Investigating the Economic Importance of Pricing-Rule Mis-Specification in Empirical Models of Electronic Auctions

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Abstract
Because of bid increments bidders in electronic auctions engage in demand shading, in contrast to the commonly assumed second-price format. We demonstrate that mis-specifying the pricing rule in this way can lead to significant bias in estimates of the latent valuation distribution and explore identification and estimation of a model with a correctly-specified pricing rule. A further challenge is that the econometrician only observes a lower bound on the number of participants in each auction. We are able to identify nonparametrically the form of an exogenous bidder arrival process, which matches potential buyers to auction listings from this observed lower bound. This then allows us to identify the private value distribution without functional form assumptions. We propose a computationally convenient sieve-type estimator of the private value distribution which involves B-splines. We also compare two parametric models of bidder participation and find that a generalized Poisson model cannot be rejected against the empirical distribution of the observables. Our structural estimates enable an exploration of information rents and optimal reserve prices on eBay. Keywords: eBay; electronic auctions; bid increments; pricing rule.

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1 Introduction

In the past two decades electronic auctions (EAs) have been an important market mechanism, reducing frictions by bringing together large numbers of buyers and sellers for an incredibly diverse array of goods. eBay, the leading host of EAs, alone averaged $4.95 billion in quarterly sales volume from auctions between Q1-2011 and Q4-2013.\footnote{Data downloaded from \url{http://www.statista.com/statistics/242267/ebays-quarterly-gross-merchandise-volume-by-sales-format} on 8/25/2015.} EAs have become of interest to economists not just for their size, but also because they present a data-rich setting wherein fundamental economic phenomena—such as market design, private information rents, market frictions, and price discovery—may be studied. They are generally modeled on traditional auction formats, but display certain characteristics due to their online implementation that present unique challenges for empirical researchers. One example is what eBay refers to as proxy bidding.

Proxy bidding is when a potential buyer reports a number to the server (which is kept private until the bidder is outbid) that represents the maximal amount he authorizes the server to bid. In turn, as competing bids arrive the server acts on the bidder’s behalf to increase his standing offer up to, but not exceeding, his reported maximum. When the EA software overtakes one bidder’s maximum bid on behalf of another, it forces the new lead bidder, whenever possible, to surpass the former lead bidder by a discrete amount, $\Delta$, called the bid increment. Bid increments are fixed by the online auction house, and openly advertised to market participants prior to bidding.

Researchers have typically modeled EAs as some variant of a second-price auction (SPA) format. This is understandable given how proxy bidding works: if one ignores the presence of bid increments, then the winner pays a price equal to the second-highest proxy bid for the item. However, if one takes bid increments into account, then the EA pricing rule becomes more complex than a simple SPA rule. As the EA software forces the lead bidder to surpass the second highest bid by $\Delta$, a necessary exception occurs when the top two proxy bids are within $\Delta$ of one another, because a jump in the full amount $\Delta$ would surpass the high bidder’s maximum authorized bid. Under this contingency, the price is set at the value of the high bid, and the winner directly determines sale price as would happen in a first-price auction (FPA).

Because EAs constitute one of the largest applications of auctions in history, understanding the impact of this non-standard pricing rule is an important and useful endeavor. In fact, there are numerous other instances in which minimum bid increments have been noted, but are ignored in how the pricing rule is modeled. For example, McAfee and McMillan\textsuperscript{[1996]} noted that the FCC spectrum auctions have minimum bid increments that are typically five to ten percent of the current price of a given license. We use simpler, single-unit EAs as an opportunity to account formally for the complications minimum bid increments introduce, and to highlight the consequences of ignoring this part of the pricing rule.

Hickman\textsuperscript{[2010]} derived theoretical implications of the EA pricing rule and found that dominant strategies cease to exist, unlike in SPAs. In equilibrium, participants engage in demand shading, similarly as in FPAs, because the winner’s bid will affect the transaction price with positive probability. Hickman also showed empirical evidence that even seemingly small bid increments may play a nontrivial role in shaping bidder behavior. Using data from a sample of laptop auctions

\footnote{Bid increments may have been originally implemented as a security measure against cyber-attacks. For example, even setting a relatively small increment of $\Delta = 2.50$ would increase the cost of automated, high-frequency bid submission by orders of magnitude, relative to the case where price is allowed to adjust by as little as a penny.}
held on eBay, where the average sale price was roughly $300 and $ was $5, he found that the final sale price was set at the value of the winner’s bid nearly one-quarter of the time.

Since structural work in auctions hinges crucially on the form of the theoretical equilibrium, the presence of bid shading in EAs implies that the traditional SPA view will lead to an incorrectly specified bidding model. This may induce significant bias in the estimated valuation distribution, which in turn may bias inference on various other questions relating to bidders’ private valuations, including information rents, optimal reservation price, or predicted revenue changes from another bidder at auction. In this paper, we develop a complete structural model of EA bidding with a correctly specified pricing rule, based on the theory of [Hickman, 2010]. The first contribution of our paper is to demonstrate the empirical relevance of the pricing rule mis-specification problem through a simple Monte Carlo exercise calibrated to resemble a host of realistic EA scenarios.

Our second contribution is then to develop a full structural model, designed for implementation using actual eBay data with possible bid shading as part of the equilibrium. We demonstrate that the model is identified from observables that are commonly available. Along the way, we also overcome another, independent challenge inherent in EA data. In many auction settings, it is notoriously difficult to measure participation at an auction precisely, and thus get an accurate view of how many bidders competed for the item. eBay (and EAs in general) is no different. There is a considerable body of work within the auctions literature to deal with this problem, but we take a novel approach to identification by explicitly modeling the process through which some bidder identities are revealed to the econometrician and others are not. Our model also takes into account the fact that within electronic platforms, the bidders and the econometrician are on a similar informational footing: neither can observe the total number of potential bidders watching an item at home with intent to bid on it. Thus, we model the number of competitors, $N$, as a random variable from the bidders’ perspectives, and we demonstrate that its distribution is nonparametrically identified from the observable lower bounds within each auction.

Even if the EA pricing rule were simply second-price, this problem of imperfectly observed participation would still represent a significant roadblock to empirical work. Previous methods have been developed to side-step the need for observing $N$ when identifying the private value distribution (for example, see [Song, 2004a] and [Song, 2004b]), but without knowing $N$ (or at least its distribution), many counterfactual simulations based on private value estimates are still impossible. Ours is the first paper to solve this problem and propose an estimator that makes counterfactual simulations possible.

Our third contribution is to propose a flexible and computationally convenient sieve-type estimator of the private value distribution, based on B-splines. Our choice of B-splines provides capability for matching crucial aspects of the theory—namely, points where the bid increment

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3One might also wonder about possible revenue implications of $Δ$ itself, but when (1) bidder valuations are independent and identically distributed (iid); (2) bidders are risk neutral; and (3) the expected payment of a bidder with a value of zero is zero, then in a symmetric, increasing equilibrium of an auction game under a variety of auction formats and pricing rules, the seller will earn the same expected revenue. This is the well-known revenue equivalence proposition—first demonstrated by [Vickrey, 1961] with uniformly-distributed valuations, and later generalized by [Riley and Samuelson, 1981] and [Myerson, 1981]. Thus, while the EA pricing rule changes how bidders behave, it may not cause changes in expected revenue if the above 3 conditions hold. Whether these assumptions are empirically relevant is beyond the scope of this paper. Instead we focus on developing and implementing an empirical bidding model that takes account of the bid shading pressure induced by positive increments in EAs.

4See [Hickman et al., 2012] for a complete survey of previous methods for coping with an imperfectly observed number of bidders.
transitions discretely from one value to another, causing kinks in the equilibrium bid function. Other popular methods, such as kernel smoothing and orthogonal polynomials, cannot do this. B-splines also provide a remarkable degree of functional flexibility and numerical stability which we explain and demonstrate below. Moreover, we incorporate Galerkin methods—commonly used in physical science applications—to solve the differential equation defined by a bidder’s first order condition. This approach substantially reduces computational cost during estimator runtime by circumventing the necessity of repeatedly solving a new differential equation every time the empirical objective function is evaluated. Instead, our proposed method requires solving the first order conditions only once during runtime.

We also consider estimation of two nested parametric models for the bidder participation process: Poisson and generalized Poisson. Although a fully nonparametric estimator is possible, a parametric approach affords computational convenience and statistical efficiency, so long as the functional form assumptions provide a reasonable fit to the data. We find that both models fit the data fairly well, but the latter, not surprisingly, does significantly better. More importantly though, we also find that the generalized Poisson model induces a distribution over observed participation levels that lay strictly within the nonparametric confidence bounds of the empirical CDF of the observables. In other words, our result suggests that there is little additional gain to be had in this case from a fully nonparametric estimator. Finally, we use our structural estimates to explore model simulations which inform us about two aspects of online market design. We quantify the distribution of winners’ information rents and explore optimal reserve prices on eBay. We find that winners on eBay are able to garner significant information rents, despite a large number of buyers participating in the market. We also resolve the puzzle of why the majority of sellers on eBay set reserve prices at zero: although the expected revenue curve (as a function of reserve price) has an interior global optimum, the difference between expected revenue at a reserve price of zero and the optimal expected revenue is a matter of mere pennies.

The remainder of this paper has the following structure: in Section 1.1 we briefly cover the related literature. In Section 2 we present the basic theory on bidding, identification, and estimation for a simplified version of the model with fixed, known $N$. This simplification facilitates a parsimonious Monte Carlo exercise to explore the impact of pricing rule mis-specification. The results of this mis-specification analysis are presented in Section 3. In Section 4 we extend the basic bidding model, identification results, and estimation method to handle more realistic aspects of real-world EAs, such as uncertainty over $N$, limited data availability, and non-constant $\Delta$. In Section 5 we implement our estimator on a sample of data from eBay laptop auctions and discuss our empirical results. In Section 6 we conclude, and in the Appendix we provide proofs for some of our results and a brief primer on B-splines.

### 1.1 Related Literature

[Lucking-Reiley](2000) has provided a guide to EAs, for economists. In surveying Internet auctions in 1998, he found that 121 of 142 Internet auctions used an ascending-price format. Lucking-Reiley recognized the use of proxy bidding, but did not distinguish between SPAs and EAs. Likewise, [Bajari and Hortaçsu](2004) discussed the prevalence of the ascending-price format in online auctions. While they noted that bid increments existed, and discussed the eBay case explicitly, Bajari and Hortaçsu implicitly assumed that EAs were SPAs. Other empirical researchers (such as [Roth and Ockenfels](2002), [Adams](2007) as well as [Zeithammer and Adams](2010)) have recognized
that bid increments exist, too, but did not model them formally. Instead, they have assumed that
EAs are SPAs.

Given the ascending-price format is predominant in EAs, a natural first step to analyzing them
might be to look at how such auctions are typically modeled. Haile and Tamer [2003] presented
what remains to be the most flexible open-auction format method available. There are two relevant
points to keep in mind in comparing their research to ours. First, the EA rules are clear enough
that we can use the structure to identify fully the model. Haile and Tamer focused on partial
identification, in part because their assumptions are quite flexible and tailored toward fitting a host
of settings that may deviate from the canonical “clock” model of an ascending auction. Second,
Haile and Tamer considered an environment where bidders may only submit bids from a discrete
grid—for example, when the auctioneer raises (or bidders call out) the price in discrete jumps and
asks for audience volunteers to pay the proposed price. In EAs, however, bidders may choose
their own bids from a continuum (or a very fine grid), and the fixed increment instead governs the
probability that they will pay their own bid upon winning.

Recent experimental work by Tukiainen [2015] complements our research. Tukiainen used
field experiments to investigate the effects of bid increments, selling gift cards at auctions in which
he varied the bid increment. He found that eBay-sized bid increments were close to optimal and
presented empirical findings that suggested EAs are not SPAs: the presence of bid shading in EAs
suggests that these increments induce bidders to behave in ways that are inconsistent with behavior
at SPAs.

We investigate the statistical and the economic importance of mis-specifying the EA pricing
rule as a second-price rule. In the next section, we present a simple game-theoretic model of bid-
ding at EAs where the number of competitors is known and constant across different auctions. We
begin with the cleanest model possible to focus attention on the effects of neglecting a minimum
bid increment in considering the pricing rule at EAs. Hortac̄su and Nielsen [2010] emphasized the
crucial identification of the correct mapping between valuations and bids. We demonstrate that this
mapping within the EA model is nonparametrically identified and we present a sensitivity analy-
sis, describing the results of several simulation experiments in which we estimated the correctly-
specified EA model as well as a mis-specified model where the researcher assumes the data are
from an SPA. The results of our simulation experiments suggest that model mis-specification can
lead to significantly different (in a statistical sense) estimates of the latent distribution. Such dif-
ferences can lead to significantly different policy prescriptions concerning optimal auction design. We
then consider model identification and estimation under more empirically realistic circumstances
when the true number of competitors is random and unknown to the econometrician.

2 Baseline Model: Fixed Participation

Consider a seller who seeks to divest a single object at the highest price. There are N potential
buyers, each of whom has a privately known value for the object for sale. Each potential buyer
views the private value of each of his competitors as a random variable, V, being an independent
draw from the cumulative distribution function \(F_V(v)\), which is twice continuously differentiable,
and has a strictly positive density function \(f_V(v)\) on a compact support \([\underline{v}, \overline{v}]\) where \(\underline{v}\) weakly ex-
ceeds zero. This information is common knowledge to all bidders, and we also assume for now that the number of potential buyers \( N \) is fixed and known. In Section 4 below we adopt a more realistic framework for \( N \) when we establish identification and estimation on actual eBay data. For current purposes, however, it will help to isolate the specific implications of pricing rule mis-specification if we begin with a more simplistic view of auction participation. This environment is often referred to as the symmetric independent private-values paradigm (IPVP). To reduce clutter later on, it will be useful to let \( Z \equiv V_{1\cdots(N-1)} = \max \{V_1, V_2, \ldots, V_{N-1}\} \) denote the maximum from a sample of \( (N-1) \) draws from \( F_V(v) \): \( Z \) is a random variable that represents the highest private value of a bidder’s \( (N-1) \) rivals at the auction. Given that valuations are distributed independently and identically, the cumulative distribution function and density of \( Z \) are \( F_Z(v) = F_V(v)^{N-1} \) and \( f_Z(v) = (N-1)F_V(v)^{N-2}f_V(v) \).

The highest bidder is declared winner, but the price is determined by a special hybrid rule where the winner pays the smaller of either his bid or the second-highest bid plus a commonly-known bid increment \( \Delta \). Below we characterize the symmetric equilibrium bid function and investigate the importance of the error that obtains from assuming the EA is an SPA.

2.1 Deriving the EA Equilibrium Bid Function

Hickman [2010] showed that a unique, monotone, pure-strategy, Bayes–Nash equilibrium exists within the IPVP. Denote by \( \beta(v) \) the symmetric bid function at an EA with bid increment \( \Delta \). At an EA two scenarios can determine the sale price: in the first, the highest losing bid is within \( \Delta \) of the winner’s bid and she therefore pays her own submitted bid (the first-price rule is used); in the second, the two top bids are further apart, and the winner pays the highest losing bid plus \( \Delta \) (a second-price-like rule is used). Note that when the winner’s bid is \( \Delta \) or less, the first-price rule is always used. We shall refer to the valuation which yields a bid of exactly \( \Delta \) as \( v_\Delta \equiv \beta^{-1}(v + \Delta) \). Thus, on the interval \([v, v_\Delta)\), the EA bid function is the same as the FPA equilibrium derived by Holt [1980] and Riley and Samuelson [1981] as the solution to

\[
\beta'(v) = \frac{[v - \beta(v)] f_Z(v)}{F_Z(v)}, \quad \beta(v) = v. \tag{1}
\]

This function can be used to solve for the cutoff type \( v_\Delta \) as well.

