

ON THE PRICING RULE IN ELECTRONIC AUCTIONS*

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ABSTRACT. [This is the working version of the paper with an additional appendix. The permanent link to the print version in *The International Journal of Industrial Organization* is <http://dx.doi.org/10.1016/j.ijindorg.2009.10.006>.](#)

Researchers and experts have typically viewed electronic auctions (such as those implemented by eBay, Amazon, and Yahoo!) as either oral, ascending-price (English) auctions or second-price, sealed-bid (Vickrey) auctions. I show that important theoretical differences exist between English and Vickrey pricing rules and those used in electronic auctions, due to the presence of bid increments. I also show, using data on eBay laptop sales, that these differences have practical significance. I explore the implications of bid increments for strategic bid selection in a static model within the symmetric independent private-values paradigm. I derive the unique symmetric equilibrium bid function, showing that the presence of bid increments can significantly alter bidder behavior. Using numerical methods, I also illustrate that these result in a highly non-linear bid function, in contrast to that predicted under either the English or the Vickrey formats.

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1. INTRODUCTION

Electronic auctions (EAs) have become important mechanisms for exchange in the modern marketplace. Three familiar examples are the publicly-traded web giants Amazon, eBay and Yahoo!, which are known for their remarkable ability to attract large numbers of buyers and sellers to one common marketplace in which a vast array of unique objects is sold. Bajari and Hortaçsu [1] have reported that in 2001 eBay, the largest online auction house, had 423 million items listed for sale in 18,000 unique categories. Items sold range from consumer electronics to automobiles to rare collectibles, like Roman coins. Since their emergence in the mid 1990s, internet auction houses have been known for extraordinary growth and volume as well. Over a five year stretch from 2001 to 2006, eBay saw an average annual growth rate of 47% in total sales; in 2006 it sold well over US\$52 billion of merchandise, with two thirds coming from auctions.¹

EAs have some features in common with traditional auction formats, and others that make them unique. For a given sale item, potential buyers report a number to an online auction server, representing the maximal amount that they authorize the server to bid for them.² It is known only to the EA software, which then competitively bids as low as possible on the bidder's behalf, up to but not exceeding his reported maximum. Different auction sites have different names for this software; *e.g.*, eBay refers to it as *proxy bidding*. When the EA software overtakes one bidder's bid on behalf of another, it forces the new lead bidder, whenever possible, to surpass the former lead bidder by a discrete amount, called a *bid increment*. Increments are fixed beforehand by the online auction house, and are known to both buyers and sellers.

For reasons that will become clear shortly, there are three game-theoretic auction models which are related to EAs: the oral, ascending-price (also known as English); the second-price, sealed-bid (also known as Vickrey) and the first-price, sealed-bid. As is commonly done in the literature, I shall broadly refer to the first two formats as second-price auctions because the rules are such that the winner pays a price equal to the second-highest bid. In contrast, the third format is known as a first-price auction because the winner pays his own bid; *i.e.*, the highest bid. This distinction between the first-price rule and the second-price rule will be central to my analysis in this paper.

Business and economic researchers have typically thought of EAs in terms of a second-price auction. The reasons are clear: given the way proxy bidding works, and assuming no bid increments exist, once all bidders have submitted their final proxy bids, the price the winner pays is his nearest opponent's bid. An important implication of this second-price view of EAs is that when each bidder's value for the object is privately known, there is a dominant strategy of simply reporting to the server the maximal price one would be willing to pay for the sale object. This understanding is reflected in advice that eBay supplies to bidders. The following quote is from eBay's help page: "Make sure that your maximum bid is the highest price that you're

¹ See figures on "Gross Merchandise Volume" and "Fixed Price Trading as % of gross merchandise volume" on eBay quarterly financial results: <http://investor.ebay.com/results.cfm>.

²Although the various internet auction houses differ in some respects, the rules which I discuss here are typical of Amazon, eBay and Yahoo! auctions.

willing to pay. Our bidding system automatically increases your bid up to the maximum price you specify. . .”³

However, the pricing rule in EAs is more complex than those in second-price auctions due to the presence of bid increments, which I shall henceforth denote by Δ . When the EA software overtakes one bidder’s bid on behalf of another, it forces the new lead bidder, whenever possible, to surpass the former lead bidder by Δ . A necessary exception to this rule is when the top two proxy bids are closer than Δ , as a jump in the full amount of Δ would surpass the high bidder’s maximal authorized bid. In this case, the price is set exactly equal to the high bid.

The following example illustrates the EA pricing rule. Consider three bidders in a widget auction with a reserve price of \$3 and $\Delta = \$2$. Suppose player 1 bids $b_1 = \$10$, meaning that the initial posted price is $P = \$3$ and player 1 is the current lead bidder. Moreover, the server will now only accept bids of at least $P + \Delta = \$5$. Suppose that player 2 submits a lower bid of $\$5 \leq b_2 \leq \10 , leaving player 1 in the lead, but with an updated price. As mentioned above, the new posted price will depend on how close b_2 is to b_1 .

- **Case 1:** $b_2 = \$7 \Rightarrow b_1 - b_2 = \$3 > \Delta \Rightarrow P = b_2 + \Delta = \9 .
- **Case 2:** $b_2 = \$9 \Rightarrow b_1 - b_2 = \$1 < \Delta \Rightarrow P = b_1 = \10 .

