial note, when any single agent deviates from σ in the limit game, no change in the aggregate variables occurs. When a single agent deviates in the finite game, it causes a change of at most $(\mathcal{C}^N)^{-1}$. Since $(\mathcal{C}^N)^{-1} \to \infty$ if $e < 1$ in any equilibrium of the finite game, these deviations do not affect the convergence arguments presented below.

First, we have that $G_{R,t+1}^N \to G_R$ by Remark 1. What remains is to show that $\frac{C_{t+1}^N}{N} \to C_{t+1}$ and $F_{V,t+1}^N \to F_{V,t+1}$. Focusing on the new potential entrants, there are $\lceil N\mu \rceil$ agents added to the game with types $(\tilde{v}_1, ..., \tilde{v}_{\lceil N\mu \rceil})$ drawn from T_V. $\frac{\lceil N\mu \rceil}{N} \to \mu$ as $N \to \infty$ and Remark 1 imply

$$\frac{1}{\lceil N\mu \rceil} \sum_{i=1}^{\lceil N\mu \rceil} \mathbf{1}\left\{v \geq \widetilde{v}_i \geq e\right\} \to T_V(v) \text{ uniformly over } v \text{ almost surely}$$

This means that the only thing we need to show is that the number and type distribution of agents continuing onto the next period in the finite game converges to the analogous measure and distribution in the limit game as $N \to \infty$. Lemma A.7 implies the following, where the convergence is in probability and uniform over v

$$\frac{1}{C_{t+1}^N}\sum_{i=1}^{C_t^N} \mathbf{1}\left\{v_i \geq e\right\}\left(1 - x_i(\beta(v))\right) \quad \to \quad \int_0^1 \mathbf{1}\left\{v_i \geq e\right\}\left(1 - \chi(\beta(v))dF_{V,t}(s)\right.$$

$$\frac{1}{C_{t+1}^N}\sum_{i=1}^{C_t^N} \mathbf{1}\left\{v \geq v_i \geq e\right\}\left(1 - x_i(\beta(v))\right) \quad \to \quad \int_0^1 \mathbf{1}\left\{v \geq v_i \geq e\right\}\left(1 - \chi(\beta(v))dF_{V,t}(s)\right.$$

Bringing these results together, we have $\left(C_{t+1}^N, F_{V,t+1}^N, G_{R,t+1}^N\right) \to \left(C_{t+1}, F_{V,t+1}, G_R\right)$ in probability as $N \to \infty$, which is equivalent to our desired result. □

Iterating Lemma A.10 immediately gives us the following:

Corollary A.11. *Consider a stationary SCE strategy σ and aggregate state (C, F_V, G_R). For any $\Delta, \gamma > 0$, we can choose $\eta > 0$ and N^* such that for all $N > N^*$ and $\left(C_t^N, F_{V,t}^N, G_{R,t}^N\right)$ such that*

$$\left\|C_t^N - C\right\| + \left\|F_{V,t}^N - F_V\right\| + \left\|G_{R,t}^N - G_R\right\| < \eta$$

we have for all $t' \in \{t, .., t+\tau\}$ with probability at least $1-\gamma$

$$\left\|C_{t'}^N - C\right\| + \left\|F_{V,t'}^N - F_V\right\| + \left\|G_{R,t'}^N - G_R\right\| < \Delta \tag{62}$$

where for $k \in \{0, .., \tau - 1\}$ we define $Q_N\left(C_{t+k}, F_{V,t+k}^N, G_{R,t+k}^N | \sigma_{-i}, \sigma_i'\right) = \left(C_{t+k+1}^N, F_{V,t+k+1}^N, G_{R,t+k+1}^N\right)$.

A.4. **No Profitable Deviations.** In this subsection, we finally prove our approximation result. We start by proving that the limit model has a continuous per-period utility function.

Lemma A.12. *$\chi(b)v - \rho(b) - \kappa$ is continuous with respect to (C, F_V, G_R) and β when F_V admits a PDF that is bounded from above*

Proof. $\chi(b)$ was proven to be continuous in Lemma A.3. We can write

$$\rho(b) = \pi_M(1;\boldsymbol{\lambda})\int_0^b u\; G_R(du) + \sum_{k=2}^{\infty}\pi(k;\boldsymbol{\lambda})\int_0^b\int_0^b \max\{u,t\}G_R(du)G_B^{k-1}(dt)$$

Lemma A.1 implies that G_B is continuous in (F_V, G_R) and β under the conditions of our lemma. Since the integrands are continuous and G_B and G_R are continuous, then the resulting integral is continuous. □

These results, together with the convergence of the utility functions, yields the following result on the convergence of value functions.