We now characterize bids on the interval \([v_\Delta, \bar{v}]\). Let \( B \) denote \((B_1, B_2, \ldots, B_N)\), the vector that collects all bids, and let \( B_{-n} \) denote \((B_1, B_2, \ldots, B_{n-1}, B_{n+1}, \ldots, B_N)\), the bid vector with only player \( n \)'s opponents. Bidder \( n \)'s optimization problem can be expressed as

\[
\max_{b_n} (v_n - b_n) \Pr(b_n - \Delta < \max \{B_{-n}\} \leq b_n) + [v_n - \mathbb{E}(\max \{B_{-n}\} | \max \{B_{-n}\} \leq b_n - \Delta) - \Delta] \Pr(\max \{B_{-n}\} \leq b_n - \Delta). \tag{2}
\]

The first term in the sum corresponds to winning the EA under a first-price rule, that is, when bidder \( n \)'s bid and the second-highest bid are within \( \Delta \) of each other. The second term corresponds to winning the EA under a second-price rule, that is, \( b_n \) exceeds the second-highest bid by at

\[\text{We use uppercase Roman letters to denote random variables and lowercase Roman letters to denote realizations of these random variables.}\]
least $\Delta$, and bidder $n$ pays $\Delta$ more than the second-highest bid. We can express the probability of winning under the first-price rule given equilibrium bid function $\beta(v)$ with inverse $\beta^{-1}(b)$ as

$$\Pr(b_n - \Delta < \max \{B_{-n}\} \leq b_n) = \Pr[\beta^{-1}(b_n - \Delta) < Z \leq \beta^{-1}(b_n)]$$

$$= F_Z[\beta^{-1}(b_n)] - F_Z[\beta^{-1}(b_n - \Delta)]$$

and, under the second-price rule, as

$$\Pr(\max \{B_{-n}\} \leq b_n - \Delta) = \Pr[ Z \leq \beta^{-1}(b_n - \Delta)]$$

$$= F_Z[\beta^{-1}(b_n - \Delta)].$$

The conditional expectation in the second term of expression (2) is

$$\mathbb{E}(\max \{B_{-n}\} | \max \{B_{-n}\} \leq b_n - \Delta) = \frac{\int_{\beta^{-1}(b_n - \Delta)}^{\beta^{-1}(b_n)} \beta(u) f_Z(u) \, du}{F_Z[\beta^{-1}(b_n - \Delta)].}$$

By expressing the problem in terms of the order statistic $Z$, we can eliminate the bidder subscript $n$ below. Maximizing a bidder’s expected surplus on the interval $[v_\Delta, \bar{v}]$ yields a differential equation

$$\beta'(v) = \frac{[v - \beta(v)] f_Z(v)}{F_Z(v) - F_Z(\beta^{-1}(v) - \Delta)},$$

along with boundary condition $\beta(v_\Delta) = v + \Delta$, which together define the function $\beta(v)$. This formulation, which was developed by Hickman [2010], is not particularly helpful to empirical researchers because the differential equation does not admit a closed-form solution. Fortunately though, after applying the inverse function theorem this differential equation can be re-cast solely in terms of the equilibrium inverse-bid function, so equation (3) can be rewritten as

$$\frac{d\beta^{-1}(b)}{db} = \frac{F_Z[\beta^{-1}(b)] - F_Z[\beta^{-1}(b - v - \Delta)]}{[\beta^{-1}(b) - b] f_Z[\beta^{-1}(b)]}$$

where we have used the fact that $\beta(v)$ equals $b$ and $\beta^{-1}(b)$ equals $v$. Thus, the EA equilibrium is characterized by a piecewise differential equation for optimal bidding on $[0, v_\Delta)$ and $[v_\Delta, \bar{v}]$.6

As an example of how equilibrium behavior at an EA differs from the standard SPA and FPA, we depict in Figure 1 the bid functions under the three pricing rules.7 In this figure, and below, we refer to the equilibrium bid function at an EA as $\beta_{EA}(v)$, the equilibrium bid function at an SPA as $\beta_{SP}(v) = v$, and the equilibrium bid function at an FPA as $\beta_{FP}(v) = \mathbb{E}[Z | Z < v]$. The important point to note is that the EA equilibrium bid function lay weakly between the SPA and the FPA bid functions. In fact, Hickman [2010] Theorem 3.6] showed that the EA model nests the SPA and FPA as special cases in the sense that, as $\Delta \to 0$, we get $\beta_{EA}(v) \to \beta_{SP}(v)$ uniformly, and as $\Delta \to \mathbb{E}[Z]$, we get $\beta_{EA}(v) \to \beta_{FP}(v)$ uniformly.

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6Despite the piecewise definition, $\beta(v)$ turns out to be continuously differentiable at the point $v_\Delta$, since the derivative in equation (4) approaches the first price derivative as $v \to v_\Delta$ from above; see Hickman [2010].

7This example was constructed using $N = 3$, $V \sim Rayleigh(1)$, truncated to the interval $[0, 1]$, and $\Delta = 0.1$. 

2.2 Nonparametric Identification and Estimation with Fixed $N$

The EA model is identified nonparametrically and can be estimated using a simple modification to the approach proposed by Guerre et al. [2000, GPV] for models of first-price, sealed-bid auctions. Interestingly, while most researchers have assumed that EAs are SPAs, nonparametric estimation involves a simple correction term relative to FPAs which accounts for bid shading due to the presence of increments.

To see why, let $G_B$ denote the cumulative distribution function of equilibrium bids, and note that $G_B(b) = F_V(v) = F_V\left[\beta^{-1}(b)\right]$. From this fact it follows that $g_B(b) = \frac{f_V(v)}{f_V(\beta^{-1}(b))} \frac{d\beta^{-1}(b)}{db}$ is the corresponding probability density function. Substituting these terms into equations (1) and (4) yields

$$v = b + \frac{G_B(b)^{N-1} - G_B(b - \Delta)^{N-1}}{(N-1)g_B(b)G_B(b)^{N-2}}$$

(5)

for $b \geq \Delta$, and

$$v = b + \frac{G_B(b)}{(N-1)g_B(b)}$$

(6)

for $b < \Delta$. This formulation of the equilibrium shows that each bidder’s private value is point identified from a sample of bids. This is particularly useful because, when mis-specifying the pricing rule as SPA, the researcher implicitly assumes that bids are private valuations, whereas, given an estimate of the bid distribution and density, equations (5) and (6) allow for a simple error correction to adjust for demand shading, or the difference between private values $v$ bids $b$.

In the following section we present a sensitivity analysis based on simulated data to compare empirical results under the (incorrect) SPA assumption, which merely takes bids as private values and estimates their density nonparametrically, and the (correct) EA assumption. For the latter case, we construct a two-stage nonparametric estimator in the spirit of GPV: in a first stage we construct empirical analogs to (5) and (6) using a kernel density estimator for $g_B$ to get a sample of private value estimates, $\{\hat{v}\}$; in a second stage we then kernel smooth the density of $\hat{v}$ to obtain...
an estimate of $f_V$. In order to avoid problems of sample trimming, we use the boundary-corrected GPV (BCGPV) estimator proposed by [Hickman and Hubbard 2015]. In order to provide a basis of comparison between the EA and SPA estimates, we also directly estimate the density $f_V$ by kernel smoothing the sample of simulated private values as well. Although such a strategy would never be available to practitioners working with real data, it provides a baseline estimate by which to judge the closeness of the other two estimators to the true distribution, given a finite sample of data with which to work.

As a precursor to our sensitivity analysis, in Figure 2 we plot three estimated probability density functions. We considered a simple case in which a researcher observes 1,000 auctions each having five bidders who draw valuations from a Weibull(0.5,4.0) defined on [0,1]. In panels (a), (b), and (c) of the figure, we display representative results under a bid increment of 0.02, 0.05, and 0.1, respectively. In each panel the solid line is the kernel-smoothed nonparametric estimate of the marginal density based on the random valuations that we generated. This is the best we could hope for from a nonparametric estimator and we denote it $\hat{f}(V)$. The other two estimators take as input the equilibrium bids from an EA that correspond with the randomly-drawn valuations. The dash-dotted line represents $\hat{f}_{EA}(V)$—the nonparametric estimate under the correctly-specified EA model. The dashed line represents $\hat{f}_{SP}(V)$—the nonparametric estimate under the mis-specified SPA model. Note that $\hat{f}_{EA}(V)$ is not visually distinct in the figure because it coincides almost exactly with $\hat{f}(V)$. The mis-specified nonparametric estimate attains a higher peak and is shifted left of the optimal estimate (and the EA nonparametric estimate), with the effect becoming more pronounced under higher $\Delta$. This occurs because, for a given valuation, the SPA model predicts a higher bid (bidders’ weakly dominant strategy is to bid their valuation) than the EA model which involves bidders shading in the hope that the item at auction is awarded under a first-price rule. As such, the valuation implied by a given bid is lower under a SPA-assumed pricing rule. It is also worth mentioning that $\hat{f}_{EA}$ is naturally handicapped against $\hat{f}_{SP}$, since the former is a two-step nonparametric estimator and has a slower convergence rate than the latter, a one-step nonparametric estimator. In the next section our sensitivity analysis will test varying sample sizes, but the figure suggests the statistical challenges a two-step nonparametric estimator faces can be less than the cost of pricing rule mis-specification, something we formally pursue in the next section.

3 Mis-Specification Analysis

In this section we adopt three alternative specifications for the private value distribution—Exponential, Power, and Rayleigh—in order to explore the implications of the non-standard pricing rule based

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8Traditional kernel density estimators are known to be inconsistent and biased at the boundary of the support. GPV proposed a solution to this problem that involved discarding data within a neighborhood of the sample extremes in order to preserve consistency within the interior of the support. In finite samples this creates several problems for inference: [Hickman and Hubbard 2015] proposed an alternative approach based on boundary-corrected kernel density estimators which are uniformly consistent on the closure of the support. They described a number of attractive features of the boundary-corrected GPV estimation strategy, but for this paper the most important benefit is perhaps preserving the entire sample of data, making the one-step SPA estimator and the two-step EA estimators more comparable.
on bid increments. The following are the formulae for the CDF functions we consider:

- **Exponential**\( (\theta) \) : \( F(v) = 1 - \exp(-v/\theta) \)
- **Power**\( (\theta) \) : \( F(v) = v^\theta \)
- **Rayleigh**\( (\theta) \) : \( F(v) = 1 - \exp(-v^2/2\theta^2) \).

For each one we assumed that \([v, \overline{v}]\) equals \([0, 1]\), and all distributions are truncated so that \(F_X(v) = \frac{F(v) - F(0)}{F(1) - F(0)} = \frac{F(v)}{F(1)}\), with corresponding probability density function \(f_X(v) = \frac{f(v)}{F(1)}\). For simplicity, in what follows we simply use the names of the untruncated distributions when referring to the truncated ones. The three truncated densities are depicted in Figure 2. We chose these three in particular for our Monte Carlo exercise in order to evaluate the effect of the probability density function having a mode at \(v\), an interior mode, and a mode at \(\overline{v}\). Unless explicitly stated, we assumed the bid increment \(\Delta\) is two percent of the highest valuation which is consistent with a major portion of eBay bid increments. Before presenting our Monte Carlo experiments, we perform two exploratory analyses to probe the realism of our three test distributions.

### 3.0.1 Error in the Equilibrium Bid Function

To quantify the effect of mis-specifying the EA pricing rule, we computed a relative measure of error in the implied bidding function. Define the relative error from modeling and solving for the equilibrium at an EA by assuming a SPA by \(\varepsilon(v) = \frac{\beta_{SP}(v) - \beta_{EA}(v)}{v} \equiv v - \frac{\beta_{EA}(v)}{v} = 1 - \frac{\beta_{EA}(v)}{v} \), which can be interpreted as the percentage error in the predicted bid for a given valuation as \(\beta_{SP}(v)\) equals \(v\).

In Table 1 we summarize the expected relative error involved in assuming an SPA—which we computed as \(\mathbb{E}[\varepsilon(V)] = \int_v \varepsilon(v)f_{\hat{v}}(v)\ dv\)—for each of the three distributions. The expected relative

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9 Bid increments at eBay auctions are discussed at the following URL: [http://pages.ebay.com/help/buy/bid-increments.html](http://pages.ebay.com/help/buy/bid-increments.html) which we accessed on 8/25/2015. Bid increments ranged from five percent for low-valued items (under $5.00) to one percent for items that are selling for between $2,500.00 and $4,999.99. As an example, items with prices between $5.00 and $24.99 have a bid increment of $0.50 which is ten percent of $5 and two percent of $25.
errors reported in Table 1 are all greater than three percent and can be as high as eleven percent. For each distribution we see that as the number of bidders increases, the expected error decreases. For an EA, the sign of this effect is not obvious: with more competition bidders behave more aggressively, but with more participants at an auction, the probability that the top two bids are within $\Delta$ of each other also increases. The numbers in the table suggest that the former competitive effect dominates the latter probabilistic effect.

### 3.0.2 Bid Increments and the Frequency of a First-Price Rule

[Hickman 2010] found, within a sample of 1,128 eBay auctions for laptop computers, that 23.05% of final sale prices were generated by a first-price rule being triggered (because the top two bids were close together), as opposed to the often assumed second-price rule. To probe our test distributions, we simulated EA auctions involving three, five, and ten bidders, with bid increments of 0.01, 0.02, and 0.05. In each case we simulated one million auctions and computed the fraction where the first-price rule determined the transaction price.

In Table 2 we present these frequencies for the simulated scenarios. The results illustrate that, for a given distribution and a fixed number of players at auction, increasing the bid increment increases the share of transaction prices determined by the first-price rule. Likewise, for a given distribution and a fixed bid increment, increasing the number of players at auction increases the

---

10By calibrating $\bar{v} = 1$, the bid increments correspond to a percentage of the highest possible valuation.
Table 2: Frequency of First-Price Rule at EA Auctions

<table>
<thead>
<tr>
<th>Distribution</th>
<th>( \Delta )</th>
<th>Exponential(2.0)</th>
<th>Rayleigh(0.3)</th>
<th>Power(1.5)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>0.01</td>
<td>2.90%</td>
<td>4.04%</td>
<td>3.80%</td>
</tr>
<tr>
<td>( N = 3 )</td>
<td>0.02</td>
<td>5.80%</td>
<td>8.04%</td>
<td>7.47%</td>
</tr>
<tr>
<td></td>
<td>0.05</td>
<td>14.65%</td>
<td>20.23%</td>
<td>18.04%</td>
</tr>
<tr>
<td></td>
<td>0.01</td>
<td>3.53%</td>
<td>5.21%</td>
<td>6.72%</td>
</tr>
<tr>
<td>( N = 5 )</td>
<td>0.02</td>
<td>7.00%</td>
<td>10.27%</td>
<td>13.07%</td>
</tr>
<tr>
<td></td>
<td>0.05</td>
<td>17.04%</td>
<td>25.35%</td>
<td>30.07%</td>
</tr>
<tr>
<td>( N = 10 )</td>
<td>0.01</td>
<td>5.04%</td>
<td>6.54%</td>
<td>13.52%</td>
</tr>
<tr>
<td></td>
<td>0.02</td>
<td>9.90%</td>
<td>12.85%</td>
<td>25.37%</td>
</tr>
<tr>
<td></td>
<td>0.05</td>
<td>23.71%</td>
<td>31.48%</td>
<td>52.88%</td>
</tr>
</tbody>
</table>

share of transaction prices determined by the first-price rule. The Power distribution dominates the Exponential one, but there is no clear ranking between these two and the Rayleigh. For a given \((N, \Delta)\) pair, the Power distribution involves a higher share of bid profiles triggering the first-price rule than the Exponential. Our purpose here is simply to illustrate that the cases we have considered for the simulations presented below involve realistic bid increments capable of generating phenomena consistent with what is observed in actual data. Moreover, we hope to convince the reader that assuming an EA is an SPA involves potentially serious consequences.

To evaluate the effects of mis-specification, we employed a nonparametric empirical model under the correct EA assumption as well as the mis-specified SPA assumption. In the first subsection, we report the results of simulation experiments in which we used different distributions to demonstrate that mis-specification can lead to significantly different estimates of the latent valuation distribution. In the second subsection, we consider the implications that model mis-specification can have, estimating optimal auctions and quantifying the economic importance of the biased policy prescriptions and predictions deriving from the SPA assumption.