In Case 1, player 2 underbid player 1 by more than Δ , so the updated price was simply $b_2 + \Delta$. In Case 2, player 2 still underbid, but this time she came closer to her competitor’s undisclosed maximum bid. In fact, the difference between their bids was less than Δ , so the server was forced to update the price by less than a full increment, raising it to exactly b_1 . To continue the example, let us assume that Case 1 has occurred, meaning that the current price is \$9, and the minimum acceptable bid is now $\$9 + \Delta = \11 . Note that player 3 knows that player 1 is the high bidder, but she has no way of knowing what player 1’s maximum bid is.⁴ Suppose that player 3 overtakes player 1 with a bid of $b_3 \geq \$11$.

- **Case 3:** $b_3 = \$15 \Rightarrow b_3 - b_1 = \$5 > \Delta \Rightarrow P = b_1 + \Delta = \12 .
- **Case 4:** $b_3 = \$11.50 \Rightarrow b_3 - b_1 = \$1.50 < \Delta \Rightarrow P = b_3 = \11.50 .

Similarly as before, the way in which the price updates depends on the difference between the two highest bids. In Case 4, if the auction were to end with no further bidding, then player 3 would win and the sale price would be her own bid.

Clearly, modeling EAs is not going to be straightforward. As the example illustrates, when the top two bids are further apart than Δ , the price is based solely on the second-highest bid, as in second-price auctions. When the top two bids are closer than Δ , the price is exactly the high bid, as in a first-price auction. Rather than a simple traditional auction, the EA game is a hybrid of the second- and first-price formats, as bid increments cause the pricing rule to exhibit elements of both. This statement begs the question of how important bid increments are in practice, if

³Quote taken from <http://pages.ebay.com/help/buy/outbid.html> on January 29, 2009.

⁴In Case 1, after updating the price the online server would publicly display that bidder 2’s maximum bid was \$7 and that a bid was submitted on player 1’s behalf in the amount of \$9. Since this amount is exactly $b_2 + \Delta$, player 3 has no way of knowing b_1 ; she only knows that $b_1 \geq \$9$. In Case 2 however, on observing that $b_2 = \$9$ and the current price is less than $\$9 + \Delta$, player 3 could deduce that b_1 is exactly equal to the current price.

they are only a small fraction of the current posted price. Below, I shall present some empirical evidence for the practical significance of the first-price component of EAs when Δ is seemingly small.

With this in mind, I pose the following question: when bid increments are included in the EA pricing rule, what are the implications for equilibrium bid selection? If bidder behavior no longer resembles the bid-your-value equilibrium as it would without increments, then the disparity cannot be ignored by researchers who wish to accurately model prices and bidding in electronic auctions.

To answer this question, I propose a new model of EA pricing in which bidders take into consideration that if they win there is a chance they might pay their own bid. I show that EA equilibrium bids lie strictly below private values. Since the EA pricing rule blends characteristics of both second-price *and* first-price rules, demand reduction occurs as it would in a first-price setting. Equilibrium bidding lies strictly between the second-price equilibrium and the first-price equilibrium, and it may closely resemble either, depending on model primitives. As it turns out, the first- and second-price auction formats are actually special cases of the EA auction game.

The remainder of this paper has the following structure: in section 2, I motivate the theoretical exercise with some empirical evidence of first-price rules in eBay laptop auctions. In section 3, I develop the theoretical framework of EAs by outlining the informational environment and the rules of the auction. I abstract away from the dynamic aspects of EAs in order to isolate the effects of the unique pricing rule on equilibrium bidding. I then characterize the symmetric Bayes-Nash equilibrium bidding strategy. Because it is difficult to obtain many closed-form results, I explore the model equilibrium using numerical methods in section 4. I conclude in section 5.

2. PRELIMINARY OBSERVATIONS: EBAY LAPTOP AUCTIONS

At this point, it is natural to question whether a rethinking of EA theory is necessary. After all, Δ is typically small in practice. On Amazon, eBay and Yahoo!, bid increments follow a schedule which produces occasional adjustments according to the current posted price. When the posted price is between \$100 and \$249.99, Δ is fixed at \$2.50, and when it is between \$250 and \$499.99, Δ is \$5.⁵ If increments are only between 1% and 2% of the current high bid, how likely is it that price adjustments follow a first-price rule? Recall that this only happens when two bids are within Δ of each other. If price adjustments rarely follow a first-price rule, then without any practical loss of generality researchers might continue to treat EAs as second-price auctions.

To answer this question, I collected data on eBay auctions for laptop computers during the months of April, May and June of 2008. The largest seller during that period was a firm called CSR Technologies (CSRT), whose product line consisted of second-hand Dell Latitude laptops, in several different configurations. Their most common configuration, which I shall refer to as “low-end,” had a 1.4GHz Pentium 4M processor, 512MB of RAM, 30GB of hard drive, and a

⁵As of the current date, a complete listing of the bid increment schedule on eBay may be found at <http://pages.ebay.com/help/buy/bid-increments.html>. Increment schedules on Amazon and Yahoo! are the same.

TABLE 1. Summary Statistics for Laptop Computer Sale Prices

Sample	# of Obs	Mean	StDev	Min	Max	% FP Rule
All Data	1228	305.29	34.62	202.50	585.03	23.05
Low-End, Used	458	296.88	32.83	227.50	450.00	22.27
Low-End, Refurbished	279	303.14	30.97	202.50	430.00	23.30
Mid-Line	363	306.05	34.6	217.50	585.03	23.42
High-End	128	337.93	29.16	265.00	455.00	24.22

DVD-ROM. Of these, 279 were listed as “refurbished” and 458 were simply described as used.⁶ CSRT’s next most common configuration, dubbed “mid-line,” had a 1.6GHz processor, 512MB of RAM, 40GB of hard drive, and a DVD-ROM with a CD burner. This sample includes 363 auctions. Finally, CSRT also offered a more powerful machine, dubbed “high-end,” with a 1.8GHz processor, 1GB of RAM, 40GB of hard drive, and a DVD-ROM/CD burner. There were 128 laptops in this sample.