Lemma A.13. *Consider a stationary SCE strategy σ and aggregate state (C, F_V, G_R). For any $\epsilon, \gamma > 0$ we can choose $\eta > 0$ and N^* such that for all $N > N^*$ and $\left(C_0^N, F_{V,0}^N, G_{R,0}^N\right)$ such that*

$$\|C_0 - C\| + \left\|F_{V,0}^N - F_V\right\| + \left\|G_{R,0}^N - G_R\right\| < \eta \tag{63}$$

we have with probability at least $1 - \gamma$

$$\textit{For all } v, \ \left\|\mathcal{V}^N\left(v, C_0^N, F_{V,0}^N, G_{R,0}^N|\sigma\right) - \mathcal{V}(v, C, F_V, G_R|\sigma)\right\| < \varepsilon$$

Proof. First note that if $v < e$, then we are done since the buyer never enters the market and receives the same payoff in either game. For the duration we assume that $v \geq e$.

Let $E_0^N[x_t]$, etc. refer to an agent's expectation in period 0 about an event that occurs in period t of the finite game. We can write the value functions as

$$\mathcal{V}^N\left(v, C_0^N, F_{V,0}^N, G_{R,0}^N|\sigma\right) = \sum_{t=0}^{\infty} \delta^t E_0^N\left[x_t\, v - p_t - \kappa|C_0^N, F_{V,0}^N, G_{R,0}^N, \sigma\right]$$

$$\mathcal{V}(v, C, F_V, G_R|\sigma) = \sum_{t=0}^{\infty} \delta^t\left(\chi v - \rho - \kappa\right)$$

Choose T such that $\delta^T < \frac{\varepsilon}{3}$, and note:

$$\sum_{t=T}^{\infty} \delta^t\left[(x_t v - p_t - \kappa)\right] < (1-\delta)\delta^T v < \frac{\varepsilon}{3}$$

From hereon, we consider only the first T periods.

Lemma A.8 implies that for any sample path of $\left\{\left(C_t^N, F_{V,t}^N, G_{R,t}^N\right)\right\}_{t=0}^{\infty}$ we can choose an N^* sufficiently large so that for all $t \in \{0, .., T\}$

$$\sup_{b,v\in[0,1]}\left\|E_t^N\left[x_t^N(b)v - p_t^N(b)|C_t^N, F_{V,t}^N, G_{R,t}^N\right] - (\chi(b)v - \rho(b))\right\| < \varepsilon \tag{64}$$

where χ and ρ are conditioned on $(C_t^N, F_{V,t}^N, G_{R,t}^N)$. Lemma A.11 implies that for any $\Delta, \gamma > 0$ we can choose η sufficiently small and N sufficiently large such that

$$\left\|C_{t+\tau}^N - C\right\| + \left\|F_{V,t+\tau}^N - F_V\right\| + \left\|G_{R,t+\tau}^N - G_R\right\| \leq \Delta \tag{65}$$

for all $\tau \leq T$ with probability at least $1 - \gamma$.

From Lemma A.12, if Equation 65 holds and Δ is sufficiently small, we have for all $t \in \{0, .., T\}$ we have:

$$\left\|E_0\left[xv - p - \kappa|C_t^N, F_{V,t}^N, G_{R,t}^N, \sigma\right] - (\chi(b)v - \rho(b))\right\| < \frac{\varepsilon}{3T}$$

where χ and ρ are conditioned on the steady state aggregate variables. Note that this result holds uniformly over v and the closed neighborhood defined by Equation 65.

Finally, in the complementary event that the sample path of $\left\{\left(C_t^N, F_{V,t}^N, G_{R,t}^N\right)\right\}_{t=0}^{\infty}$ is not close to (C, F_V, G_R) (i.e., Equation 65 fails to hold for Δ sufficiently small) we have:

$$\left\|E_0^N\left[(xv-p-\kappa)\,|C_t^N, F_{V,t}^N, G_{R,t}^N, \sigma\right] - E_0\left[(xv-p-\kappa)\,|C, F_V, G_R, \sigma\right]\right\| < 1$$

Therefore, for Δ (and hence η) sufficiently small and $\gamma < \frac{\varepsilon}{3(T+1)}$, we have uniformly over v

$$\left\|\mathcal{V}^N(v, C_t^N, F_{V,t}^N, G_{R,t}^N|\sigma) - \mathcal{V}(v, C, F_V, G_R|\sigma)\right\| < T * \frac{\varepsilon}{3T} + \frac{\varepsilon}{3} + (T+1)\gamma < \varepsilon$$

where the first error term refers to errors that occur in the approximation when Equation 65 holds, the second term includes errors accruing in periods after T, and the final term is the expected error from the event when equation CloseEqn fails to hold. □

Now we prove our main approximation results.