### 3.1 Simulation Experiments

We conducted a series of simulation experiments in which we varied model components, including \( F_V \in \{ \text{Exponential}(2.0), \text{Rayleigh}(0.3), \text{Power}(1.5) \} \), \( N \in \{3, 5, 10\} \), and \( \Delta \in \{0.02, 0.05, 0.10\} \). Moreover, we also varied the sample size of auctions \( T \in \{100, 300\} \). We simulated each instance \( S = 1,000 \) times and allowed the econometrician to observe the bids from all potential bidders. For each simulation \( s \), we did the following:

---

11In earlier versions, we considered a parametric exercise under a best-case scenario where we assumed that the econometrician knew the correct family of the valuation distribution, but not the specific parameter value(s) of the distribution. The advantage of a parametric model is that there is a clear way of comparing the two sets of estimates—the difference in two parameters (or two parameter vectors). We have relegated such results to an online appendix and chosen to present only the results of the nonparametric simulation experiments as this is the relevant practical case. The parametric simulation experiments showed even more of an advantage to correctly specifying the pricing rule as the rate of convergence of the estimators is far quicker in that setting.

12The parametric simulation routines were similar in spirit, but involved a quasi maximum-likelihood estimator for the SPA model and a maximum-likelihood estimator built on a nested fixed-point algorithm for the EA model. In
1. generated $T N$-tuples of valuations from a given distribution;

2. used the true EA bid function to map these valuations into bids which we assume the researcher actually observes;

3. assumed the bids came from an SPA and estimated the model via the one-step nonparametric estimator described in the previous section;

4. assumed the bids came from an EA auction and estimated the EA model via the two-step nonparametric estimator described in the previous section.

To evaluate the statistical performance of the estimators, we constructed tests of the null hypothesis that the sample of estimated pseudo-values recovered under the SPA and EA assumptions, respectively, came from the same distribution as the actual sample of simulated valuations. Specifically, we used a two-sample Kolmogorov–Smirnov test as well as an Anderson–Darling test. We present only results from the Anderson–Darling tests (based on Scholz and Stephens [1987]) since the results of the Kolmogorov–Smirnov tests are nearly identical.

In Table 6 in the Appendix, we present results from this simulation exercise by reporting the number of null hypotheses rejected as well as the median $p$-value of the Anderson–Darling test statistic for the instances involving different distributions, bid increments, number of bidders at auction, and sample sizes. The most robust result is that the two-step EA estimator always outperforms the one-step, albeit mis-specified, SPA estimator. For the exponential and Rayleigh distribution cases, a null hypothesis is never rejected under the EA estimation. For a given bid increment, as either the number of auctions or the number of bidders increases, the number of times the SPA-based null hypothesis is rejected is increasing. Moreover, once the bid increment is five percent, nearly every simulation involving the SPA estimates allows the null hypothesis to be rejected. Regardless of the distribution, we never reject the null hypothesis for the two percent bid increment cases involving the EA-based estimates. The power-distribution case, however, is notably difficult for both estimators. The power distribution is the case which puts an extreme amount of weight on high valuations, where curvature in the bid function is most prevalent. This could explain why the null hypothesis is rejected frequently for the largest sample sizes and ten percent bid increment, even for the correctly-specified EA estimator.

Under the SPA assumption, the private-value distribution is the same as the bid distribution. Thus, structural estimation methods which employ the second-price rule will uncover the population bid distribution $G_B(b)$ as the sample size gets large. That is, if we denote the estimated valuation distribution under an SPA assumption given a sample of $T$ auctions by $\hat{F}_{SP}^T(V)$, then as the number of auctions in the sample increases, we have the following:

$$\operatorname{plim}_{T \to \infty} \hat{F}_{SP}^T(V) = G_B(b) = F_V(v) \circ \beta^{-1}(b).$$

Since $\beta(v)$ does not equal $v$ when $\Delta$ is positive, it is clear that $\hat{F}_{SP}(V)$ will fail to converge in probability to $F_V(v)$. As such, the estimated demand functions under the SPA assumption will always lie to the left of the estimated demand function under the EA assumption.

short, because the EA model is an hybrid of an SPA and an FPA, it fails to satisfy a necessary regularity condition as the support of the distribution of the equilibrium strategies depends on the parameters of the distribution of the latent characteristics to be estimated. Donald and Paarsch [1996] proposed a solution in the form of a constrained optimization approach to maximum-likelihood estimation. Details are described in the online appendix.
3.2 Economic Importance of Mis-Specification

To investigate the economic importance of the bias that obtains when a researcher estimates an EA under the SPA assumption, we considered two exercises which an econometrician might be asked to conduct: recommending a reserve price and predicting anticipated revenues if another bidder were to enter the auction. First, we computed the implied optimal auctions (involving optimally-chosen reserve prices) corresponding to each estimated distribution (based off the EA and the SPA assumptions), for each simulation $s$ for each instance of our experiment. Denote by $\Omega$ the set of all auctions at which: (1) any bidder can submit a bid as long as it is greater than some value $r^*$; (2) the buyer submitting the highest bid above $r^*$ is awarded the object; (3) auction rules are anonymous in that each bidder is treated in the same way; and (4) there exists a monotone, symmetric, pure-strategy, Bayes–Nash equilibrium. At any auction satisfying these four conditions, the optimal reserve price $r^*$ must satisfy

$$r^* = v_0 + \frac{1 - F_V(r^*)}{f_V(r^*)}$$

where $v_0$ is the seller’s valuation for the item at auction; see, Riley and Samuelson [1981] as well as Myerson [1981]. In our simulation experiments, we assumed $v_0$ is zero and computed the optimal reserve price implied by the EA and SPA estimates for each simulation experiment.

In Table 7 in the Appendix, we present the mean reservation price $\hat{r}_{EA}$ and $\hat{r}_{SP}$ implied by the estimates of the latent valuation distribution and density under each assumption, EA and SPA, respectively, along with their standard errors $\hat{\sigma}_{r_{EA}}$ and $\hat{\sigma}_{r_{SP}}$. The true optimal reserve price for the Exponential(2.0), Rayleigh(0.3), and Power(1.5) cases are 0.36077, 0.29905, and 0.54288, respectively. The mean reservation price under the EA auction is always within a standard deviation of the true optimal reservation price. In contrast, under the SPA assumption, the mean reservation price is regularly at least two standard deviations away from the truth for large enough sample size and especially for the larger bid increments, suggesting that, from a policy perspective, mis-specification has important effects.

In our second exercise, we used the estimated latent value distributions to estimate what revenues would be were another bidder to participate at auction. This consideration is motivated by researchers who have pointed out that adding another bidder to the auction is often far more valuable than getting the reservation price exactly right. For example, Bulow and Klemperer [1996] showed that, in a world with a fixed number of participants, the optimal auction with a specific number of bidders provides less revenue than an auction with no reserve price, but one additional bidder. We consider an econometrician who might be asked to predict the expected revenues were another bidder to show up at auction. To do this, we appeal to the revenue equivalence theorem and note that we need only draw with replacement from each estimated valuation distributions and compute the average values of the second highest valuation a sufficient number of times. In this way, the root cause of any discrepancy is entirely attributed to the error deriving from the original mis-specification in recovering the valuation distribution from the observed bids.

In Table 8 (see Appendix), we present the expected revenue an econometrician would predict were another bidder to enter the auction. For the reasons documented earlier, the SPA-estimated specification always underpredicts expected revenue. Note that, since our simultaneous EA model

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13Recall that the optimal reserve price does not depend on the number of bidders at auction, although we present estimates in Table 7 for which the number of bidders at auction varied according to the cases we considered in our simulation experiments.
satisfies the assumptions of the revenue equivalence theorem, a practitioner should predict the same expected revenue regardless of the minimum bid increment \( \Delta \) corresponding to the data from which the underlying distribution was estimated from. The results in the table illustrate the substantial variation in predictions from the SPA-based estimations for fixed values of \( N \) and \( T \) across the sections of the table in which \( \Delta \) is changing. Specifically, as \( \Delta \) increases, a practitioner assuming a SPA pricing rule would underpredict the expected revenue by larger amounts.

Our simulation results suggest that mis-specifying the pricing rule can result in significantly different estimates of the latent distribution of valuations. These differences carryover to policy recommendations because the SPA-estimated valuation distributions suggest significantly different policy estimates—optimal reserve prices. There are also consequences in applying estimates from the mis-specified model to predict expected revenues; for example, if another bidder were to participate at auction. In contrast, concerns about the two-step estimation process required of a correctly-specified EA model do not appear to cause issues in practice as the estimator performs quite well in these dimensions.

4 Identification and Estimation Under Stochastic Participation

We now extend our model to make it more compatible with real-world data from eBay. The principal challenges in empirical applications are three-fold: first, the econometrician does not observe all bids, but rather a selected sample of bids which may or may not be consistent with equilibrium play. Second, the total number of bidders participating in an auction is unobserved to the econometrician, and instead, only a lower bound on total participation can be gleaned from data. Third, real-world electronic auctions do not use constant bid increments, but bid increment schedules that are piecewise constant (that is, they discretely jump at specific, predetermined points in bid space). In this section, we provide tractable solutions to these three problems, and demonstrate that a model with a correctly-specified pricing rule is still nonparametrically identified under these more empirically realistic conditions. We then propose a sieve-type estimation strategy, based on B-splines, that we implement in the next section using data from eBay laptop auctions.

4.1 A Bidding Model With Stochastic \( N \)

One common characteristic of EAs is that the web-based interface makes it impossible to observe precisely the number of potential competitors within a given auction; that is, the number of users who are following an item with intent to bid on it. Difficulty in measuring \( N \) has long been a principal challenge within the empirical auctions literature, particularly when the bidders are able to observe \( N \) and adjust their bidding strategies with information unavailable to the econometrician. At an EA, however, the researcher and the bidders are on the same footing in that neither observes \( N \). Therefore, we shall model participation from the perspective of a bidder as a stable stochastic process that exogenously allocates bidders to a given auction.

Specifically, let bidders view \( N \) as a random variable with probability mass function \( \rho_N(n; \lambda) \equiv \Pr(N = n) \) indexed by a parameter vector \( \lambda \) and assume that they do not know the realization of \( N \) \textit{ex ante}, when they compute their strategic bids.\(^\text{14}\) In formulating the exogenous participation

\(^{14}\) Song [2004b] was the first to propose a bidding model of first-price auctions where bidders view unknown \( N \) as a random variable. She then showed that assuming \( N \) is distributed Poisson makes it possible to identify the distribution.
process this way, we allow for $\lambda$ to be infinite dimensional so that the distribution of $N$ may be fully nonparametric if, for example, $\lambda = \{\lambda_0, \lambda_1, \lambda_2, \ldots\}$, where $\lambda_n = \Pr(N = n)$.

Once again, consider the auction from bidder 1’s perspective, and let $M \equiv N - 1$ denote the number of opponents he faces. We also define

$$\rho_M(m; \lambda) \equiv \Pr(M = m|N \geq 2) = \frac{\rho_N(m + 1)}{1 - \rho_N(0) - \rho_N(1)}, \quad m \in \{1, 2, 3, \ldots\}$$

as the probability that bidder 1 faces exactly $m \geq 1$ opponents. Just as before when $N$ was known ex ante, a bidder’s strategic decision problem within an auction is how to respond optimally to his highest rival bid. We denote the highest rival valuation and bid as random variables $V_M$ and $B_M$, respectively, and we denote their respective distributions as

$$F_M(V_M) = \sum_{m=2}^{\infty} \rho_M(m; \lambda) F_V(V_M)^m, \quad \text{and} \quad G_M(B_M) = \sum_{m=2}^{\infty} \rho_M(m; \lambda) G_B(B_M)^m. \quad (7)$$

As such, $F_M$ and $G_M$ are weighted sums of powers of their respective parent distributions, where the weights represent the probability of a given realization for the number of potential bidders.

With these adjustments to notation, the bidding model based on a hybrid pricing rule as proposed by [Hickman 2010] can be easily extended to handle stochastic exogenous participation. By inserting $F_M$ into equations (1) and (3) above in place of $F_Z$, we get a new set of equations to redefine the equilibrium $\beta$ for the stochastic participation case; namely,

$$\beta(v) = v, \quad \beta'(v) = \frac{[v - \beta(v)] f_M(v)}{F_M(v)}, \quad v \in [v_1, v_\Delta], \quad \text{and} \quad \beta'(v) = \frac{[v - \beta(v)] f_M(v)}{F_M(v) - F_M(\xi[\beta(v) - \Delta])}, \quad v \geq v_\Delta. \quad (8)$$

Intuitively, whether $N$ is known or stochastic, the highest rival bid is still a random variable to which each bidder is best responding. The only difference now is that randomness comes from two separate sources: a given opponent’s valuation is unknown, and the quantity of opponents is also unknown.

Going forward we denote the inverse bidding function by $\xi(b) \equiv \beta^{-1}(b) : [b, \bar{b}] \rightarrow [v_1, v]$. As before, these equations can be transformed to express the inverse bidding relationship as

$$\xi(b) = v = b + \frac{G_M(b)}{g_M(b)}, \quad b \leq \Delta, \quad \text{and} \quad \xi(b) = v = b + \frac{G_M(b) - G_M(b - \Delta)}{g_M(b)}, \quad b \geq \Delta. \quad (10)$$

of private valuations without knowing the exogenous arrival rate of bidders. However, knowledge of $\lambda$ is needed for counterfactuals and revenue projections using structural estimates. We take a different approach here which aims to identify both the private value distribution and the distribution of $N$ from observable data. The advantages of our approach are that parametric assumptions are unnecessary and structural counterfactual simulations are well defined.

Conditioning on the event that $N \geq 2$ comes from two facts. First, if bidder 1 exists, then he knows $N$ is at least 1. Second, if $N = 1$ so that bidder 1 faces no opponent, then he wins the object at a price of $\Delta$. But since his own bid has no bearing on the likelihood of this possibility, the $N = 1$ scenario does not enter his decision making.
where \( g_M(b) \) is the density corresponding to the distribution of the highest rival bid. These two equations establish that the form of the inverse bid function can be inferred, so long as \( \lambda \) and the parent distribution of bids \( G_B \) can be identified. If \( \xi \) and \( G_B \) are known, then \( F_V \) is also known since \( F_V(v) = G_B[\xi(b)] \). Thus, structural identification now hinges crucially on recovering the distribution of \( N \) from data. This we address in the next subsection.

4.2 Identifying Exogenous Participation Rates

The set of observables available from an electronic platform like eBay, however, presents several challenges. We assume the econometrician does not observe all bids, from which the parent distribution \( G_B \) could be easily estimated. Rather, he observes a selected subset, being only the second order statistic from a sample of stochastic size. Note that on eBay, the maximal bid is only observed at auctions where the first-price rule was triggered, roughly one in every five in our data. There may also be reasons to doubt whether the third-highest and other observed bid submissions are generated by equations (9)–(10), as we discuss below in Section 5.1. If the researcher has access to a broader subset of bids than what we describe here, then the estimator resulting from our identification strategy will have more statistical power. But in order to be conservative on the capabilities of our proposed method, we use only the highest losing bid (the second order statistic).

Furthermore, the econometrician does not observe \( N \), the total number of bidders, directly, but rather, she sees only the number of bidders who submit tenders to the server, call it \( \tilde{N} \). This number we argue is merely a lower bound: some bidders who watch an item with intent to bid may find that their planned bid was surpassed before they get around to submitting it. Thus, the list of actual participants is passed through a natural “filter process” which withholds some of them from view before the econometrician is allowed to see the list of observed participants.

4.2.1 The Filter Process

Underlying this idea is an assumption of simple intra-auction dynamics in the sense that ordering of bidders’ submission times is random\(^{16}\). We assume that, prior to the auction, nature generates a list of bidders, indexed \( \{1, 2, \ldots, n\} \), where \( n \) follows known distribution \( \rho_N(n; \lambda) \), but each bidder is confined to an enclosed cubicle so that she cannot observe the realization of \( n \). For each \( i \) nature generates an iid private valuation \( v_i \) from \( F_V \). Each bidder then formulates his strategic sealed bid, \( \beta(v_i) \), and waits for nature to come collect it from him. Nature then visits each bidder in order of his index within the list, to record his bid tender. If however, the highest two bids from previous tenders both exceed \( \beta(v_i) \), then nature skips bidder \( i \)’s submission, discarding it as if it never happened. At the conclusion, nature reports to the econometrician the number of recorded bidders.