Aside from the above distinctions, all other characteristics, including screen size (14.1”), operating system (Windows XP), shipping fees (\$36), reserve price (\$0.01), bidding duration (24 hours) and seller, were the same. I chose these data because of the large number of available observations and also for the remarkable degree of homogeneity among the sale items.

For each auction I have detailed bidding information, including the timing and amount of each reported maximum bid, as well as the identity of the bidder who submitted the bid.⁷ From these data, I constructed a measure of how often the final sale price was generated by a first-price rule simply by observing the final price adjustment in each auction. When this final adjustment is strictly less than Δ , it means that the sale price was generated by a first-price rule.

Table 1 displays summary statistics on sale prices for each of the four data sets. The most interesting column in the table is the final one, where I report the percentage of auctions within each sample where the sale price was generated via a first-price rule. In each sample, this was true for more than one in five auctions. These observations provide suggestive evidence that the first-price component of the EA pricing rule should not be ignored. In contrast to a second-price auction, rational EA bidders will take into account the positive probability of paying their own bid after winning.

3. THE STATIC EA GAME

I begin constructing the theoretical model by abstracting away from the dynamic aspects of EAs in order to isolate the effects of the unique pricing rule on equilibrium bidding. Hereafter, the term EA will refer to the static version.

⁶A two-sample t-test and a Kolmogorov-Smirnov test revealed that the distributions of sale prices for the two samples differed significantly, so I treated them separately.

⁷To be complete, I should mention that the amounts of maximum bid reports were only revealed after the price weakly exceeded them. Therefore, the maximum bid submitted by the winning bidder was only revealed in cases where the second-highest bid is within Δ . Otherwise, the winning bid listed in the bid history was the second highest bid plus Δ , which is only a lower bound on the winning bidder’s actual maximum bid.

There is a single object up for bids, with N bidders vying for it. For notational ease, I denote the number of opponents that each bidder faces by $M \equiv N - 1$. Each bidder has a privately known *value*, representing maximal willingness to pay. Private values, denoted v_i for the i^{th} bidder, belong to the interval $[0, \bar{v}]$, where $\bar{v} < \infty$. The information available to each bidder follows an IPV paradigm. That is, each player views his opponents' private values as independent draws from a commonly known distribution $F_V(v)$.⁸ Moreover, I assume that $F_V(v)$ has a continuous density $f_V(v)$ that is strictly positive on the interval $(0, \bar{v})$.

Henceforth, I adopt the notational convention that upper-case letters denote random variables and lower-case letters denote realizations of random variables. Thus, from bidder i 's perspective, his own private value is v_i and the (unknown) private value of his j^{th} opponent is V_j . Also, for a random sample of size M ordered in ascending fashion, I denote the k^{th} order statistic by $V_{(k:M)}$. Thus, for any player the highest opponent private value is $V_{(M:M)}$.

Each bidder is allowed to submit one sealed bid $b \in \mathbb{R}_+$ to the auctioneer, who then allocates the object to the highest bidder and sets a price.⁹ There is a constant bid increment $\Delta > 0$. The winner is the highest bidder and is the only one who pays a non-zero amount. For a full set of N bids, one from each player, the auctioneer sets a price equal to either the highest bid or the second-highest bid plus Δ , whichever is less. This implies the following pricing rule:

$$P(b_{(N:N)}, b_{(M:N)}) = \min \left\{ b_{(N:N)}, b_{(M:N)} + \Delta \right\}.$$

If player i outbids all M opponents, then his payoff is

$$\pi_i(v_i, b_i, b_{(M:M)}) = v_i - P(b_i, b_{(M:M)})$$

and if he is outbid he gets zero utility.

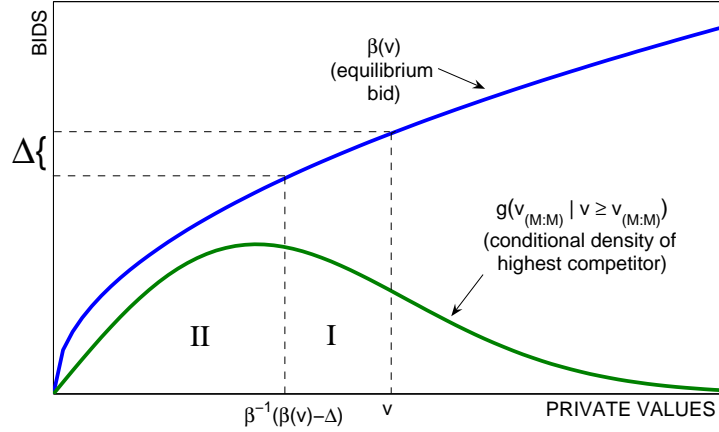
3.1. Equilibrium Analysis. I now derive the symmetric equilibrium bidding strategy in the EA game, which I denote by β . A *symmetric Bayesian Nash equilibrium in pure strategies* is a function mapping private values into bids, such that for each i , a bid of $b_i = \beta(v_i)$ maximizes player i 's expected payoff, given that his opponents' bids are $b_j = \beta(v_j)$, $j \neq i$. I shall assume for now that β exists, and that it is increasing and differentiable. These assumptions allow me to derive the equilibrium from the first-order condition (FOC) of an arbitrary bidder's objective function, but the derivation will be merely heuristic until they are verified. Accordingly, I shall denote the derived *candidate* equilibrium by $\tilde{\beta}$, and to preserve expositional continuity I shall prove existence, monotonicity and differentiability later, when I show that $\tilde{\beta}$ constitutes an equilibrium.