Proposition 6.3. Consider a SCE (σ, C, F_V, G_R) where $e(C, F_V, G_R) < 1$. For any $\varepsilon > 0$ we can choose $\eta > 0$ and N^* such that for all $N > N^*$ and $\gamma > 0$, σ is an $\varepsilon - BNE$ strategy if the state in the first period of the N-agent game, $\left(C_0^N, F_{V,0}^N, G_{R,0}^N\right)$, satisfies

$$\left\|C_0^N - C\right\| + \left\|F_{V,0}^N - F_V\right\| + \left\|G_{R,0}^N - G_R\right\| < \eta$$

Proof. From the one-step deviation principle, it suffices to consider a deviation by a single agent in a single period. Without loss of generality, let us assume the deviation occurs in period 0 by a bidder that chooses *Enter.* Lemma A.10 implies that for any $\eta, \gamma > 0$ we can choose η sufficiently small that with probability $1-\gamma$

$$\left\|C_1^N - C\right\| + \left\|F_{V,1}^N - F_V\right\| + \left\|G_{R,1}^N - G_R\right\| < \eta \tag{66}$$

even if a single agent deviates from the SCE strategy in period 1. Lemma A.13 implies that for any $\epsilon > 0$ we can choose $\eta, \gamma > 0$ sufficiently small and N sufficiently large that with probability $1-\gamma$

$$\left\|\mathcal{V}^N\left(v, C_1^N, F_{V,1}^N, G_{R,1}^N\middle|\,\sigma\right) - \mathcal{V}(v, C, F_V, G_R|\sigma)\right\| < \frac{\epsilon}{4} \tag{67}$$

for $\left(C_1^N, F_{V,1}^N, G_{R,1}^N\right)$ that satisfy equation 66. Equation 67 implies that the effect of the current deviation on future periods is small.

Let $\beta_{dev}(v)$ denote the optimal deviation for type v at $\left(C_0^N, F_{V,0}^N, G_{R,0}^N\right)$. From Lemma A.2 and Assumption 2.3 we have that for any $\delta \geq 0$ and all v that we can choose η sufficiently small that

$$\|\beta_{dev}(v) - \beta(v)\| < \delta v \tag{68}$$

From Lemma A.8 we have for N sufficiently large:

$$\sup_{b,v\in[0,1]} \left\| E_0^N \left[x(b)v - p(b) \left| C_0^N, F_{V,0}^N, G_{R,0}^N \right. \right] - E_0 \left[x(b)v - p(b) \left| C_0^N, F_{V,0}^N, G_{R,0}^N \right. \right] \right\| < \frac{\varepsilon}{6}$$

Combining this with Lemma A.12 yields for η sufficiently small and N sufficiently large

$$\left\| E_0^N \left[x(\beta_{dev}(v))v - p(\beta_{dev}(v)) | C_1, F_{V,0}^N, G_{R,0}^N \right] - E_0 \left[x(b)v - p(b) \left| C, F_V, G_R \right. \right] \right\| < \frac{\varepsilon}{4} \quad (69)$$

Equations 67 and 69 and the fact that the SCE strategy is weakly optimal in the limit game given the SCE aggregate values of C, F_V, and G_R we get

$$\mathcal{V}^N \left(v,\ C_0^N, F_{V,0}^N, G_{R,0}^N \middle| \sigma \right) + \varepsilon \geq E_0^N \left[(x(\beta_{dev}(v))v_i - p(\beta_{dev}(v)) - \kappa) + \right.$$
$$\left. (1 - x(\beta_{dev}(v)))\delta \mathcal{V}^N \left(v_i, C_1^N, F_{V,1}^N, G_{R,1}^N | \sigma \right) \middle| \sigma, C_0^N, F_{V,0}^N, G_{R,0}^N \right]$$

which implies that following the SCE strategy is an ϵ-BNE for $\eta > 0$ sufficiently small. □