Within this simple environment, for each bidder \( i \geq 3 \), whenever the second-highest bid from among \( \{\beta(v_1), \ldots, \beta(v_{i-1})\} \) exceeds \( \beta(v_i) \), then \( i \) will not appear to have participated, even though she may have intended, \( ex \ ante \), to submit a bid. Observing only a subset of potential bidders presents a challenge to the econometrician, but by explicitly modeling the filter process we can overcome it and still identify \( \lambda \) from observed lower bounds \( \tilde{n} \). Moreover, if one is interested solely in modeling

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\(^{16}\text{Note that our assumption allows us to be agnostic concerning how agents individually decide on bidding early or late within the auction. We do rule out, however, the possibility of coordination on some observable aspect of the auction so that the relative ordering of their bid submission times is systematic, rather than random.}\)
auction participation, then the filter process can be further simplified. Since equilibrium bidding is monotone and bidder visibility depends on the relation between rank ordering of bids and bid timing, we can re-cast the filter process in equivalent terms where nature endows each bidder \( i \) with an iid quantile rank \( Q_i \sim \text{Uniform}(0,1) \), rather than a private valuation. Nature then walks through the (unordered) list \( q = \{q_1, q_2, \ldots, q_n\} \) (for a given value of \( n \)) and reports the following number to the econometrician:

\[
\tilde{n} = \begin{cases} 
  n & \text{if } n \leq 2, \\
  2 + \sum_{i=3}^{n} 1(q_i^* < q_i) & \text{if } n \geq 3,
\end{cases}
\]

where \( q_i^* \) is the second-highest from among \( \{q_1, q_2, \ldots, q_{i-1}\} \), and \( 1(\cdot) \) is an indicator function. This observation facilitates simulation of the filter process without knowing \( F_V \) \textit{ex ante}, which in turn makes it possible to separate identification/estimation of \( \lambda \) and \( F_V \).

From the above description, it is easy to see that the distribution of \( \tilde{N} \) for a given value of \( N \) is invariant to changes in \( \lambda \). Therefore, the filter process can be repeatedly simulated for arbitrary hypothetical values of \( n \), and we can treat the conditional probabilities \( \Pr(\tilde{n}|n) \) as known quantities for arbitrary \((\tilde{n}, n)\) pairs. Since \( \tilde{N} \) is observable, we can in turn treat its probability mass function, denoted \( \tilde{\rho}_N(\tilde{n}) \), as an observable since it can be directly estimated from data. Moreover, by the law of total probability, we have the following relationship which establishes identification of the exogenous participation process:

\[
\tilde{\rho}_N(\tilde{n}) = \sum_{n=0}^{\infty} \Pr(\tilde{n}|n) \times \rho_N(n; \lambda). \tag{12}
\]

Since the above argument does not rely on an assumption that \( \lambda \) is finite-dimensional, our identification result is, in fact, \textit{nonparametric}. In other words, observed participation together with our model of the filter process are enough to identify the distribution of \( N \) on their own, without appealing to specific functional form assumptions on \( \rho_N \). In practice, additional parametric assumptions (for example, specifying \( N \) as Poisson) may provide benefits such as statistical efficiency or numerical tractability, but they are not fundamentally necessary from an identification standpoint. Below in our empirical application, we shall see that the parametric generalized Poisson model (Consul and Jain [1973]), a two-parameter distribution, provides a remarkably tight fit to the data, leaving very little room (given our finite sample) for additional improvements to fit through functional form relaxations.

### 4.3 Identifying \( F_V \)

With the above result in hand, nonparametric identification of the remainder of the structural model is straightforward. Let \( H(b) \) denote the \textit{ex ante} distribution of the highest losing bid within an auction (the second order statistic from a sample of stochastic size \( N \)). Once again, since the highest losing bid is observable, we can treat \( H \) as observable since it can be directly estimated from data. It relates to the parent distribution of bids via the following mapping

\[
H(b) = \sum_{n=2}^{\infty} \frac{\rho_N(n; \lambda)}{1 - \rho_N(0; \lambda) - \rho_N(1; \lambda)} \left( G_B(b)^n + nG_B(b)^{n-1} \left[1 - G_B(b)\right] \right). \tag{13}
\]

For fixed \( \lambda \) this mapping is a bijection for each \( b \) in the bid support. Therefore knowing \( \lambda \) and \( H \) implies that the parent distribution of bids is identified.
With the above arguments in place, we can state formally our identification result. In order to fix notation, we define a model as a set of (potentially nonparametric) arrival probabilities \( \rho_N(n; \lambda) \), \( n = 0, 1, 2, \ldots \) and a private value distribution \( F_V \). Moreover, we assume that the observables available to the econometrician include \( H(b) \), the distribution of the highest losing bid, and \( \hat{\rho}_N(n) \), \( n = 0, 1, 2, \ldots \), the probability mass function for observed participation \( \hat{N} \) (which is a lower bound on actual participation).

**Proposition 4.1.** Under the assumptions of Section 4.2.1, the bidding model \( \left( \rho_N(n; \lambda)^\infty_{n=0}, F_V \right) \) is nonparametrically identified from the observables \( \left( H(b), \hat{\rho}_N(n)^\infty_{n=0} \right) \).

**Proof:** Equation (12) establishes identification of the nonparametric bidder arrival probabilities \( \{\rho_N(n; \lambda)\}^\infty_{n=0} \) from the distribution of observed lower bounds under the model of the filter process described in Section 4.2.1. Given known arrival probabilities, the bijectivity of the mapping (13) establishes that the parent distribution of bids \( G_B \) is nonparametrically identified from the observables.

This in turn means that we can now construct \( G_M \) from equation (7) using the parent bid distribution and the bidder arrival probabilities. Moreover, if \( G_M \) is known then we can re-construct the inverse bidding function \( \xi(b) = \beta^{-1}(v) \) from equations (9) and (10). Finally, this also implies that the private value distribution \( F_V \) is nonparametrically identified through the relationship \( F_V(v) = G_B(\beta(v)) \). □

### 4.4 Model Extension: Nonconstant \( \Delta \)

Having presented the basic identification strategy, we now extend our model to handle a final challenge. Above, we assumed that the bid increment \( \Delta \) is constant, but at most real-world EAs \( \Delta \) changes at pre-determined transition points. For simplicity, consider a single transition point, as the extension generalizes straightforwardly for two or more transition points.

Suppose we have a transition point, denote it \( b^* \), and two bid increments: \( \Delta_1 \) applies for bids \( b \in [v^*, b^*) \), and \( \Delta_2 \) applies for bids \( b \geq b^* \).\(^{18}\) Through a minor adjustment to existing theoretical results by [Hickman 2010] it follows that the equilibrium bid function \( \beta \) is still continuous in this case\(^{19}\). Given continuity, Hickman’s Proposition 3.3 still establishes that the right-hand and left-hand derivatives exist and must be the same everywhere but at \( v^* \). Therefore, to characterize the

\(^{17}\)Note that \( H(b) \), the distribution of the highest of \( (N - 1) \) draws, and \( G_M(b) \), the distribution of the second highest of \( N \) draws, are not the same distribution. To see why, consider the task of repeatedly simulating order statistics based on fixed \( N \). For each simulation, \( N \) iid realizations are generated from some distribution and stored in an unordered list. To compute the value of the second highest from the list of \( N \), we find the maximum, discard it, and take the maximum of the remaining \( (N - 1) \) draws; that is, in this case, the highest draw from the original list of \( N \) is discarded with certainty. To compute the highest draw from a sample of size \( (N - 1) \), we merely discard the first observation of the unordered list and then find the maximum of the remaining \( (N - 1) \) draws. But this procedure only discards the maximum of the original \( N \) draws with probability \( (1/N) \). Therefore, the two random variables cannot have the same distribution.

\(^{18}\)We shall also assume for simplicity that \( v + \Delta_1 < b^* \).

\(^{19}\)To see why, let \( v^* \) denote the type such that \( \beta(v^*) = b^* \). Lemma 3.3.1 of [Hickman 2010] still establishes continuity on the set \( [v, v^*) \cup (v^*, \bar{v}] \). The proof of the lemma proceeds by first arguing for left continuity and then for right continuity; the intuition being that if either is violated then types just to the right of the discontinuity could profit by reducing their bids. If we suppose that \( \beta \) is discontinuous at \( v^* \) and then merely insert \( \Delta_1 \) into Hickman’s left-continuity argument, and \( \Delta_2 \) into the right-continuity argument, then the same contradictions follow and the logic of the proof holds at \( v^* \) as well. Thus, the equilibrium must be continuous even at the transition point.
equilibrium on the interval \([v, v^*]\) we solve out the same first-order conditions (FOCs) as before for the decision problem under \(\Delta_1\). Then, the continuity property gives us a boundary condition for the interval \([v^*, \bar{v}]\), where we can solve for the equilibrium using the FOCs for a bidder’s decision problem under \(\Delta_2\). Thus, if we segment the function by \(\beta_0 : [v, \Delta_1] \to \mathbb{R}, \beta_1 : (\Delta_1, v^*) \to \mathbb{R}, \) and \(\beta_2 : [v^*, \bar{v}] \to \mathbb{R},\) we get the following set of ordinary differential equations:

\[
\beta(v) = v,
\]

\[
\beta'(v) = \begin{cases} 
\beta_0'(v) & v < v_\Delta, \\
\beta_1'(v) & v_\Delta < v < v^*, \text{ and} \\
\beta_2'(v) & v \geq v^*,
\end{cases}
\] (14)

where

\[
\beta_0'(v) = \frac{[v - \beta_0(v)] f_M(v)}{F_M(v)},
\]

\[
\beta_1'(v) = \frac{[v - \beta_1(v)] f_M(v)}{F_M(v) - F_M(\xi_1 [\beta_1(v) - \Delta_1])}, \quad \text{and}
\]

\[
\beta_2'(v) = \frac{[v - \beta_2(v)] f_M(v)}{F_M(v) - F_M(\xi_2 [\beta_2(v) - \Delta_2])}.
\] (17)

From these, we can see why \(\beta\) must be kinked at \(v^*\): since \(F_M \in C^2\) and \(\beta\) is continuous, the left-hand derivative \(\beta_1'(v^*)\) and the right-hand derivative \(\beta_2'(v^*)\) cannot be equal if \(\Delta_2 \neq \Delta_1\).

Note, too, that since \(F_v\) is \(C^2\) (by assumption), the model places restrictions on the derivatives of \(G_B\) at the transition point \((v^*, b^*)\). To see why, we first display the piecewise-defined inverse bid function

\[
\xi_0(b) = b + \frac{G_M(b)}{g_{M_1}(b)},
\]

\[
\xi_1(b) = b + \frac{G_M(b) - G_M(b - \Delta_1)}{g_{M_1}(b)}, \quad \text{and}
\]

\[
\xi_2(b) = b + \frac{G_M(b) - G_M(b - \Delta_2)}{g_{M_2}(b)},
\] (20)

where the notation \(g_{M_1} : [b, b^*) \to \mathbb{R}\) and \(g_{M_2} : [b^*, \bar{b}] \to \mathbb{R}\) will be used to distinguish between the left-hand and right-hand derivatives of \(G_M\) at the transition point \(b^*\), and

\[
\xi(b) = \begin{cases} 
\xi_0(b) & \text{if } b \leq \Delta_1, \\
\xi_1(b) & \text{if } \Delta_1 < b < b^*, \\
\xi_2(b) & \text{if } b^* \leq b.
\end{cases}
\] (21)

Now, recall that \(F_v(v) = G_B[\beta(v)]\); therefore, \(F_v\) being continuous at \(v^*\) implies

\[
\lim_{y \to b^*} G_B(y) = \lim_{y \to b^*} G_B(y) = G_B(b^*)
\] (22)

and \(\beta_1^{-1}(b^*) = \beta_2^{-1}(b^*)\) as well. Substituting equations (19) and (20) into this expression and rearranging gives the following

\[
\frac{G_M(b^*) - G_M(b^* - \Delta_1)}{g_{M_1}(b^*)} = \frac{G_M(b^*) - G_M(b^* - \Delta_2)}{g_{M_2}(b^*)}.
\] (23)
Intuitively, (23) says that the bid shading factor must be the same on either side of the transition point in order for $\beta$ to be continuous. Equations (22) and (25) follow directly from continuity of $F_V$: they convey that $G_B$ must be continuous but kinked at $b^*$, and they indicate the size of the discontinuity in $g_B$ at the transition point.

Likewise, knowing that $F_V$ has two continuous derivatives at $v^*$ gives us two additional conditions on the second and third derivatives of $G_B$. Note that $f_V(v^*) = g_B[\beta(v^*)]'/\beta'(v^*) = g_B(b^*)/\xi'(b^*)$. In order for $f_V(v^*)$ to exist, the left-hand and right-hand derivatives must both agree at $b^*$, or

$$\frac{g_B(b^*)}{\xi'(b^*)} = \frac{g_B(b^*)}{\xi'(b^*)},$$  \hspace{1cm} (24)

where, from taking derivatives in equations (19) and (20) we get

$$\xi_j''(b^*) = 1 + \frac{g_{M_2}(b^*) - g_{M_1}(b^* - \Delta_j)}{g_{M_1}(b^*)} - \frac{G_M(b^*) - G_M(b^* - \Delta_j)}{g_{M_1}(b^*)^2} g_{M_j}'(b^*), \hspace{1cm} j = 1, 2. \hspace{1cm} (25)$$

Equations (24) and (25) pin down the size of the discontinuity in the second derivative of $G_B$ at $b^*$. Finally, note that $f_V''(v^*) = g_B[\beta(v^*)]/\beta'(v^*)^2 + g_B[\beta(v^*)]$$\beta''(v^*)$. Using properties of inverse functions and their higher order derivatives, this can be re-written as a condition equating the left- and right-hand derivatives:

$$\frac{g_B'(b^*)}{\xi'_1(b^*)^2} - g_B(b^*) \left[ \frac{\xi''_1(b^*)}{\xi'_1(b^*)^3} \right] = \frac{g_B'(b^*)}{\xi'_2(b^*)^2} - g_B(b^*) \left[ \frac{\xi''_2(b^*)}{\xi'_2(b^*)^3} \right],$$  \hspace{1cm} (26)

where

$$\xi_j''(b^*) = \left[ \frac{g_{M_1}(b^*) - g_{M_1}(b^* - \Delta_j)}{g_{M_1}(b^*)} \right] - 2 \left[ \frac{g_{M_1}(b^*) - g_{M_1}(b^* - \Delta_j)}{g_{M_1}(b^*)^2} \right] g_{M_j}'(b^*) + 2 \left[ \frac{G_M(b^*) - G_M(b^* - \Delta_j)}{g_{M_1}(b^*)^2} \right] g_{M_j}'(b^*)^2 - \left[ \frac{G_M(b^*) - G_M(b^* - \Delta_j)}{g_{M_1}(b^*)^2} \right] g_{M_j}'(b^*)^2, \hspace{1cm} j = 1, 2. \hspace{1cm} (27)$$

Equations (26) and (27) pin down the size of the discontinuity in the third derivative of $G_B$ at $b^*$.

At the end of the day though, equations (18–20) establish nonparametric identification of the bidding model with transition points as well, with the intuition being nearly identical to what it was before. The principal challenge that transition points will pose is on the implementation of an estimator, which we discuss in the following section.

### 4.5 Two-Stage Estimator

In this section, we propose a simple estimator for the bidder arrival process, as well as a sieve-type estimator of the private value distribution $F_V$ based on B-splines. Our choice of B-splines is motivated partly by their ability to accommodate elements of the empirical model flexibly, such as the kink in the bid function at $b^*$, which alternative methods such as kernel smoothing or global polynomials cannot easily do.