For notational ease I drop the subscripts and view the decision problem from player 1's perspective. Before moving on, it will also be useful to define some additional notation. Let the

⁸The analysis done in this section extends straightforwardly to the more general case of affiliated private values (APV), where the joint density is log-supermodular and values are correlated. Although this is likely a more realistic model, the notation and derivations under APV are considerably more cumbersome, whereas the special case of IPV allows for a relatively simpler exposition of the main ideas I wish to explore. If private values are correlated, then the tendency to reduce bids away from one's value will become more pronounced, as the likelihood of an opponent's value being close to one's own will be greater. Therefore, the arguments in this paper may be interpreted as a conservative view of the effects of bid increments on demand reduction.

⁹In the event of a tie, the auctioneer awards the object with equal probability to each bidder involved in the tie.

FIGURE 1. Decision Problem



distribution and density of the highest opponent private value be

$$G(v) \equiv F_V(v)^M \quad \text{and}$$

$$g(v) \equiv M F_V(v)^{M-1} f_V(v),$$

respectively. Finally, I denote the symmetric equilibrium bidding strategy in a second-price auction by $\beta^{II}(v) = v$, and that in a first-price auction by $\beta^I(v) = E[V_{(M:M)} | V_{(M:M)} \leq v] = \frac{1}{G(v)} \int_0^v xg(x)dx$.

CASE 1: Consider first the case when player 1's equilibrium bid will be less than Δ ; *i.e.*, assume $v < v_\Delta \equiv \tilde{\beta}^{-1}(\Delta)$. Since $\tilde{\beta}$ is increasing, player 1's probability of winning is the probability of his private value being highest, or $G(v)$. Player 1 knows that if he wins, he will surely pay his bid. Hence, his decision problem is identical to the one in a first-price auction:

$$(1) \quad \max_b \left\{ (v - b)G(\tilde{\beta}^{-1}(b)) \right\}.$$

Therefore, on the interval $[0, v_\Delta)$ we have

$$(2) \quad \tilde{\beta}(v) = \beta^I(v) = E[V_{(M:M)} | V_{(M:M)} \leq v] = \frac{1}{G(v)} \int_0^v xg(x)dx.$$

This function also provides a solution for $v_\Delta = (\beta^I)^{-1}(\Delta)$, the cutoff between types that will bid less than Δ and those that will bid more.

CASE 2: Now consider the case when player 1's equilibrium bid is greater than Δ ; *i.e.*, when $v \in [v_\Delta, \bar{v}]$. Figure 1 provides a useful aid for visualizing the components of player 1's decision problem in this case. The upper curve is a hypothetical equilibrium bid function and the lower curve is a hypothetical density of the highest opponent private value, conditional on v being the highest. In a monotonic equilibrium player 1 will bid $b = \beta(v)$, and the probability of paying his own bid is the mass of bidder types between v and $\beta^{-1}(\beta(v) - \Delta)$. This mass is represented

graphically by region I in Figure 1 and mathematically (assuming existence and monotonicity) by

$$(3) \quad \begin{aligned} \Pr [b - \Delta \leq B_{(M:M)} \leq b] &= \Pr [\tilde{\beta}^{-1}(b - \Delta) \leq V_{(M:M)} \leq \tilde{\beta}^{-1}(b)] \\ &= G(\tilde{\beta}^{-1}(b)) - G(\tilde{\beta}^{-1}(b - \Delta)). \end{aligned}$$

In this situation, his utility is

$$(4) \quad v - b.$$

He also knows that the probability of an opponent's bid determining the sale price is the mass below $\beta^{-1}(\beta(v) - \Delta)$. Graphically, this is region II in Figure 1, and mathematically it is

$$(5) \quad \begin{aligned} \Pr [B_{(M:M)} \leq b - \Delta] &= \Pr [V_{(M:M)} \leq \tilde{\beta}^{-1}(b - \Delta)] \\ &= G(\tilde{\beta}^{-1}(b - \Delta)). \end{aligned}$$

Under this circumstance, his utility is

$$(6) \quad v - \mathbb{E} [B_{(M:M)} | B_{(M:M)} \leq b - \Delta] - \Delta.$$

This gives rise to the following maximization problem:

$$(7) \quad \begin{aligned} \max_b \left\{ (v - b) \Pr [b - \Delta \leq B_{(M:M)} \leq b] \right. \\ \left. + \left(v - \mathbb{E} [B_{(M:M)} | B_{(M:M)} \leq b - \Delta] - \Delta \right) \Pr [B_{(M:M)} \leq b - \Delta] \right\}, \end{aligned}$$

where the conditional expectation in the second term can be expressed as

$$(8) \quad \begin{aligned} \mathbb{E} [B_{(M:M)} | B_{(M:M)} \leq b - \Delta] &= \mathbb{E} [\tilde{\beta}(V_{(M:M)}) | V_{(M:M)} \leq \tilde{\beta}^{-1}(b - \Delta)] \\ &= \frac{\int_0^{\tilde{\beta}^{-1}(b - \Delta)} \tilde{\beta}(v_{(M:M)}) g(v_{(M:M)}) dv_{(M:M)}}{G(\tilde{\beta}^{-1}(b - \Delta))}. \end{aligned}$$