Appendix B. ONLINE SUPPLEMENTAL MATERIALS FOR *How Efficient are Decentralized Auction Platforms?* by Aaron L. Bodoh-Creed, Jörn Boehnke, and Brent R. Hickman EMPIRICS: ADDITIONAL DETAILS AND FIGURES

B.1. **GMM Standard Errors.** In this section we relate our stage I estimator to well-known GMM theory, and we present our calculations for standard errors in stages I and II. Let $\boldsymbol{\alpha} = (\lambda_1, \lambda_2, \alpha_{b2}, \ldots, \alpha_{bI_b-1}, \alpha_{r1}, \ldots, \alpha_{rI_r-1})^\top$ denote a column vector of all stage-I free parameters, let $I_1 \equiv 2 + I_b + 1 + I_r + 2$ denote its dimension, and let $w_l = (\tilde{k}_l, y_l, r_l)$ denote the observables for the l^{th} auction, $l = 1, \ldots, L$. Moreover, let $\boldsymbol{m}(w_t; \boldsymbol{\alpha})$ denote our $(3L \times 1)$-dimensional moment condition function, for which the n^{th} component is

$$m_n(w_t; \boldsymbol{\alpha}) = \begin{cases} \tilde{\pi}(\tilde{k}_n | r_n; \boldsymbol{\lambda}, \boldsymbol{\alpha}_b) - L\mathbb{1}\left(\tilde{k}_t = \tilde{k}_n\right) \dfrac{\mathcal{K}\left(\frac{r_t - r_n}{h_R}\right)}{\sum_{u=1}^{L} \mathcal{K}\left(\frac{r_t - r_u}{h_R}\right)}, & n \in \{1, \ldots, L\} \\ H(y_n; \boldsymbol{\lambda}, \boldsymbol{\alpha}_b) - \mathbb{1}(y_t \leq y_{n-L}), & n \in \{L+1, \ldots, 2L\}, \text{ and} \\ \hat{G}_R(r_n; \boldsymbol{\alpha}_r) - \mathbb{1}(r_t \leq r_{n-2L}), & n \in \{2L+1, \ldots, 3L\}, \end{cases}$$

and satisfying $\mathrm{E}\left[\boldsymbol{m}(w_t; \boldsymbol{\alpha})\right] = \mathbf{0}$. Note that a consistent estimator of the asymptotic variance of $\boldsymbol{m}(w_t; \boldsymbol{\alpha})$, denoted $\hat{\boldsymbol{\Sigma}}$, a $(3L \times 3L)$-dimensional matrix, is given by $\hat{\boldsymbol{\Sigma}} = L^{-1} \sum_{t=1}^{L} \boldsymbol{m}(w_t; \hat{\boldsymbol{\alpha}}) \boldsymbol{m}(w_t; \hat{\boldsymbol{\alpha}})^\top$. The analogous empirical moment conditions are $\boldsymbol{m}_L(\boldsymbol{\alpha}) = L^{-1} \sum_{t=1}^{L} \boldsymbol{m}(w_t; \boldsymbol{\alpha})$, and note that our estimators (27) and (28) are equivalent to finding the standard GMM estimator $\hat{\boldsymbol{\alpha}} = \arg\min_{\boldsymbol{\alpha}} L\boldsymbol{m}_L(\boldsymbol{\alpha})^\top \boldsymbol{m}_L(\boldsymbol{\alpha})$.[47] Finally, let $\hat{\boldsymbol{M}}$ denote the $(3L \times I_1)$-dimensional matrix of (estimated) first derivatives, where the $(i,j)^{th}$ element is defined as $\hat{M}_{ij} \equiv L^{-1} \sum_{t=1}^{L} \frac{\partial m_i(w_t; \hat{\boldsymbol{\alpha}})}{\partial \theta_j}$, where