To fix notation, let $\{y_t, n_t\}_{t=1}^T$ denote a sample of auctions; for each we observe the highest losing bid, $y_t$, and the number of observed participants, $n_t$. Although we have demonstrated nonparametric identification of the bidder arrival process, we shall focus our discussion on estimation of a model
in which $\lambda$ is finite-dimensional. In our empirical application, we shall specify the distribution of $N$ as generalized Poisson, or

$$\rho_n(n; \lambda) = \frac{\lambda_1(\lambda_1 + n\lambda_2)^{n-1}}{n!} e^{-\lambda_1 - n\lambda_2}, \quad 0 < \lambda_1, \quad 0 \leq \lambda_2 < 1.$$  

Later, we show that this two-parameter model leaves little room for further improvements to data fitting through more flexible functional forms: the generalized Poisson model is able to generate a distribution over the observables $\tilde{\rho}_N(\tilde{n}; \lambda)$ that lay within the nonparametric 95% confidence bounds of the empirical distribution of $\tilde{N}$. For the present discussion, however, it suffices to consider any known parametric family $\rho_N(n; \lambda)$ which is indexed by a finite-dimensional parameter vector, $\lambda$. Where appropriate, we shall discuss further concerns and complications that would arise if the parametric assumptions are relaxed.

### 4.5.1 First Stage: Estimating $\lambda$

We begin by constructing a simulation routine that mimics the filter process and allows us to estimate $Pr(\tilde{n}|n)$. Fix a finite upper bound $N$ and for each $n \in \{0, 1, 2, \ldots, N\}$ simulate $s = 1, 2, \ldots, S$ auctions wherein a list of $n_s$ independent (unordered) quantile ranks $q_{ns} = \{q_{1s}, \ldots, q_{ns}\}$ are drawn from the Uniform(0, 1) distribution. For each such list, we then compute $\tilde{n}_s$ according to the definition in equation (11). For each $n$, the simulated conditional frequencies are then computed as

$$\hat{Pr}(\tilde{n}|n) = \frac{1}{S} \sum_{s=1}^{S} I(\tilde{n}_s = \tilde{n}).$$

Note that the simulation frequencies are zero whenever $n < \tilde{n}$.

In a slight change of notation, we now redefine the model-generated frequencies of $\tilde{N}$ as

$$\hat{\rho}_N(\tilde{n}; \lambda) = \sum_{n=0}^{N} \hat{Pr}(\tilde{n}|n) \times \rho_N(n; \lambda), \quad (28)$$

and define the empirical frequencies as $\hat{\rho}_N(\tilde{n}) \equiv \frac{1}{T} \sum_{t=1}^{T} I(\tilde{n}_t = \tilde{n})$. Finally, letting $\{\tilde{n}_1, \tilde{n}_2, \ldots, \tilde{n}_L\}$ denote the complete set of unique observed values of $\tilde{n}$ in the data, we define a nonlinear least squares (NLS) estimator as the optimizer of the following objective function:

$$\hat{\lambda} = \underset{\lambda \in \mathbb{R}^k}{\text{argmin}} \left\{ \sum_{l=1}^{L} \left[ \hat{\rho}_N(\tilde{n}_l; \lambda) - \hat{\rho}_N(\tilde{n}_l) \right]^2 \right\} \quad (29)$$

In words, the estimate $\hat{\lambda}$ is chosen to make the model-generated frequencies of observed bidders as close to the empirical frequencies as possible.

As a practical matter, specifying $N$ and $S$ involves a trade-off between computational cost and numerical accuracy. For the former, we judged $N = 100$ to be a sensible choice for several reasons. An analogous nonparametric estimator could be similar, but with additional complications. Reverting back to the case where $\lambda = \{\lambda_0, \lambda_1, \lambda_2, \ldots\}$, $\lambda_n = Pr(N = n)$, is infinite dimensional. The main challenge now is that only finitely many elements of $\lambda$ can be estimated with finitely much data. Therefore, for finite sample size $T$, we begin by
First, note that the Poisson probability $\rho_N(100; 40) \approx 7.315 \times 10^{-16}$, so the terms truncated out of the infinite sum in (28) will be on or below the order of machine precision whenever $N$ is roughly Poisson with a parameter that is weakly less than 40. Second, while eBay auctions are known for high participation rates, 40 is still quite a large number. For example, in our empirical application with laptop computer data we observed a maximum of 11 participants in any given auction. Thus, our choice of $\bar{N} = 100$ ensures that the finite truncation which we must impose on equation (11) will have no discernible effect for a wide array of eBay datasets. In turn, a relatively low truncation point allowed us to simulate a large number of auctions, or $S = 10^{10}$, which delivers at least 5 (and up to 6) digits of accuracy in each cell of the matrix $\hat{\Pr}(\tilde{n}|n)$. In other words, if the conditional probabilities in equation (28) above are expressed as percentages, then our simulation estimates are accurate to within one one-thousandth of a percentage point. One advantage of our approach is that these need only be simulated once and, given our choice of $\bar{N}$ and $S$, can be re-used with any dataset for which the average participation rate is less than 40. A copy of the matrix of simulated conditional probabilities $\hat{\Pr}(\tilde{n}|n)$, $n = 1, \ldots, 100$, as well as MATLAB code implementation of the estimator, is available from the authors upon request.

4.5.2 Second Stage: B-spline Estimator of $F_V$

Our identification argument above states that knowledge of $G_M$ is sufficient to recover the private value distribution. A seemingly natural way to proceed then would be to estimate $G_B(b)$ directly from the observables and then reverse-engineer $F_V$. Implementing this approach can be difficult. From the discussion above, it is clear that the estimator $\hat{G}_B$ must respect the system of nonlinear equations (22)–(27) in order to ensure that the resulting estimator $\hat{F}_V \in C^2$. Moreover, $\hat{G}_B$ must also ensure that the mappings implied by equations (18)–(20) are monotone in order to be con-

Choosing an upper bound, $\bar{N}_T < \infty$, after which we restrict $\lambda_n = 0$ whenever $(n - 1) > \bar{N}$ and define the following:

$$\{\hat{\rho}_N(n; \lambda)\}_{n=0}^{\bar{N}_T} = \arg\min_{\lambda \in \mathbb{R}_+} \left\{ \sum_{i=1}^L \left[ \hat{\rho}_N(\hat{r}_i; \lambda) \right] \right\}$$

subject to $\sum_{n=0}^{\bar{N}_T} \hat{\rho}_N(n; \lambda) = 1.$

Choice of $\bar{N}_T$ involves the usual variance-bias tradeoff. For larger $\bar{N}_T$, less bias arises from setting high-order elements of $\lambda$ to zero, but as $(\bar{N}_T/T)$ gets large the variance of the estimator will increase as well. A second challenge involves specifying the rate at which $\bar{N}_T$ should optimally grow as $T \to \infty$. However, our empirical application suggests strongly that solving these problems would produce little benefit above available finite-dimensional parametric methods, so we do not address them here.

21The probabilities $\Pr(\tilde{n} | n)$ are computed as the sample mean of a Bernoulli random variable $1(\bar{N} = \tilde{n}|N = n)$. Since the sample mean is known to converge at rate $\sqrt{S}$, our simulation error is on the order of $1/\sqrt{10^{10}} = 10^{-5}$, but may be even less. Simulation was performed in 100 blocks of $10^8$ simulations. As a check on accuracy, these can be used to compute 100 different estimates of the conditional probability matrix. Taking standard deviations across all 100 estimates for each $(\tilde{n}, n)$ pair (and excluding pairs that trivially render a conditional probability of zero), we get mean and maximum standard deviations of $1.47 \times 10^{-3}$ and $5.55 \times 10^{-3}$, respectively. Of course, averaging across these 100 estimates (as our final conditional probability matrix does) should further improve the precision for each $(\tilde{n}, n)$ pair, reducing the mean and maximum standard deviations further to roughly $1.47 \times 10^{-4}$ and $5.55 \times 10^{-4}$, respectively.

22In all, we simulated the filter process for $10^{12}$ separate auctions. Computation was performed in parallel using a cluster of 150 MATLAB workers for 310 hours. We periodically reset the seed so as to avoid surpassing the periodicity of the random number generator.
sistent with equilibrium bidding. These requirements rule out kernel density estimators, in favor of sieve estimation where a finite-dimensional parametric restriction is imposed and then gradually relaxed as the sample size increases. Even then, parameterizing $G_B$ and then optimizing some empirical criterion function subject to the mentioned constraints above, along with enforcing monotonicity and boundary conditions of $G_B$ itself, poses a formidable numerical challenge. The resulting constraint set becomes very complicated and highly nonconvex, making it difficult to compute admissible initial guesses and converge to global optima afterward.

We propose an alternative approach whereby we directly parameterize the private value distribution $F_V$ as a flexible B-spline function which is twice continuously differentiable by construction. Still, since $G_B(b) = F_V[\xi(b)]$, building and optimizing an empirical criterion function of the observables—order statistics of bids—requires finding the solution to a differential equation based on $F_V$ given in equations (15)–(17), or equivalently, equations (18)–(20). To solve this piece of the puzzle, we employ the Galerkin method which is commonly used to solve differential equations numerically in physical sciences applications. This approach involves parameterizing the inverse bid function $\xi$ as a B-spline as well, and afterward enforcing its adherence to the conditions of the boundary value problem defined by the equilibrium FOCs. This is done by defining a grid of points on the domain (where the number of grid points is at least as large as the number of free parameters in the B-spline function), and then augmenting the estimator objective function with extra terms that penalize it for deviations from equilibrium. Our approach of augmenting an extremum estimator with the Galerkin method has the added benefit of being relatively inexpensive to compute: rather than repeatedly updating parameter values for $F_V$ and then solving a differential equation in sequence, modeling $\xi$ as a B-spline allows us to fit $F_V$ to the data while simultaneously adjusting $\xi$ to conform to the equilibrium conditions required by theory. We explain our approach concretely below, but first a brief word on B-splines is in order.

B-splines have many attractive properties that are well-adapted to our application. First, they behave identically to piecewise splines and so, by virtue of the Stone–Weierstrass theorem, they can fit a broad class of nonparametric curves to arbitrary precision, given a fine enough partition of the domain. They also retain attractive local properties of piecewise splines: being locally low-dimensional they are numerically stable; adjusting parameters to improve model fit at one point will have little or no effect on behavior of the B-spline function at points outside of a relatively small neighborhood. This is in contrast to global polynomials, where adjusting parameters to improve fit at one point may have drastic consequences for the behavior of the function at points far away.

On the other hand, like standard polynomial bases, B-spline basis functions are globally defined, making their functional values, derivatives, and integrals less cumbersome to compute than piecewise splines. In addition, B-splines are more flexible than global polynomials in that they can also naturally accommodate a finite set of interior points where the underlying nonparametric curve is known to be kinked or even discontinuous. This property will be particularly useful in dealing with transition points where the bid increment discretely shifts. For a brief, but more detailed, primer on how we constructed our B-spline functions, see the Appendix.

To outline our estimator, let $j_J = \{j_1 < j_2 < \cdots < j_J < j_{J+1}\}$ denote a set of unique “knots”

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23See Zienkiewicz and Morgan [2006] for details on Galerkin method approximation. Hubbard and Paarsch [2014] discuss and compare various ways in which researchers have solved the differential equation(s) characterizing equilibrium behavior at auctions.
which partition the private value support into $J$ subintervals, with $j_1 = v$ and $j_{J+1} = \bar{v}$, and let $k_K = \{k_1 < k_2 < \cdots < k_K < k_{K+1}\}$ denote a partition of the bid support into $K$ subintervals, with $k_1 = b = v$ and $k_{K+1} = \bar{b}$. These knot vectors uniquely define a set of $(J + 3)$ and $(K + 3)$ 4th-order (cubic) B-spline basis functions $\{V_j(\cdot)\}_{j=1}^{J+3}$ and $\{B_k(\cdot)\}_{k=1}^{K+3}$, where $V_j : [v, \bar{v}] \to \mathbb{R}$ for each $j$ and $B_k : [b, \bar{b}] \to \mathbb{R}$ for each $k$.

The basis functions are uniquely defined from their respective knot vector through the Cox–de Boor recursion relation formula with concurrent boundary knots (see the Appendix for details). The resulting basis behaves the same as a set of piecewise cubic splines that are constrained to be $C^2$ at the endpoints of the subintervals defined by the knot vector. Each basis function is $C^2$ everywhere on the global domain, with the $C^2$ conditions at the knots being built into the recursion formula. Moreover, each one is nonzero on at most four of the $K$ subintervals (though some are nonzero on fewer than four), and exactly four of the cubic basis functions are nonzero on each of the $K$ subintervals, which explains the name “4th-order B-spline,” and also why there are $(K + 3)$ basis functions in total. Figure 4 depicts a Basis on the interval $[-1, 1]$ partitioned by a uniform knot vector with $K = 3$.

Letting $\alpha = \{\alpha_1, \ldots, \alpha_{K+3}\} \subset \mathbb{R}^{K+3}$ and $\mu = \{\mu_1, \ldots, \mu_{J+3}\} \subset \mathbb{R}^{J+3}$ denote sets of weights, we can now parameterize $F_V$ and $\xi$ as

$$\hat{F}_V(v; \mu) = \sum_{j=1}^{J+3} \mu_j V_j(b),$$

$$\hat{\xi}(b; \alpha) = \sum_{k=1}^{K+3} \alpha_k B_k(b).$$

Given this parameterization, the bid distribution is $\hat{G}_B(b; \mu, \alpha) = \hat{F}_V(\hat{\xi}(b; \alpha); \mu)$, and, in a slight
change of notation, we can redefine the model-generated distribution of the highest losing bid as

\[
H(b; \mu, \alpha, \lambda) = \sum_{n=2}^N \rho_n(n|N > 2; \hat{\lambda}) \left( \hat{G}_B(b; \mu, \alpha)^n + n\hat{G}_B(b; \mu, \alpha)^{n-1} \left[ 1 - \hat{G}_B(b; \mu, \alpha) \right] \right).
\]  

(31)

In what follows, for notational parsimony we shall take the dependence on the first-stage estimate \( \lambda \) as given, and drop it as an argument unless additional specificity is needed. This function can now be directly compared to its empirical analog as \( \hat{H}(b) \equiv \frac{1}{T} \sum_{t=1}^T I(y_t > b) \), in order to build a criterion function that will form the basis of an estimator. While optimizing the empirical criterion function, we must also impose the equilibrium conditions to ensure that \( \hat{F}_V \) and \( \hat{\xi} \) will be jointly consistent with the theory model. In other words, our final estimate for \( \hat{\xi} \) must constitute a valid solution to the boundary value problem defined in (14). In order to accomplish this, we specify a uniform grid of checkpoints \( \{b_i\}_{i=1}^L \subset [b, \bar{b}] \), with \( L \geq K + 3 \), and we introduce three additional parameters, \( \varepsilon_0, \varepsilon_1 \), and \( \varepsilon_2 \), as well as a residual function based on equations (18)–(20):

\[
R(b_i; \mu, \alpha) = \begin{cases} 
(\hat{\xi}(b_i; \alpha) - b_i) \hat{g}_M(b_i; \mu, \alpha) - \hat{G}_M(b_i; \mu, \alpha), & b_i \leq b + \Delta_1, \\
(\hat{\xi}(b_i; \alpha) - b_i) \hat{g}_M(b_i; \mu, \alpha) - \hat{G}_M(b_i - \Delta_1; \mu, \alpha), & b + \Delta_1 < b_i < b^*, & b^* < b_i, \\
(\hat{\xi}(b_i; \alpha) - b_i) \hat{g}_M(b_i; \mu, \alpha) + \hat{G}_M(b_i - \Delta_2; \mu, \alpha), & b^* \leq b_i,
\end{cases}
\]

where \( \hat{G}_M(b_i; \mu, \alpha) = \sum_{n=2}^N \rho_n(n|N \geq 2; \hat{\lambda}) \hat{G}_B(b_i; \mu, \alpha)^{n-1} \). With that, we can now define a constrained NLS estimator as

\[
(\hat{\mu}, \hat{\alpha}) = \arg\min_{(\mu, \alpha) \in \mathbb{R}^{L+K+6} \times \mathbb{R}^3} \left\{ \sum_{i=1}^T \left[ H(x_i; \mu, \alpha, \hat{\lambda}) - \hat{H}(x_i) \right]^2 + P_0 \varepsilon_0 + P_1 \varepsilon_1 + P_2 \varepsilon_2 \right\}
\]

subject to

\[
\hat{F}_V(\nu; \mu) = 0, \quad \hat{F}_V(\bar{\nu}; \mu) = 1, \\
\hat{f}_V(\nu; \mu) > 0, \quad \nu \in [\nu, \bar{\nu}], \\
\hat{\xi}(k_1; \alpha) = \nu, \quad \hat{\xi}(k_{K+3}; \alpha) = \bar{\nu}, \\
\hat{\xi}'(b; \alpha) > 0, \quad b \in [\nu, \bar{\nu}], \\
N \left[ (R(b_i; \mu, \alpha))_{i=1}^L \right] \leq \varepsilon_0, \quad b_i \leq \nu + \Delta_1, \\
N \left[ (R(b_i; \mu, \alpha))_{i=1}^L \right] \leq \varepsilon_1, \quad \nu + \Delta_1 < b_i < b^*, \\
N \left[ (R(b_i; \mu, \alpha))_{i=1}^L \right] \leq \varepsilon_2, \quad b^* \leq b_i,
\]

where \( N[\cdot] \) is a norm function such as the \( L_{\infty} \)-norm (the sup-norm), \( \sup_{i \in \{1, 2, \ldots, L\}} |R(b_i; \mu, \alpha)| \), or the \( L^2 \)-norm, \( \left( \sum_{i=1}^L R(b_i; \mu, \alpha)^2 \right)^{1/2} \), and \( (P_0, P_1, P_2) \) is a set of pre-specified penalty parameters.\(^{24}\)