Combining expressions (3)–(6) and (8), (7) can be rewritten as

$$(9) \quad \begin{aligned} \max_b \left\{ (v - b) [G(\tilde{\beta}^{-1}(b)) - G(\tilde{\beta}^{-1}(b - \Delta))] \right. \\ \left. + (v - \Delta) G(\tilde{\beta}^{-1}(b - \Delta)) \right. \\ \left. - \int_0^{\tilde{\beta}^{-1}(b - \Delta)} \tilde{\beta}(v_{(M:M)}) g(v_{(M:M)}) dv_{(M:M)} \right\}, \end{aligned}$$

and the FOC is the following:

$$\begin{aligned} G(\tilde{\beta}^{-1}(b)) - G(\tilde{\beta}^{-1}(b - \Delta)) &= (v - b) \left[\frac{g(\tilde{\beta}^{-1}(b))}{\tilde{\beta}'(\tilde{\beta}^{-1}(b))} - \frac{g(\tilde{\beta}^{-1}(b - \Delta))}{\tilde{\beta}'(\tilde{\beta}^{-1}(b - \Delta))} \right] \\ &+ (v - \Delta) \frac{g(\tilde{\beta}^{-1}(b - \Delta))}{\tilde{\beta}'(\tilde{\beta}^{-1}(b - \Delta))} - (b - \Delta) \frac{g(\tilde{\beta}^{-1}(b - \Delta))}{\tilde{\beta}'(\tilde{\beta}^{-1}(b - \Delta))}. \end{aligned}$$

Canceling terms and using the fact that in equilibrium $\tilde{\beta}^{-1}(b) = v$, the above reduces to

$$(10) \quad \tilde{\beta}'(v) = \frac{(v - \tilde{\beta}(v)) g(v)}{G(v) - G(\tilde{\beta}^{-1}(\tilde{\beta}(v) - \Delta))}.$$

This differential equation partially defines the solution for $\tilde{\beta}(\cdot)$ on the interval $[v_\Delta, \bar{v}]$, but I also need a boundary condition. By continuity, the boundary condition is

$$(11) \quad \tilde{\beta}(v_\Delta) = \Delta.$$

Hence, I have a complete solution for $\tilde{\beta}$ defined by (2), (10) and (11). Inspecting (10) reveals that $\tilde{\beta}'(v) > 0$ if and only if $\tilde{\beta}(v) < v$. Conceptually, it can be shown that the actual equilibrium is strictly monotonic (see Proposition 3.2 below) and satisfies $\beta(v) < v$ (see Appendix, Lemma 3.2.2), but I must confirm these properties for the candidate equilibrium in order to validate its derivation. It is also important to know that $\tilde{\beta}$ is strictly monotonic because otherwise, the right-hand side of equation (10) would define a correspondence, rather than a function. This leads to the following lemma:

Lemma 3.1. *Any solution to equation (2), differential equation (10) and boundary condition (11) lies below the 45°-line and is strictly monotonic everywhere on $[v_\Delta, \bar{v}]$.*

Proof: First note that by equation (2), I have $\tilde{\beta}(v_\Delta) = \Delta < v_\Delta$, from which it follows that $\tilde{\beta}$ is strictly below the 45°-line at the lower boundary. Suppose for a contradiction that $\tilde{\beta}$ touches or crosses the 45°-line at some private value, and let v^* denote the infimum of all such points. Since $\tilde{\beta}$ touches or crosses the 45°-line from below at v^* , it follows that the slope of $\tilde{\beta}$ must be no less than the slope of the 45°-line at the crossing point, or $\tilde{\beta}'(v^*) \geq 1$. Equation (10) indicates that $\tilde{\beta}$ has a strictly positive slope at all private values v' where $\tilde{\beta}(v') < v'$ and $\tilde{\beta}(\cdot)$ is strictly increasing on $[\tilde{\beta}^{-1}(\tilde{\beta}(v') - \Delta), v']$. Since $\tilde{\beta}$ is strictly increasing at all private values below v_Δ (see equation (2)), it follows that it must have a strictly positive slope at all private values below v^* as well.¹⁰ Then combining this observation with the fact that $\tilde{\beta}(v^*) = v^*$, equation (10) implies that $\tilde{\beta}'(v^*) = 0$, a contradiction. Thus, $\tilde{\beta}$ never touches the 45°-line and the result follows. ■

3.2. Existence, Differentiability and Verification of the Equilibrium. The remaining task is to show that the function $\tilde{\beta}$ defined by equations (2), (10) and (11) constitutes an equilibrium of the EA game. In order to do so, I must prove existence, monotonicity and differentiability of the equilibrium. It is important to note that the results I prove below are direct statements about the actual equilibrium β , and do not rely on any of the derivations of the candidate equilibrium $\tilde{\beta}$, in order to avoid logical circularities.

¹⁰To see why, suppose that there is some private value between v_Δ and v^* with a non-positive derivative, and let \hat{v} be the infimum of all such points. But since $\tilde{\beta}(\hat{v}) < \hat{v}$ and $\tilde{\beta}$ is strictly increasing below \hat{v} , $\tilde{\beta}'(\hat{v})$ must be strictly positive by equation (10), a contradiction.

Proposition 3.2. (Existence) *In the EA game there exists a symmetric, strictly increasing equilibrium in pure undominated strategies.*¹¹

Proof: This Proposition follows immediately from results proven in Jackson and Swinkels [4, henceforth JS] and Jackson, Simon, Swinkels and Zame [5, henceforth JSSZ]. The main result of JS is their Theorem 6 (p. 114), where the authors establish existence of an equilibrium in undominated pure strategies where bidding ties occur with zero probability for a general class of private-value auctions. One version of their proof invokes Theorem 1 of JSSZ (p. 1720), which shows that games with indeterminate outcomes may be augmented with a tie-breaking rule so that existence of an equilibrium is guaranteed. JSSZ also prove another result (Theorem 2, p. 1722) from which JSSZ Theorem 1 follows, and which also shows that when bidders are symmetric, a symmetric solution exists to the augmented game. Hence, by imposing symmetry and invoking JSSZ Theorem 2 in the proof of JS Theorem 6, I have symmetry as well. Finally, JS Corollary 14 (p. 123) delivers monotonicity in addition to the above properties. In the current setup, a zero tie probability plus monotonicity imply *strict* monotonicity. Since imposing symmetry does not alter the proof of JS Corollary 14, and since the EA game defined above satisfies the assumptions of JS, the result follows.¹² ■

Now that existence has been established, I can exploit the fact that β is an equilibrium in order to prove differentiability. Verifying differentiability allows me to infer that the differential equation arising from the FOC of the bidder decision problem is a necessary condition for optimality: if the equilibrium involves a differentiable bidding strategy, then it must satisfy the FOC. This is a vital step because the form of differential equation (10) renders standard methods of verifying the equilibrium inapplicable.