$$\frac{\partial m_i(w_t; \hat{\boldsymbol{\alpha}})}{\partial \theta_j} = \begin{cases} \frac{\partial \tilde{\pi}(\tilde{k}_i | r_i; \boldsymbol{\lambda}, \boldsymbol{\alpha}_b)}{\partial \lambda_j} & i = 1, \ldots, L, \quad j = 1, 2, \\ \frac{\partial H(y_{i-L}; \boldsymbol{\lambda}, \boldsymbol{\alpha}_b)}{\partial \lambda_j} & i = L+1, \ldots, 2L, \quad j = 1, 2, \\ 0 & i = 2L+1, \ldots, 3L, \quad j = 1, 2, \\ & \\ \frac{\partial \tilde{\pi}(\tilde{k}_i | r_i; \boldsymbol{\lambda}, \boldsymbol{\alpha}_b)}{\partial \alpha_{b,j-2}} & i = 1, \ldots, L, \quad j = 3, \ldots, K_b + 5, \\ \frac{\partial H(y_{i-L}; \boldsymbol{\lambda}, \boldsymbol{\alpha}_b)}{\partial \alpha_{b,j-2}} & i = L+1, \ldots, 2L, \quad j = 3, \ldots, K_b + 5, \\ 0 & i = 2L+1, \ldots, 3L, \quad j = 3, \ldots, K_b + 5, \\ & \\ 0 & i = 1, \ldots, 2L, \quad j = K_b + 6, \ldots, K_b + K_r + 8, \\ \frac{\partial \hat{G}_R(r_{i-2L}; \boldsymbol{\alpha}_r)}{\partial \alpha_{r,j-K_b-5}} & i = 2L+1, \ldots, 3L, \quad j = K_b + 6, \ldots, K_b + K_r + 8. \end{cases}$$

[47] Note that throughout this section we take the choice of I_b, I_r, and knot vectors $\mathbf{n}_b$ and $\mathbf{n}_r$ as fixed, and we adopt the identity matrix for weighting in the GMM objective function.

With these definitions, it follows from well-known econometric theory (see [Hayashi, 2000, Chapter 7]) that $\hat{\boldsymbol{\alpha}}$ is asymptotically normal, converging at rate $\sqrt{L}$, and a consistent estimator of its $(I_1 \times I_1)$-dimensional asymptotic variance-covariance matrix is

$$\widehat{avar}(\hat{\boldsymbol{\alpha}}) = L^{-1} \left(\hat{\boldsymbol{M}}^\top \hat{\boldsymbol{M}}\right)^{-1} \hat{\boldsymbol{M}}^\top \hat{\boldsymbol{\Sigma}} \hat{\boldsymbol{M}} \left(\hat{\boldsymbol{M}}^\top \hat{\boldsymbol{M}}\right)^{-1}.$$

B.1.1. *Stage II Standard Errors.* Given this result, a straightforward approach to computing standard errors for stage II objects—including χ, ρ, κ, $\mathcal{V}(v)$, $\tilde{\beta}$, β, and F_V, all functions of $\hat{\boldsymbol{\alpha}}$—is the delta method. Let $\boldsymbol{v}(\boldsymbol{\alpha}) = (v_1(\boldsymbol{\alpha}), \ldots, v_d(\boldsymbol{\alpha}))^\top$ denote an I_2-dimensional, smooth, vector-valued function of the Stage I free parameters, and let $\mathbf{Y}(\boldsymbol{\alpha})$ denote its $I_2 \times I_1$-dimensional partial derivative matrix, where the $(i,j)^{th}$ element is given by

$$\mathrm{Y}_{ij}(\boldsymbol{\alpha}) = \frac{\partial v_i(\boldsymbol{\alpha})}{\partial \theta_j}, \;\; i = 1, \ldots, I_2, \;\; j = 1, \ldots, I_1.$$

Since $\hat{\boldsymbol{\alpha}}$ is approximately distributed as multivariate normal with mean $\boldsymbol{\alpha}$ and variance $avar(\hat{\boldsymbol{\alpha}})$, and linear transformations of normal random variables are normal, and since smooth functions are approximately locally linear, it follows that

$$\boldsymbol{v}(\hat{\boldsymbol{\alpha}}) - \boldsymbol{v}(\boldsymbol{\alpha}) \sim \mathcal{N}\left(\mathbf{0}, \mathbf{Y}(\boldsymbol{\alpha}) avar(\hat{\boldsymbol{\alpha}}) \mathbf{Y}(\boldsymbol{\alpha})^\top\right).$$

Therefore, the variance of $\boldsymbol{v}(\boldsymbol{\alpha})$ can be consistently estimated by $\mathbf{Y}(\hat{\boldsymbol{\alpha}})\widehat{avar}(\hat{\boldsymbol{\alpha}})\mathbf{Y}(\hat{\boldsymbol{\alpha}})^\top$. In principle, this method could be used for both parameters (such as κ) and functionals (such as t_V), if we define $\boldsymbol{v}$ as a vector of functional values on a grid of d domain points.