Intuitively, the residual function equals zero when the private value distribution \( \hat{F}_V \) together with the numerical bid function \( \hat{\xi} \) exactly conform to the first-order conditions of a bidder’s decision problem, and the constraint that \( \hat{\xi}(k_1; \alpha) = \nu \) enforces the boundary condition. Thus, the

\(^{24}\)In more general settings, the estimation method proposed here allows for an additional dimension of flexibility: the least squares component of the objective function could be replaced with a log-likelihood function instead to produce an alternative estimator, provided that the regularity conditions for maximum likelihood estimation are satisfied.
vector \((e_0, e_1, e_2)\) controls the degree of numerical error in the approximated solution to the piecewise differential equation. Moreover, by fitting the parameterized distribution \(F_V\) to the data while simultaneously searching over parameter values for \(\hat{\xi}\) that conform to the first order conditions, given \(F_V\), we avoid the computational cost involved in repeatedly solving differential equations each time the objective function is evaluated, which can easily number in the tens of thousands. In this sense, our NLS estimator with Galerkin differential equation solution is in the spirit of the MPEC (Mathematical Programming with Equilibrium Constraints) method pioneered by Su and Judd [2012]: we choose the parameters of the parent distribution \(G_B\) so that the model-generated order statistic quantiles match the empirical order statistic quantiles as closely as possible, while penalizing the objective function for deviations from the equilibrium conditions in (18)–(20).

The main difference between our proposal here and the way Galerkin methods are typically implemented stems from how the equilibrium conditions are enforced. The most common implementation is to define the residuals \(R(b)\) on a grid of checkpoints \(\{b_1, \ldots, b_L\}\) with \(L = K + 2\). This, plus a boundary condition, produces a square system of \(K + 3\) equations in \(K + 3\) unknowns, which can be solved exactly on the grid of checkpoints (see, e.g., Hulme [1972] for further discussion). However, following this standard approach during estimator runtime would once again necessitate solving the differential equation once for every objective function evaluation, which can easily number in the tens of thousands. This is why we opt for the objective function penalization approach described above (with over-fitting, or \(L \geq K + 2\)), which allows for the parameters of \(\hat{F}_V\) and \(\hat{\xi}\) to move independently during runtime. This essentially means that the differential equation need only be solved once. As a check on output though, having obtained the point estimate \(\hat{F}_V(v; \hat{\mu})\), the researcher can easily verify the solution for \(\hat{\xi}(b; \hat{\mu})\) by re-solving the equilibrium boundary value problem in (18)–(20) using the standard Galerkin solution method based on a square system of nonlinear equations, holding \(\hat{\mu}\) fixed, and even with a finer knot vector than \(j_J\), if desired.

Before moving on, a final attractive property of B-splines is their ability to facilitate known shape restrictions on the latent function they parameterize. Note from Figure 4 that, by construction, the extremal basis functions are the only ones to attain nonzero values at the boundaries, and they both equal one exactly at their respective endpoints. Therefore, \(\hat{F}_V(v) = 0 \Rightarrow \mu_1 = 0\) and \(\hat{F}_V(\bar{v}) = 1 \Rightarrow \mu_{J+3} = 1\), which automatically reduces the number of free parameters (and hence computational cost) by two. Moreover, because B-splines are linear in parameters, we also have that \(\hat{f}_V(v) > 0 \Rightarrow \sum_{j=1}^{J+3} \mu_j V_j'(v) > 0\); since \(V_j'(v)\) is simply a third-order B-spline, which must be computed first in order to get \(V_j(v)\) (see the Appendix for details), enforcing bounds on derivatives involves relatively little extra computational cost over computing the basis functions themselves.

### 4.5.3 Accomodating Discontinuous Derivatives

Given the above derivations, we know that our parameterization of \(\hat{\xi}\) must be flexible enough to accommodate a continuous function with the first three derivatives being discontinuous at the transition point \(b^*\). We compute our basis functions using the Cox–de Boor recursion formula (see the Appendix) with concurrent boundary knots, which was developed with such a contingency in mind. Recall from above that we defined knot vector \(k_k = \{k_1 < k_2 < \cdots < k_K < k_{K+1}\}\) in bid space, which in turn defined \((K + 3), C^2\) basis functions. We further assume here that \(b^*\) lay strictly between two knot points, and for ease of discussion, let \(K > 2\) and \(k_2 < b^* < k_3\). We now modify
the knot vector by inserting 4 copies of $b^*$, so that we now get a knot vector with $(K + 5)$ elements

$$k'_K = \{k_1 < k_2 < b^* = b^* = b^* = < k_3 < \cdots < k_{K+1}\}.$$

This insertion will have several effects on the recursion formula, which we briefly outline here; the interested reader is once again directed to the Appendix for additional details. First, we increase the number of B-spline basis functions by four, and in particular, exactly eight basis functions will be nonzero on the subinterval $[k_2, k_3]$ now. To facilitate discussion it will be convenient to give special names to these eight basis functions, four of which will be positive to the left of $b^*$ and four of which will be positive to the right, see Figure 5. Due to knot stacking, of these eight basis functions two will be discontinuous at the transition point $b^*$—one will be left continuous, denote it $B^*_{-1}(\cdot)$, and the other will be right continuous, denote it $B^*_1(\cdot)$; two will have discontinuous first and second derivatives at $b^*$—one will have a left-continuous second derivative, denote it $B^*_{-2}(\cdot)$, and the other will have a right-continuous second derivative, denote it $B^*_2(\cdot)$; two will have discontinuous first derivatives at $b^*$—one will have a left-continuous first derivative, denote it $B^*_{-3}(\cdot)$, and the other will have a right-continuous first derivative, denote it $B^*_3(\cdot)$; the final two will be $C^2$—one being positive to the left of $b^*$, denote it $B^*_{-4}(\cdot)$, and the other being positive to the right, denote it $B^*_4(\cdot)$. Figure 5 depicts the B-spline basis resulting from inserting four copies of the number zero into the knot vector from Figure 4.

This method has exactly enough flexibility to fulfill the above conditions on $G_B$, which guarantee that $\hat{\xi}$ is consistent with a $C^2$ private value distribution $F_V$. Conceptually, this could also be done with a piecewise global polynomial series (for example, Laguerre or Chebyshev polynomials), but once again the locality properties of B-splines dramatically simplify things: fulfilling the equilibrium conditions at the transition point $b^*$ involves only comparisons between a small subset of model parameters, rather than all model parameters at the same time. To see why, we shall denote the weights on $B^*_1, B^*_2, B^*_3, B^*_4$, and $B^*_1, B^*_2, B^*_3, B^*_4$ as $\alpha^*_1, \alpha^*_2, \alpha^*_3, \alpha^*_4,$ and $\alpha^*_1, \alpha^*_2, \alpha^*_3, \alpha^*_4$, respectively, and we shall argue that allowing $\hat{\xi}$ to respect both equilibrium bidding

Figure 5: Fourth-Degree (Cubic) B-Spline Basis Functions with Concurrent Knots
and the $C^2$ property of $\hat{F}_V$ involves only a simple set of pairwise comparisons between $\alpha^*_i$ and $\alpha_i^*$, $i = 1, 2, 3, 4$.

In order to facilitate discussion, we shall denote the inverse bid function to the left of $b^*$ as $\hat{\xi}_1(b) \equiv \alpha_{-1}^* B_{-1}^*(b) + \alpha_{-2}^* B_{-2}^*(b) + \alpha_{-3}^* B_{-3}^*(b) + \alpha_{-4}^* B_{-4}^*(b)$ and we denote the inverse bid function to the right of $b^*$ as $\hat{\xi}_2(b) \equiv \alpha_1^* B_1^*(b) + \alpha_2^* B_2^*(b) + \alpha_3^* B_3^*(b) + \alpha_4^* B_4^*(b)$. It is easy to see from Figure 5 that, by construction,

$$B_{-3}^*(b^*) = B_{-4}^*(b^*) = B_3^*(b^*) = B_2^*(b^*) = 0, \quad \text{and} \quad B_{-1}^*(b^*) = B_1^*(b^*).$$

From this, it follows that $\hat{\xi}_1(b^*) = \alpha_{-1}^* B_{-1}^*(b^*)$ and $\hat{\xi}_2(b^*) = \alpha_1^* B_1^*(b^*)$, and equation (22) is satisfied if and only if $\alpha_{-1}^* = \alpha_1^*$. Furthermore, by construction of the B-spline basis functions, we also know

$$B_{-3}^*(b^*) = B_{-4}''(b^*) = B_4''(b^*) = B_3''(b^*) = 0,$$

from which it follows that $\hat{\xi}_1''(b^*) = \alpha_{-1}^* B_{-1}''(b^*) + \alpha_{-2}^* B_{-2}''(b^*)$, and $\hat{\xi}_2''(b^*) = \alpha_1^* B_1''(b^*) + \alpha_2^* B_2''(b^*)$. Since the relationship between the first two parameters is already pinned down, it follows that the second continuity condition (23) reduces to a simple restriction on the relationship between $\alpha_{-2}^*$ and $\alpha_2^*$. It also turns out that

$$B_{-4}''(b^*) = B_4'''(b^*) = 0,$$

so $\hat{\xi}_1''(b^*) = \alpha_{-1}^* B_{-1}'''(b^*) + \alpha_{-2}^* B_{-2}'''(b^*) + \alpha_{-3}^* B_{-3}'''(b^*)$, and $\hat{\xi}_2''(b^*) = \alpha_1^* B_1'''(b^*) + \alpha_2^* B_2'''(b^*) + \alpha_3^* B_3'''(b^*)$. Similar to before, since the relationships between the first two parameter pairs are already pinned down, the $C^1$ equilibrium condition (24)–(25) reduces to a single additional restriction on the relationship between $\alpha_{-3}^*$ and $\alpha_3^*$. Finally, once the first six parameters are known, the only remaining free parameters that enter the $C^2$ equilibrium condition (26)–(27) are $\alpha_{-4}^*$ and $\alpha_4^*$. From this it is clear that our B-spline formulation dramatically simplifies implementation of equilibrium constraints, say relative to global polynomials where equations (22)–(27) would be complicated functions of \textit{all model parameters at once}. As for implementation, in order to accommodate a transition point the researcher need only add to the constraint set in (33) the condition that $\alpha_{-1}^* = \alpha_1^*$ (i.e., $\hat{\xi}$ is continuous at $b^*$). The other three constraints discussed here are automatically enforced by the existing norm constraints which push the implied numerical residuals near $b^*$ toward zero.

One final point is worth discussion before moving on. One must remember that the purpose of constraining $(\alpha_{-1}^*, \ldots, \alpha_4^*)$ is to ensure proper behavior of the left- and right-hand derivatives at the transition point $b^*$, but these conditions are not meant to profoundly impact the behavior of the estimator at points outside a neighborhood of $b^*$. Once again, using properties of B-splines, this problem also has a simple remedy. For some relatively small number $\epsilon$, one can insert an additional set of knots $(b^* + \epsilon, b^* + 2\epsilon, \ldots, b^* + 8\epsilon)$. This extra inclusion ensures that $(\alpha_{-1}^*, \ldots, \alpha_4^*)$ completely determine behavior of $\hat{\xi}(b; \alpha)$ only on the subinterval $(b^*, b^* + \epsilon)$, with their influence vanishing as one approaches $b^* + 8\epsilon$ from the left. This is because the basis functions operating above $b^* + 8\epsilon$ will have no nonzero overlap with those determining the right-hand derivative at $b^*$, since cubic B-spline basis functions are nonzero on at most four subintervals.

## 5 Empirical Application

### 5.1 Data

We now consider a sample of laptop auctions collected between April and June of 2008 from the eBay website, which provides extensive information on item characteristics and bid histories. This
application highlights various challenges present in real-world data, including those addressed in the previous section. The largest seller during the data collection period was a firm by the name of CSR Technologies (henceforth CSRT), which purchased large quantities of second-hand laptops for resale on eBay. CSRT’s product line was mostly made up of Dell Latitude laptops, which come in several different configurations. Because laptop and auction characteristics can be important determinants of the price a computer will receive, we attempt to homogenize our sample as much as possible.

CSRT’s most common laptop configuration, comprising 733 total auction listings, included an Intel Pentium 4M processor with a clock speed of 1.4GHz, 512MB of RAM, 30GB of hard drive, a DVD-ROM optical drive, a 14.1-inch screen, and with the Windows XP Professional operating system installed. All laptops in this sample were described by the seller as either “refurbished” or “used,” and all corresponding auctions lasted for 24 hours. We restrict our sample to only laptops sold by CSRT, which ensures constant seller reputation, exchange and upgrade policy, flat shipping rate ($36), auction setup and so on. Moreover, CSRT used a template format for the display of each of its auction listings on eBay, making appearance during the auction uniform as well. After these restrictions, the only auction characteristics which vary are the bidding data themselves. This unique and homogeneous sample of auctions will allow us to abstract away from further complications (such as unobserved heterogeneity) as we develop an empirical methodology to tackle the already formidable challenges inherent in ideal eBay data.

5.2 Sample Paring

For each auction, we have detailed bidding information, including the timing and amount of each bid submitted to the eBay server, as well as the identity of the bidder who submitted it. Looking into the bid history allows us to calculate the number of observed bid submissions and the number of observed unique bidders. The first empirical challenge we encounter is that bidders may play different, potentially non-equilibrium actions at various points in the auction; for example, submitting low, cheap-talk bids early on, and then later bidding based on best-response calculations resembling those in a sealed-bid auction. Empirically, a significant fraction of observed bid amounts, particularly those submitted early on in the life of the auction, fall too far below realistic transaction prices to be taken seriously.

On the other hand, in the vast majority of cases, the top two bids arrive within the final 30 minutes of the auction. Table 3 shows that the average time to end when the winning bid arrives is 24.04 minutes, and the median time to end is 1.43 minutes. For the highest losing bid, the mean and median time remaining are 24.29 and 3.10 minutes, respectively. Previous empirical work on eBay has established these phenomena as empirical regularities and our data are no different.

5.2.1 Intra-Auction Dynamics

This discussion hints at a need to deal with the issue of intra-auction dynamics on eBay in some way. We adopt an approach similar to that of Bajari and Hortacsu [2003] by partitioning the auction run time into two stages. Taking the total auction time to be $T$, we treat the first stage as an open-
exit ascending auction played until \( T - \varepsilon \); the final (terminal) stage, of length \( \varepsilon < T \), is treated as a sealed-bid auction.\(^{26}\) During the first stage of the auction bidders submit cheap-talk bids which convey little, if any, information on the likely sale price which will result from the auction.