The standard methods begin by assuming differentiability of the equilibrium, and then they focus on proving sufficiency of the FOC by one of two techniques. Both involve simplifying the FOC via analytic integration, after which either (i) optimality is directly verified (e.g., see Krishna [6, Ch. 2]), or (ii) key properties of the candidate equilibrium are verified—including differentiability—which allow one to invoke a sufficiency theorem (e.g., see Milgrom [9, Theorem 4.2]). Both of these options hinge on one’s ability to analytically integrate the differential equation arising from the FOC. For example, in a sealed-bid, first-price auction, integration produces a simple expression defining the equilibrium in terms of model primitives only.

Here, however, equation (10) does not fall within the limited class of differential equations which admit analytic integration, and there is no way of which I am aware to directly verify either optimality or differentiability using equation (10).¹³ However, if one can show that a

¹¹An “undominated” strategy profile is a list of strategies, one for each player, such that for each $i \in \{1, \dots, N\}$ the i^{th} component strategy is not strictly dominated by another strategy in player i ’s strategy set. In the context of the EA game, an undominated pure strategy profile is one in which no player bids strictly more than his private value.

¹²For a description of the assumptions required for the results in JS [4] and JSSZ [5], along with a verification of those assumptions in the EA game, the interested reader is directed to the working version of this paper, available from the author upon request.

¹³A function $y(t)$ can be obtained by analytic integration only if the differential equation that defines it can be expressed as $y' = J(t)y + K(t)$, where J and K are known functions of t (see Boyce and DiPrima [2]).

symmetric equilibrium is differentiable, then it immediately follows that the FOC applies, and since it is optimal to choose a bid of $b = \beta(v)$ (by definition of the equilibrium), then β must satisfy the FOC.

Proposition 3.3. (Differentiability) *Any symmetric, monotonic equilibrium of the EA game is differentiable everywhere on the interval $[0, \bar{v}]$.*

I have left the proof of Proposition 3.3 to the Appendix, as it is quite involved. With that out of the way, I now have all of the necessary information to prove that my derivations characterize the unique symmetric EA equilibrium.

Theorem 3.4. *$\tilde{\beta}$ as defined by equations (2), (10) and (11) is the unique, symmetric, Bayes-Nash equilibrium, β , of the EA game.*

Proof: First note that by definition of the equilibrium, it is optimal for each bidder to select a bid according to β for any $v \in [0, \bar{v}]$. Therefore, β must satisfy the FOC, which defines $\tilde{\beta}$. At this point, there is a well-known result in differential equations theory, sometimes called the Fundamental Theorem of Ordinary Differential Equations (FTODE), which establishes conditions for both existence and uniqueness of solutions to standard initial value problems (IVPs).

However, equation (10) with boundary condition (11) is not a standard IVP when applied to the entire interval $[v_\Delta, \bar{v}]$, due to the presence of *two* unknown functions, $\tilde{\beta}$ and $\tilde{\beta}^{-1}(\tilde{\beta}(v) - \Delta)$ on the right-hand side of (10). This problem can be addressed by breaking up $[0, \bar{v}]$ into a set of adjoining sub-intervals, where the t^{th} interval is of the form $[\tilde{\beta}^{-1}((t-1)\Delta), \tilde{\beta}^{-1}(t\Delta)]$, $t \geq 1$.¹⁴ Moreover, I shall denote the solutions for bidding and inverse bidding on the t^{th} interval by $\tilde{\beta}_t(v)$ and $\gamma_t(v) \equiv \tilde{\beta}_t^{-1}(b)$, respectively. For $t = 1$, the solution to $\tilde{\beta}_1$ is characterized by the (known) solution to equation (2), which in turn characterizes γ_1 on the first sub-interval. Now note that the IVP which defines $\tilde{\beta}$ on the second sub-interval depends only on model primitives and the known solution to γ_1 . Similarly, equation (10) can be re-written on the t^{th} interval as

$$(12) \quad \tilde{\beta}'_t(v) = \frac{(v - \tilde{\beta}_t(v)) g(v)}{G(v) - G(\gamma_{t-1}(\tilde{\beta}_t(v) - \Delta))}, \quad t \geq 2,$$

so that the IVP defining $\tilde{\beta}_t$ depends only on model primitives and γ_{t-1} , which is known by induction.

Finally, by inductively applying the FTODE to each successive IVP defined by equation (12) on the t^{th} sub-interval, $t \geq 2$, and the boundary condition given by $\tilde{\beta}_{t-1}$, I can conclude that a unique solution exists to equations (2), (10) and (11) on $[0, \bar{v}]$.¹⁵ Since any symmetric equilibrium must be consistent with (2), (10) and (11), and since these define a unique function, it follows that β is the unique symmetric equilibrium of the EA game. ■

¹⁴If $\tilde{\beta}^{-1}(t\Delta)$ does not exist for some t (i.e., if bidding never reaches $t\Delta$), then simply re-define the upper boundary of the t^{th} interval as \bar{v} .