B.2. **Additional Figures.** Figure 9 provides intuition about the nature of the B-spline functions we use to fit the variables of our model. The top panel of the figure displays the observed distribution of the highest losing bids ($H(y)$) as well as the underlying parent distribution of bids ($G_B(b; \alpha_b)$). The center panel provides the distribution of the reserve prices. Since both the top and center panels describe observable variables, we have included the empirical CDFs as well. A comparison between the empirical CDFs and the B-spline fit shows a very close correspondence. Finally, the bottom panel describes the distribution values that we infer from the observables. To illustrate the underlying components of our B-spline functions, we have included the locations of the knots and the basis functions in the plot as well. The knot locations are described by the vertical lines extending below the top of each panel. The families of basis functions are drawn at the bottom of each panel.

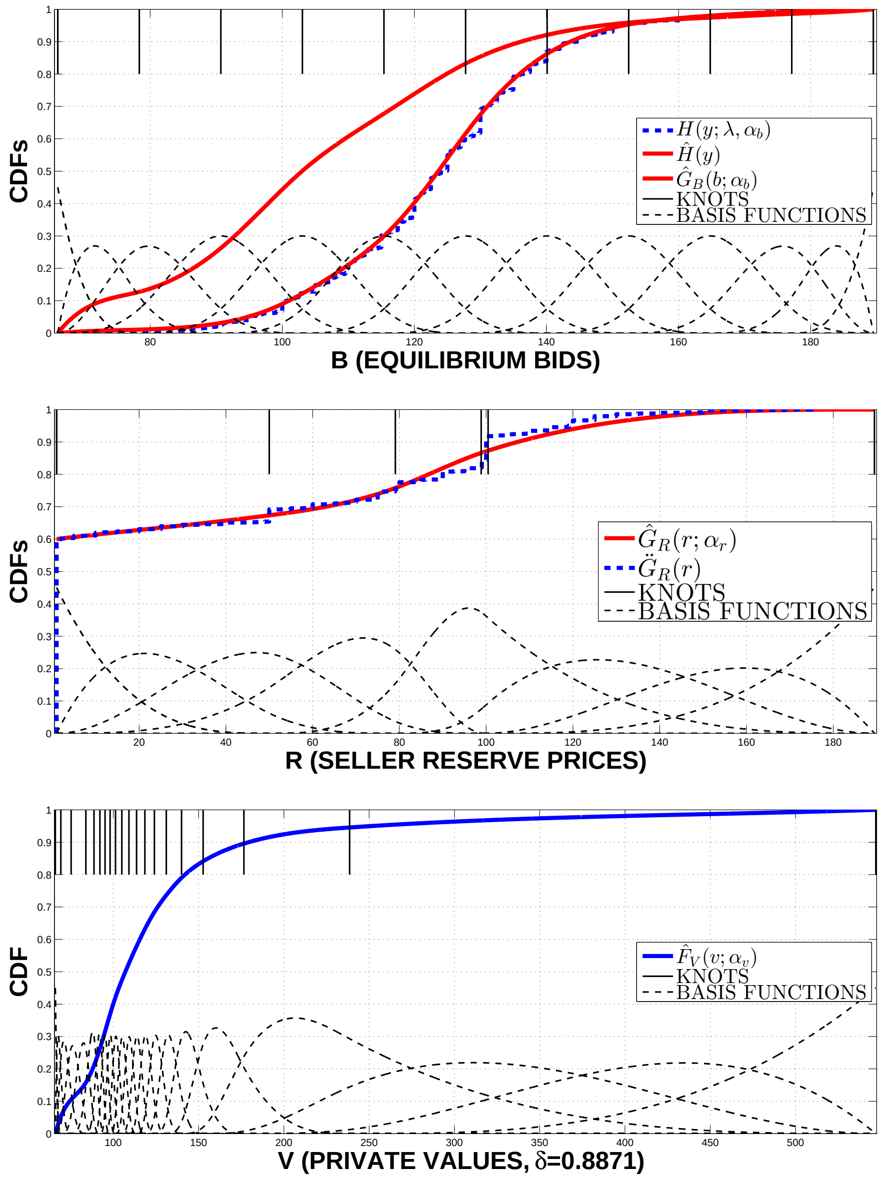

Figure 9. Stage I Estimates