As in Bajari and Hortacsu [2003], bidders formulate a strategic sealed bid for submission during the terminal stage. We assume that this strategic bid ignores the outcome of the initial cheap-talk stage and the terminal stage is like a sealed-bid auction in the sense that bidders do not re-optimize their strategic bids in response to observed price-path dynamics during the terminal stage. Bidders may, however, submit these bids directly to the server to take advantage of eBay’s automated proxy bidding capability, or they may choose to incrementally increase their bid submissions up to the level of their strategic bid on their own. In other words, we assume bidders formulate strategic bids based on the distribution of private values and their expectation of \( N \) at the beginning of the terminal period, and afterward they stick to their planned strategic bid through the end of the auction. Finally, consistent with the previous section, price-path dynamics within the terminal period are assumed to be simple in the sense that ordering of bidders’ submission times is random rather than coordinated.

### 5.2.2 Serious Bidders and Strategic Bids

This partitioning of the auction into an initial cheap-talk period followed by a terminal sealed-bid auction leads to the following definition: a **serious bid** is one which affects the price path within the terminal period; likewise, a **serious bidder** is one who is observed to submit at least one serious bid. This distinction allows for the possibility that some observed bidders early on in the life of the auction were merely “fishing for a steal” or casually dabbling, rather than seriously vying to win a laptop like other bidders who remain active close to the end. Of course, the possibility always exists that some bidders who are determined to be non-serious by the above criterion had serious intent to compete for the item, but were priced out before submitting a serious bid. This, however, is just part of the problem that our proposed estimator for \( \lambda \) based on our explicit model for the filter process is meant to solve: recall that observed participation is merely treated as a lower bound on actual participation when identifying the arrival rate of bidders within an auction.

In our empirical application, we specify the terminal period as the last 30 minutes of an auction. During this period, we see an average of 4.69 observed serious bidders. Figure\(^6\) provides a justification for this choice. It shows two empirical distribution functions: one for time remaining when the highest losing bid is submitted, and one for time remaining across all serious bids. Note the highest losing bid arrives with fewer than 30 minutes left in over 80% of the auctions in our sample. Note also that 20.98% of all serious bids are submitted prior to the final 30 minutes of the auction. This possibility is naturally built into our serious bidder criterion, so as to avoid making too sharp a distinction between the two stages of the auction. Whenever there are at least two bidders over the life of an auction, the two highest bids during the cheap talk phase are the ones which set the initial price for the final 30 minutes. Therefore, the individuals who submitted them will be logged as serious bidders, even if they are not observed to actively bid during the final 30

\(^{26}\) Some other work within the related literature, including Nekipelov [2007], has attempted to develop an explicit empirical model for determining both levels and timing of bids within an auction. One challenge to such an undertaking is that fully formed equilibrium theories of bid timing within an auction are rare. Such an exercise would introduce considerable complexity and is beyond the scope of our current purposes. Thus, we follow the standard approach of adopting assumptions which simplify intra-auction dynamics for tractability.
The majority of serious bidders only submit a single proxy bid, but just under one third of serious bidders are observed to raise their own bid levels incrementally with multiple submissions during the terminal stage. This discrepancy motivates our final data restriction: in order to be conservative we assume only that the highest losing submission (that is, the second-highest overall bid) is reflective of a strategic equilibrium bid consistent with the model from Section 4.1 above. If some bidders choose to raise bid levels on their own, rather than avail themselves of the automated proxy bidding system, then many submissions by serious bidders may still represent only lower bounds on their equilibrium strategy. By appealing to the Haile and Tamer [2003] assumption that bidders will never allow an opponent to win at a price they would be willing to pay, one can be confident that the highest losing submission is reflective of a full strategic bid, and it will always be available since it is always recorded by nature as the filter process runs its course. Of course, if the researcher is able to incorporate more bidding data per auction in a reliable way, then it would improve the statistical precision of resulting estimates.

In total, our various sample restrictions leave us with 733 observed highest loser bids (that is, all auctions in our data had at least two serious bidders). Table 3 displays descriptive statistics on bid timing, observed participation, sale prices, and highest losing bids. With probability 0.2224

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27Nature also always records the attendance of the winner, but the eBay bidding system only reports exact amounts for bids that were surpassed by the next lowest bid plus $\Delta$. Thus, the winning bid itself is only observable to the econometrician for auctions where price is generated by a first-price rule, which occurs in roughly 22.24% of the auctions in our sample. In principle, inclusion of these bids may improve the statistical efficiency of the estimator, but it would come with added complexity and computational cost. For simplicity sake, we ignore them here. Despite this additional data loss, the confidence bounds we get on our estimates are still remarkably tight.

28In addition, we used one final sample restriction in order to avoid numerical instability issues in the upper tail of the private value distribution. The original dataset contained 736 observations fitting the description above, but for three of these, the highest losing bid was four standard deviations or more above the mean. We drop these three auctions from the sample, leaving us with 733 total auctions.
Table 3: Descriptive Statistics

<table>
<thead>
<tr>
<th>Variable</th>
<th>Mean</th>
<th>Median</th>
<th>St. Dev.</th>
<th>Min</th>
<th>Max</th>
<th># Obs</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Time Remaining (minutes)</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Winning Bid Submission:</td>
<td>24.04</td>
<td>1.43</td>
<td>97.50</td>
<td>0.00</td>
<td>1,392.65</td>
<td>733</td>
</tr>
<tr>
<td>High Loser Bid Submission:</td>
<td>24.29</td>
<td>3.10</td>
<td>61.76</td>
<td>0.00</td>
<td>633.40</td>
<td>733</td>
</tr>
<tr>
<td><strong>Observed Participation</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>(\hat{N}) (serious bidders only):</td>
<td>4.69</td>
<td>4</td>
<td>1.70</td>
<td>2</td>
<td>11</td>
<td>733</td>
</tr>
<tr>
<td><strong>Monetary Outcomes</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Sale Price (w/o shipping):</td>
<td>$299.63</td>
<td>$300</td>
<td>$31.31</td>
<td>$202.50</td>
<td>$405</td>
<td>733</td>
</tr>
<tr>
<td>Highest Losing Bid:</td>
<td>$295.42</td>
<td>$299</td>
<td>$31.20</td>
<td>$200</td>
<td>$400</td>
<td>733</td>
</tr>
<tr>
<td>First-Price Frequency:</td>
<td>22.24%</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
</tr>
</tbody>
</table>

the sale price takes on a value of exactly the highest bid for our sample. Note, too, that bids and sale prices do not include the flat shipping fee of $36.

5.2.3 Practical Matters

As a final practical matter, we had to choose two knot vectors \(j_J\) and \(k_K\) in order for the B-spline basis functions \(V_j, j = 1, \ldots, J+3\) and \(B_k, k = 1, \ldots, K+3\) to be well-defined. The first challenge is to specify the endpoints. Since the sample extrema are superconvergent estimators of the strictly positive support endpoints, and since theory requires that \(\beta(v) = v\), we set \(j_1 = k_1 = \min_t \{y_t\}\) and \(k_{K+1} = \max_t \{y_t\}\). Because the bid function does not touch the 45\(^\circ\)-line at the upper end, we can only bound \(v\) from below and must, therefore, make a guess at an appropriate value. We chose \(j_{J+1} = \bar{b} + 2\Delta_2\) as an approximate value for the upper bound of the private value support. Our point estimates, discussed below, suggest that this was a reasonable choice which does not seem to drive results in any meaningful way.

The next step is to select the number and placement of the knots. The primary concern in selecting \(K\) is to minimize numerical error in the approximate solution to the equilibrium differential equation. We chose a uniform grid of 19 knots on the lower interval \([\bar{b}, b^*]\) (18 subintervals), an intermediate knot grid \([b^*, b^* + \epsilon, b^* + 2\epsilon, \ldots, b^* + 7\epsilon]\), with \(\epsilon = 1\), and a uniform grid of 49 knots on the upper interval \([b^* + 8\epsilon, \bar{b}]\) (48 subintervals), for a total of \(K = 78\) total knots (four of them concurrent at \(b^*\)) and 80 total basis functions for \(\hat{\xi}\). The vector of points where equilibrium conditions were enforced was a uniform grid of 500 points on \([\bar{b}, b^*]\) and 500 uniform points on \([b^*, \bar{b}]\), or in other words, \(L = 1000\). From experimentation, this configuration of the three uniform knot sub-vectors and checkpoints seemed to deliver a reasonable trade-off between computational cost and numerical accuracy: larger values of either \(K\) or \(L\) improve numerical accuracy only very little.

As for the principal knot vector \(j_J\), the primary concern is goodness of fit to the distribution of the observables. One challenge is first to reduce the dimensionality of the decision problem,
in a data-driven way, if possible. Our proposal is to specify a uniform grid of \( J \) quantile ranks \( \{q_1, \ldots, q_{J-1}, q_J+1\} \) spanning \([0, 1]\), and then to map them into bid space using the empirical quantile function \( \tilde{H}^{-1}(q) \). We then replace the uppermost knot \( j_{J+1} \) with a value of \( (\hat{v} + 2\Delta_2) \), as mentioned above, in order to account for the fact that these knots will govern the behavior of \( \tilde{F}_V \) in private value space. Finally, because our highest loser bid distribution is skewed to the right with a long, curved upper tail, we insert an additional knot \( j_{J+1} = (j_{J-1} + j_{J+1})/2 \) to provide some added flexibility in this region. The advantage of this approach is that it reduces the \((J + 1)\)-dimensional decision of knot choice to a single decision of \( J \), the number of knots, while letting the data determine the placement of the knots. On the other hand, this method concentrates knots more densely within higher quantiles of \( \tilde{F}_V \), since we are choosing knots at the quantiles of highest losing bids. However, this is a natural problem inherent in estimating a parent distribution from any dataset consisting of order statistics: observations selected from the within-auction sample extremes will always be more informative of the higher quantiles of the parent distribution. Thus, in finite samples, knot choice involves the familiar bias-variance tradeoff. We propose the above method because it allows the observables to determine knot location. Statistically optimal knot choice, while an interesting problem, is beyond the scope of the current exercise, and is therefore left to future work.\(^{29}\)

Finally, when enforcing the equilibrium conditions, we found the \( L_\infty \)-norm most effective. The problem with the other \( L_p \) norms (with finite \( p \); for example, the \( L_2 \)-norm) is that they allow for a small number of drastic deviations from the FOCs, so long as the solution to \( \tilde{\xi} \) is well-behaved at most domain points. This can lead to poor performance of the solution method. Although both methods produce a spline estimate with the same overall trend, the \( L_\infty \)-norm, by directly disciplining the worst-behaved segments of the spline, is able to achieve a better overall fit, while avoiding wild oscillations around the bidder optimality condition. As an illustration of this point, Figure 19 in the Appendix plots the relative approximation error of the B-spline approximation, \( R(b)/\tilde{\xi}(b; \hat{\alpha}) \) under the \( L_\infty \) and \( L_2 \) norms. The plot shows that the approximation error under the \( L_\infty \) norm is several orders of magnitude smaller along much of the functional domain. Moreover, few remedies exist to improve this comparison without drastically increasing computational load; for example, the picture remains virtually unchanged if we double the number of domain checkpoints and increase the penalties \( P_1 \) and \( P_2 \) by a factor of ten. Thus, we recommend the \( L_\infty \) norm for practical use. We chose values of \( P_0 = P_1 = P_2 = 10 \) as this allowed for a good balance of least squares fit and small numerical error. As a rule of thumb, it is best to rely principally on lower penalties where possible, and instead default toward achieving better fit/numerical accuracy through adding knots to the B-spline for \( \tilde{F}_V \).

### 5.3 Empirical Results

We first separately estimated two parametric models of the bidder arrival process—a generalized Poisson model indexed by \((\lambda_1^{GP}, \lambda_2^{GP})\) and a standard Poisson model indexed by \(\lambda^p\) (where the second parameter in the generalized model is restricted to be zero). Table 4 reports our point estimates and

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\(^{29}\)By experimenting with alternatives where knots are placed uniformly in \( V \) space instead, we found that the shape of the point estimate for \( f_V \) changed somewhat near the lower end, but the bootstrapped confidence bounds were also substantially wider and oddly shaped. The simple intuition is that with uniform knots in \( V \) space there are several model parameters whose values are being determined by sparse data near the lower extreme of the sample, which is likely leading to higher mean squared error.
bootstrapped standard errors for both. Under the generalized Poisson model we get both a higher mean, 10.22, and standard deviation, 5.89, of the random variable $N$ as compared to the Poisson model with 8.98 and 2.99, respectively. Figure 7 depicts a comparison of the empirical distribution of $\tilde{N}$ versus the two parametric models. Both fit the data fairly well, but the generalized Poisson model emerges as the clear winner: everywhere on the sample domain, the model-generated distribution of observables under the generalized Poisson model is within the nonparametric 95% confidence bounds of the empirical distribution of observed bidders. This implies that, although there still may be asymptotic gains from a more flexible functional form, the current sample size precludes further improvement without more data.

For the private value distribution we settled on a value of $J = 9$, meaning ten knots in $V$ space and 12 total parameters, with ten of them being free parameters after we enforce the boundary conditions on $\hat{F}_V$. This choice seemed to provide a high degree of flexibility for fitting patterns in the data, while producing a high degree of numerical accuracy. Table 4 displays our chosen (uniform) knots $\{j_1, \ldots, j_{10}\}$, parameter point estimates, and bootstrapped standard errors in parentheses. We also present the sup-norm numerical error and the sum of squared residuals from the NLS fit to the empirical distribution $\hat{H}$ for the preferred specification with $J = 9$ and for three alternative models with $J = 8, 10, 11$\(^{30}\). All four specifications considered provided a remarkably tight fit to the data (see Figure 8), but the preferred specification resulted in sup-norm numerical error that was at least an order of magnitude better than the others.

Figure 8 illustrates the fit between the empirical CDF $\hat{H}(b)$ (thick dashed line) and the model-generated version under varying choices for $J$. The thick solid line corresponds to our preferred specification with $J = 9$. Figures 9 and 10 depict the estimated private value distribution and

\(^{30}\)For each of these four specifications of $\hat{F}_V$, we held fixed $\hat{\lambda}$, the other knot vector $k_K$, the grid of 1000 check points, and the three penalty parameters so that each would be comparable. Note from the table that the sum of squared residuals need not move monotonically for small changes to the knot vector. This is possible because incrementing $J$ by one results in a sequence of non-nested models, because each uses a different set of basis functions which are uniquely determined by choice of the knot vector.
Table 4: Parameter Estimates and Standard Errors

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Estimate</th>
<th>Standard Error</th>
</tr>
</thead>
<tbody>
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<td>$\hat{\lambda}^p$</td>
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<tr>
<td>$\mu_{11}$</td>
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Table 5: Absolute and Relative Differences Between Private Values and Bids

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<th>Percentile</th>
<th>High Loser Bid</th>
<th>Bid Shading: $V - \beta(V)$</th>
<th>Buyer Information Rents</th>
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<td>$310.18$</td>
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inverse bid function with bootstrapped 95% confidence bounds\textsuperscript{31} In each figure, the thin vertical lines indicate knots. The inverse bid function plot is zoomed in below the 90\textsuperscript{th} percentile of the private value distribution so that the features of the function can be seen more clearly. Note how the derivative $\hat{\xi}'(b)$ jumps slightly at the transition point $b^* = 250$, marked with a vertical line. The confidence bounds on the inverse bid function are remarkably tight because within the relevant ranges for $\Delta$, private value distributions that significantly differ tend to still largely agree on the appropriate degree of demand shading. For completeness, Figure 17 in the Appendix presents the estimate and bounds on the entire support, along with the relative approximation error, or the

\textsuperscript{31}To compute our bounds we estimated the model parameters $\lambda$, $\mu$, and $\alpha$ on 1,000 bootstrapped samples, holding the knot vectors $k_K$ and $j_J$ fixed.
ratio of the residual function $R(b)$ to the units of $\xi(b; \hat{\alpha})$. As suggested in Section 4.5.2 above, we also compute a final check on our solution for $\hat{\xi}(b; \hat{\alpha})$ by holding the structural parameters $\hat{\lambda}$ and $\hat{\mu}$ fixed, and computing a standard Galerkin approximant, call it $\tilde{\xi}(b; \tilde{\alpha})$, by solving a square system of non-linear equations on an appropriately chosen grid of checkpoints. This we do with a knot vector $\tilde{J}$ where $\tilde{J} = 3J$ for improved accuracy (under the distributional parameters), and we then compare the two solutions. The sup-norm distance between our point estimate $\hat{\xi}(b; \hat{\alpha})$ and the standard Galerkin implementation $\tilde{\xi}(b; \tilde{\alpha})$ turns out to be less than a tenth of a penny, providing confirmation that our point estimates are based on a differential equation solution with an acceptable level of numerical error.