¹⁵For completeness, in order to invoke this theorem I must first verify a Lipschitz condition. On the second sub-interval, letting $h_2(v, y_2)$ represent the right-hand side of equation (12), where $y_2 = \tilde{\beta}_2(v)$, the Lipschitz condition requires that the derivative of h_2 with respect to y_2 must be uniformly bounded on the (convex) set of all possible

3.3. The Relation Between β , β^I and β^{II} . Now that I have pinned down the equilibrium of the EA game, I shall investigate its properties, with emphasis on its relation to the equilibria of the first- and second-price auction formats. Before moving on, it will be useful to observe that by Theorems 3.3 and 3.4, it turns out that β is not only differentiable, but *continuously* differentiable.¹⁶ This fact will come in handy for proving the following theorems.

Theorem 3.5. *For any $\Delta \in (0, E[V_{(M:M)}])$, the following is true:*

- (i) $\beta(v) < \beta^{II}(v)$, $v \in (0, \bar{v}]$, and
- (ii) $\beta^I(v) < \beta(v)$, $v \in (v_\Delta, \underline{v}]$.

Proof: (i) follows immediately from either Lemma 3.1, or Lemma 3.3.2 (see Appendix). As for (ii), first note that the differential equation which defines β^I is

$$(13) \quad \frac{d\beta^I(v)}{dv} = \frac{(v - \beta^I(v))g(v)}{G(v)},^{17}$$

which is similar to the differential equation defining β ,

$$(14) \quad \frac{d\beta(v)}{dv} = \frac{(v - \beta(v))g(v)}{G(v) - G(\beta^{-1}(\beta(v) - \Delta))}'$$

except for the presence of an additional term in the denominator. I begin by showing that β and β^I can have at most a single crossing point above v_Δ . Consider a hypothetical crossing point or a tangency at some $v' \in (v_\Delta, \bar{v}]$ and observe that the denominator of (14) would be strictly smaller than that of (13) at v' because $G(\beta^{-1}(\beta(v') - \Delta)) > 0$. Moreover, the numerators are the same, so we have $\beta'(v') > (\beta^I)'(v')$. Since the EA equilibrium slope is strictly higher at any crossing point, β can only cross β^I from below (if at all), and the two functions cross at most only once.

Now, if (ii) is not true then there exists some $v' \in (v_\Delta, \bar{v}]$ where $\beta^I(v') \geq \beta(v')$ and $\beta^I(v'') > \beta(v'')$ for each $v'' \in (v_\Delta, v')$. Define $\Gamma : [v_\Delta, v'] \rightarrow \mathbb{R}$ by $\Gamma(v) \equiv \beta^I(v) - \beta(v)$ and note that Γ is differentiable. Since $\beta^I(v_\Delta) = \beta(v_\Delta)$ and $\beta^I(v') \geq \beta(v')$, then by the mean value theorem there is some $v^* \in (v_\Delta, v')$ where $\Gamma'(v^*) \geq 0$, or $(\beta^I)'(v^*) \geq \beta'(v^*)$. However, since $G(\beta^{-1}(\beta(v^*) - \Delta)) > 0$, the denominator of (14) is strictly smaller than that of (13). Moreover, since the numerator of (14) is greater than that of (13) at v^* , it follows that $(\beta^I)'(v^*) < \beta'(v^*)$, a contradiction. ■

pairs (v, y_2) . More precisely, there must exist $L > 0$ such that

$$\left| \frac{\partial h_2(v, y_2)}{\partial y_2} \right| = \left| \frac{-g(v)}{G(v) - G(\tilde{\beta}^{-1}(y_2 - \Delta))} + \frac{(v - y_2)g(v)(G(\tilde{\beta}^{-1}(y_2 - \Delta)) - G(\tilde{\beta}^{-1}(y_2 - 2\Delta)))}{(G(v) - G(\tilde{\beta}^{-1}(y_2 - \Delta)))^2 (\tilde{\beta}^{-1}(y_2 - \Delta) - (y_2 - \Delta))} \right| \leq L$$

for all (v, y_2) such that $y_2 \in [\Delta, 2\Delta]$ and $y_2 \leq v \leq \gamma_2(2\Delta)$ (see Burden and Faires [3, theorems 5.3 and 5.4]). Using Lemma 3.1 and the properties of g , it is easy to see that the numerators of the terms inside the absolute value are bounded and the denominators are strictly bounded away from zero, implying that L exists. A similar argument verifies the Lipschitz condition on intervals $t = 3, 4, \dots$

¹⁶In fact, it is easy to see from equations (2) and (10) that β has as many continuous derivatives as F_V does.

¹⁷This can be seen by applying the maximization problem in (1) to the entire interval $[0, \bar{v}]$ and then deriving the FOC.

Theorem 3.5 shows, in contradiction of advice supplied to EA bidders, that they should *not* ensure that the maximum bid they enter is the highest price they're willing to pay. Since bid increments create a positive probability of paying one's bid after winning, reporting one's full value places positive probability on receiving zero surplus from a win. Thus, optimality requires at least some degree demand shaving.¹⁸ At the same time, Theorem 3.5 (ii) also shows that EA bidders shave demand by a smaller margin than in a first-price auction. While there is a chance that the winner will pay his bid, it is not a sure thing, so the incentives for demand reduction are less.

One might also ask how changes in Δ affect the relationship between β , β^I , and β^{II} . The remainder of the paper is geared toward answering this question.