It is interesting to note that both the point estimate $\hat{\xi}(b; \hat{\alpha})$ and the lower confidence bound lay significantly above the $45^\circ$-line, which represents the hypothetical inverse bid function under a second-price equilibrium. To put the picture into context, the upper panel of Table 5 summarizes the degree of estimated demand shading for bids occurring at various quantiles of the observable distribution $H(b)$. Within the 90-10 range, the difference between the estimated EA bid function and the second-price bid function ranges from $3.93$ to $5.80$, which amounts to between 13% and 19% of a standard deviation of $H(b)$, the distribution of the highest loser bid. It is also worth mentioning that if we compute an alternative B-spline estimator, call it $\tilde{F}_V$, using the same knot vector as our preferred specification (with $J = 9$) and the generalized Poisson point estimate $\hat{\lambda}_{gp}$, but under the second-price assumption (where we just interpret observed bids as private values), then we get a CDF estimate which is strongly rejected by the EA bidding model. Specifically, the estimator $\tilde{F}_V$ is shifted to the left and parts of it lay outside of the bootstrapped 99% confidence bounds of $\hat{F}_V(v; \hat{\mu})$.
5.4 Model Simulations and Counterfactual Analysis

We begin by exploring auction winners’ market rents arising from private information on their willingness to pay. We define information rents as the following random variable

\[ I \equiv V_{(1;N)} - P = V_{(1;N)} - \min\{B_{(2;N)} + \Delta, B_{(1;N)}\}, \]

or the winner’s private valuation minus the price she pays. Note that the randomness in \( I \) comes from it being a function of three separate random variables: \( N \), \( V_{(1;N)} \), and \( B_{(2;N)} \). In our case, where we only occasionally observe the highest bid, and where we never directly observe \( N \), our structural model estimates are needed in order to simulate the distribution of \( I \).

To do so, we proceed in three steps: first, we generate a random value of \( N \) from the generalized Poisson distribution with our point estimate \( \hat{\lambda}_{gp} \); second, we generate \( N \) random draws from the private value distribution \( \hat{F}_V(v; \hat{\mu}) \) and map them into bid space using a cubic interpolant of the inverse of \( \hat{\xi}(b; \hat{\alpha}) \); finally, we collect the highest private valuation and the two highest bids to compute a simulated value for \( I \). We followed this process 1,000,000 times to get a large enough sample to reliably represent the moments of the distribution of \( I \). The lower panel of Table 5 summarizes several quantiles. On average, the winner in an auction from our sample retained an information rent of $28.55, but the distribution is highly skewed to the right with a large standard deviation of $23.88. From a seller’s perspective, one of the main concerns is how to maximize their own revenues by extracting as much of these information rents as possible through auction design. We now conclude our discussion with a counterfactual exploration of optimal reserve prices.

One puzzling regularity within eBay data is that the vast majority of sellers set reserve prices at or near zero. In the 733 auctions from our empirical application, a reserve price is never used, but this could represent default seller policy (given that all of these auctions come from the same seller, CSRT). Nevertheless, in a broader set of used laptop data from the same time period, with 13,193 separate auctions, only 724 of them, or 5.5% overall, involved a reserve price that exceeded $1; 359, or 2.6%, had a reserve price that exceeded $10; and 156, or 1.1%, had a reserve price that exceeded $100. The sparse use of reserve prices might be surprising given the literature on
optimal mechanisms in which a positive reserve price is suggested even for a seller who values the item at $0. Equipped with our estimates of the private value distribution and the parameters of the generalized Poisson distribution, we can consider counterfactual experiments to understand why.

First though, following McAfee and McMillan [1987] and Harstad et al. [1990], observe that because we have a symmetric, independent private-values model with an unknown number of risk-neutral bidders, standard auctions will be revenue equivalent so long as bidders have the same beliefs about the number of potential bidders. Our EA model has the same equilibrium allocation rule as the canonical first- and second-price formats—the bidder with the highest valuation always wins the object (efficiency)—and the expected payoff of the lowest possible type $v$ is zero; therefore, the revenue equivalence principle applies. Second, optimal auction design does not depend on the distribution of $N$; see McAfee and McMillan [1987]. Taken together, these facts imply that the optimal auction can be implemented using a first- or second-price format, or via an EA.

This simplifies computation of counterfactuals: we can generate the correct expected revenues by considering a second-price auction, which avoids the need for solving the EA bid function using our estimated valuation distribution for every possible choice of reserve or value of the participation parameters. Let $R$ denote revenue, and $r$ denote reserve price. Expected revenues are given by

$$E(R|r) = r[1 - F_V(r)] \sum_{n=1}^{\infty} \rho_N(n; \lambda)nF_V(r)^{n-1}$$

$$+ \int_r^\infty v[1 - F_V(v)]f_V(v) \sum_{n=1}^{\infty} \rho_N(n; \lambda)n(n - 1)F_V(v)^{n-2} \, dv,$$

where the integral is solved using an adaptive recursive Simpson’s rule. The maximizer, $r^*$, of this expression satisfies the equation

$$r^* = v_0 + \frac{1 - F_V(r^*)}{f_V(r^*)},$$

first derived by Myerson [1981], where $v_0$ is the seller’s valuation. For the purpose of this discussion, we shall assume throughout that $v_0 = 0$. 

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**Figure 10: Estimated Equilibrium Inverse Bidding Function**

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38
Figure 11: Counterfactual Revenue Curve Comparisons

Figure 11 depicts three expected revenue curves as a function of reserve price $r$: one deriving from model point estimates (solid line), and two hypothetical curves where the generalized Poisson parameters are altered. For both of these we compute an alternative $\lambda$ vector so that Var$[N]$ is held constant while for one, call it $\lambda_+$, E$[N]$ increases by one relative to the point estimates, and for the other, call it $\lambda_-$, E$[N]$ decreases by one. A well-known result concerning reserve prices in auctions by Bulow and Klemperer [1996] is that the benefit from attracting an additional bidder to an auction exceeds any possible gain by optimizing the reserve price. This result plays out strongly in the figure: the impact on expected revenues from changing E$[N]$ by one is 63–80 times greater than the impact of moving from $r = 0$ to $r = r^*$. However, our empirical results say something even stronger about optimal reserve prices on eBay: they are, in fact, almost entirely irrelevant to begin with! Although the expected revenue curves in the figure have an interior global optimum, under our model estimates, the maximum benefit a seller may reap by optimizing the reserve price is estimated to be $0.0745. To put this number in perspective, if we assume that an eBay seller values her time at a conservative $10$/hour, then if it takes her longer than a mere 27 seconds to decide what the optimal reserve price should be, she would be better off by simply setting $r = 0$ and allocating her time toward some other, more profitable use. This resolves the puzzle of why the vast majority of eBay sellers choose non-binding reserve prices: the online auction house has done its job of attracting buyers to the market well enough so there is no longer any need to worry about this aspect of auction design.

## 6 Discussion and Conclusion

EAs are continually an important market mechanism today, with eBay alone accounting for a total sales volume of $25$ billion in 2010 from auction listings alone.$^{32}$ This paper has yielded four
contributions to research on EAs: first, we have documented how failing to account for the non-standard EA pricing rule can introduce bias into empirical estimates. Second, we have demonstrated that a realistic EA bidding model is nonparametrically identified from readily available observables. As part of this task we have solved another independent problem by proposing a new identification strategy for inferring the distribution of \( N \) using observable lower bounds on bidder participation. Third, we have proposed an estimation strategy to recover model parameters from actual eBay data in a flexible, yet computationally convenient way. The methodology we employ, though conceptually complex, is easily implementable using a MATLAB toolbox, which is available from the authors on request. This fills two important methodological gaps in the literature: researchers need no longer (incorrectly) assume that eBay bids are private valuations; and having estimates of both the private value distribution \( F_V \) and the ex-ante participation rate will allow for model simulation, whereas previous work (for example, Song [2004a]) had only managed to identify \( F_V \) alone.

Finally, we have used our methodology to shed light on empirically relevant areas for inquiry in online auction design. Our model estimates rationalize why reserve prices are commonly overlooked by sellers: given the number of bidders that typically participate they play very little role in determining revenues. Varying the expected number of bidders who participate, on the other hand, can still play a significant role under observed market conditions. This suggests that the most relevant aspects of online market design take place at the level of the auction house itself, where policy levers exist to produce movements on the margin of market-wide buyer-seller mix. Future work to investigate this question further will yield fruitful new insights, and will benefit from the methodological advances we develop here.

7 Appendix

7.1 A Brief Primer on B-Splines

As mentioned above, B-splines have many attractive qualities as a tool for flexible parametric curve fitting. They can be constructed to fit any continuous curve to arbitrary precision, they are locally low-dimensional, giving nice stability properties, they are convenient to compute, and can even be adapted to allow for discontinuous derivatives at a point. Here, we provide the reader with a brief primer on constructing B-spline functions and their associated bases.

Consider the problem of fitting a continuous nonparametric function \( f : [x, \bar{x}] \to \mathbb{R} \) with an \( O \)-order B-spline, \( O \in \mathbb{N} \). We shall compute our B-spline basis functions according to the Cox–de Boor recursion formula (see de Boor [2001], Chapter IX, equations (11)–(15)) with coincident boundary knots. This formula requires us to pre-specify a knot vector \( k_K = \{k_1 < k_2 < \ldots < k_K < k_{K+1}\} \) that partitions the domain into \( K \) subintervals, with \( k_1 = x \) and \( k_{K+1} = \bar{x} \). Given knot vector \( k_K \) and order \( O \), we wish to compute a set of \( (K + O - 1) \) basis functions of degree \( D = (O - 1) \), denoted \( \mathcal{B}_{k,D} = \{B_i,D : [x, \bar{x}] \to \mathbb{R}, i = 1, \ldots, K + D\} \). We begin by computing an extended knot
The Cox–de Boor recursion formula requires that in order to define basis \( \mathbb{B}_{k_d,d} \), we must first compute all lower-order bases \( \mathbb{B}_{k_d,d} \) for each \( d = 0, \ldots, D \). We initiate the recursion by defining the zero-degree basis functions as simply piecewise constant, or, for each \( i \)

\[
\bar{K}_K = \{ k_{1-D} = \ldots = k_1 < k_2 < \ldots < k_K < k_{K+1} = \ldots = k_{K+1+D} \}.
\]

Note from the above formula that there are exactly \( K \) of them. With this beginning, we can now define, for each \( d = 1, \ldots, D \), and for each \( i = 1, \ldots, K + d \),

\[
\mathcal{B}_{i,d}(x) = \begin{cases} 
\mathcal{B}_{i,d}(x) \left( \frac{x-k_{i-d}}{k_{i-d+1}-k_{i-d}} \right) + \mathcal{B}_{i+1,d}(x) \left( \frac{k_{i+1-d}-x}{k_{i+1-d+1}-k_{i-d+1}} \right) & \text{if } (k_{i+1} - k_{i-d})(k_{i+1} - k_{i-d+1}) \neq 0, \\
\mathcal{B}_{i+1,d}(x) \left( \frac{k_{i+1-d}-x}{k_{i+1-d+1}-k_{i-d+1}} \right) & \text{if } (k_{i+1} - k_{i-d+1}) = 0, \\
\mathcal{B}_{i+1,d}(x) & \text{if } (k_i - k_{i-d}) = 0.
\end{cases}
\]

It is easy to see from this recurrence relation that the order 2 (degree 1) basis \( \mathbb{B}_{k_1,1} \) results in a set of \( K + 1 \) piecewise linear functions. Likewise, the order 3 (degree 2) basis \( \mathbb{B}_{k_2,2} \) results in a set of \( K + 2 \) piecewise quadratic functions, and the order 4 (degree 3) basis \( \mathbb{B}_{k_3,3} \) results in a set of \( K + 3 \) piecewise cubic functions. Figures [12][15] display an example of a set of bases designed for knot vector \( k_5 = \{-1, -0.6, -0.2, 0.2, 0.6, 1\} \).

Other properties of the basis functions are also worth pointing out. Note that each basis function is globally defined, but any given function in \( \mathbb{B}_{k_5,D} \) is nonzero on at most \( O = (D + 1) \) subintervals defined by \( k_K \). Moreover, on each subinterval, exactly \( O \) of the \((K + D)\) basis functions are nonzero. This is what gives B-splines their attractive locality properties: for a B-spline function of the form

\[
f(x; \alpha) = \sum_{i=1}^{K+D} \alpha_i \mathcal{B}_{i,d}(x),
\]

adjusting parameter values relevant to one subinterval will have little or no effect for subintervals sufficiently far away.

Finally, it is easy to see from the above expression that since B-spline functions are linear combinations of B-spline bases, their derivatives are also linear combinations of B-spline basis functions. Specifically, like global polynomials, the derivative of a \( D \)th-order B-spline is merely a \((D - 1)\)st-order B-spline, and its integral is an \((D + 1)\)st-order B-spline: for each \( d \geq 1 \) and \( i = 1, \ldots, (K + d) \) we have

\[
\mathcal{B}'_{i,d}(x) = \begin{cases} 
(d - 1) \left( \frac{\mathcal{B}_{i,d-1}(x)}{k_{i-d+1}} + \frac{\mathcal{B}_{i+1,d-1}(x)}{k_{i+1-d+2}} \right) & \text{if } (k_i - k_{i-d+2})(k_{i+1} - k_{i-d+1}) \neq 0, \\
(d - 1)\mathcal{B}_{i,d-1}(x)/(k_{i+1}-k_{i-d+1}) & \text{if } (k_{i+1} - k_{i-d+1}) = 0, \\
(d - 1)\mathcal{B}_{i+1,d-1}(x)/(k_{i+1}-k_{i-d+2}) & \text{if } (k_{i+1} - k_{i-d+2}) = 0.
\end{cases}
\]

Figure [16] illustrates \( \mathcal{B}_{5,3}(x) \) from Figure [15] along with its three nontrivial derivatives. This picture illustrates another important property of B-splines: \( \mathcal{B}_{i,D}(x) \) has \((D - 1)\) continuous derivatives on the interval \([x, x]\). In fact, the Cox–de Boor recursion formula above was specifically constructed to deliver this property.
Figure 12: Order One (Degree Zero) B-Spline Basis Functions

7.2 Extra Tables and Figures
Figure 13: Order Two (Degree One) B-Spline Basis Functions

Figure 14: Order Three (Degree Two) B-Spline Basis Functions
Figure 15: Order Four (Degree Three) B-Spline Basis Functions

Figure 16: Basis Function $B_{5,3}(x)$ Example, and its Derivatives
Table 6: Anderson–Darling Test Results
Number of Null Hypotheses Rejected and Median $p$-Values

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Table 7: Estimates of Optimal Reservation Price for Distributions Considered

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Table 8: Expected Revenues for an Auction with \((N + 1)\) Bidders

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<th>(N)</th>
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<th>(\text{Rayleigh}(0.3))</th>
<th>(\text{Power}(1.5))</th>
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<td>0.49433 (0.01208)</td>
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<td>0.64255 (0.01524)</td>
<td>0.58881 (0.01009)</td>
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<td>0.40179 (0.01235)</td>
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Figure 17: Full Inverse Bid Function Solution with Relative Numerical Error

Figure 18: APPROXIMATION ERROR COMPARISON for Various Values of $J$

Figure 19: APPROXIMATION ERROR COMPARISON: $L_{\infty}$ $\hat{\xi}$ Estimate vs $L_2$ $\hat{\xi}$ Estimate (using the preferred model specification for $F_V$)
References


Unjy Song. Nonparametric estimation of an eBay auction model with an unknown number of bidders, typescript, Department of Economics, University of British Columbia, 2004a.

Unjy Song. Nonparametric identification and estimation of a first-price auction model with an uncertain number of bidders, typescript, Department of Economics, University of British Columbia, 2004b.