Theorem 3.6. *Given a sequence $\{\Delta_k\}_{k=1}^\infty$, let the EA equilibrium under Δ_k be denoted by $\beta_k(\cdot)$ for each k . Then the following statements are true:*

- (i) *If $\{\Delta_k\}_{k=1}^\infty \rightarrow E[V_{(M:M)}]$, $\beta_k(v) \rightarrow \beta^I(v)$ uniformly, and*
- (ii) *if $\{\Delta_k\}_{k=1}^\infty \rightarrow 0$, $\beta_k(v) \rightarrow \beta^{II}(v)$ uniformly.*

Proof: To prove (i), pointwise convergence is easy to establish on the interval $[0, \bar{v}]$, because $v_\Delta \rightarrow \bar{v}$ as $\Delta \rightarrow E[V_{(M:M)}]$, since β^I is continuous, $v_\Delta = (\beta^I)^{-1}(\Delta)$, and $\beta^I(\bar{v}) = E[V_{(M:M)}]$. Therefore, for any $v \in [0, \bar{v}]$ and any sequence $\{\Delta_k\}_{k=1}^\infty \rightarrow E[V_{(M:M)}]$, there exists finite K such that $\beta_k(v) = \beta^I(v)$, $k \geq K$. As for pointwise convergence at \bar{v} , suppose for a contradiction that there is some $\varepsilon > 0$ and some subsequence where $\{\beta_{k_j}(\bar{v})\}_{j=1}^\infty \rightarrow \beta^I(\bar{v}) + \varepsilon$. Passing to that subsequence, suppose that player type \bar{v} reduces his bid to $\beta^I(\bar{v})$. The cost of the reduction is no more than $(\bar{v} - \beta_k(\bar{v})) [1 - G(v_{\Delta_k})]$, and comes from the fact that he will be defeated by opponents in some subset of the interval (v_{Δ_k}, \bar{v}) . By continuity of G , the cost is vanishing as k gets large, because $\{v_{\Delta_k}\}_{k=1}^\infty \rightarrow \bar{v}$. The benefit of the bid reduction comes from an expected price reduction. For example, in the event that $\beta^I(\bar{v}) - \Delta_k < \beta_k(V_{(M:M)}) < \beta^I(\bar{v})$, the price drops from $\min \left\{ \beta_k(V_{(M:M)}) + \Delta_k, \beta_k(\bar{v}) \right\}$ to $\beta^I(\bar{v})$. Given this fact, it is easy to see that the limiting benefit is strictly positive, meaning that $\beta_k(\bar{v})$ is sub-optimal for some large k , a contradiction. Finally, uniform convergence follows from the fact that $\{\beta_k(v)\}_{k=1}^\infty$ is a sequence of monotonic functions converging pointwise to a continuous limit on a compact set.

As for the proof of (ii), I begin by claiming that the denominator of the right-hand side of equation (10) is tending toward zero as k gets large, or $G \left[\beta_k^{-1}(\beta_k(v) - \Delta_k) \right] \rightarrow G(v)$, $\forall v \in (0, \bar{v}]$. To prove this claim, suppose for a contradiction that there exists $v^* \in (0, \bar{v}]$, $\varepsilon > 0$, and a subsequence such that $|\beta_{k_j}^{-1}(\beta_{k_j}(v^*) - \Delta_{k_j}) - v^*| \geq \varepsilon$, $\forall j \in \mathbb{N}$. Passing to that subsequence, by

¹⁸As a referee pointed out, another reason for sub-optimality of truth-telling might be the presence of other auctions for items that are close substitutes. Rather than bidding truthfully and risking competition with another bidder who values the item highly, there may exist an equilibrium of the dynamic game in which players bid less than their full value, while justifying the added risk of losing with the fact that there are plenty of other fish in the sea. Alternatively, Nekipelov [10] characterizes an equilibrium of the dynamic game in which concurrent auctions for similar items simply give rise to an entry stage, after which bidding is unaffected by outside options.

monotonicity the supposition implies that there is an interval of player types, $[v^* - \varepsilon, v^*]$ whose bids are within Δ_k of one another, for all k . Suppose that for some $\varepsilon' \in (0, \varepsilon)$, player type $v^* - \varepsilon'$ increases his bid by the amount of $\delta_k \equiv \beta_k(v^*) - \beta_k(v^* - \varepsilon') < \Delta_k$; this will lead to a gain of $[v^* - \varepsilon' - \beta_k(v^* - \varepsilon') - \Delta_k] [G(v^*) - G(v^* - \varepsilon')]$, because by doing so he will beat all opponents between himself and v^* . Since $v^* - \varepsilon - \beta_k(v^* - \varepsilon) > 0$ for each k , then by the supposition it follows that for all $\Delta_k \leq \frac{\varepsilon - \varepsilon'}{4}$, we have $v^* - \varepsilon' - \beta_k(v^* - \varepsilon') > \varepsilon - \varepsilon' - \Delta_k \geq (\varepsilon - \varepsilon') \frac{3}{4}$. Moreover, there exists finite K such that $\Delta_k \leq \frac{\varepsilon - \varepsilon'}{4}$, $\forall k \geq K$, so $v^* - \varepsilon' - \beta_k(v^* - \varepsilon') - \Delta_k > \frac{\varepsilon - \varepsilon'}{2}$ for each $k > K$, and the benefit remains strictly positive in the limit. On the other hand, the proposed bid increase is vanishing with k , so the resulting expected price increase is also vanishing, meaning that for some $K^* \geq K$ the bid increase is profitable, a contradiction. Thus, the denominator of (10) must approach zero for each $v \in (0, \bar{v}]$ as k gets large.

Finally, to prove pointwise convergence in (ii), suppose for a contradiction that there exists $v^* \in (0, \bar{v}]$, $\varepsilon > 0$, and a subsequence such that $\{\beta_{k_j}(v^*)\}_{j=1}^\infty \rightarrow v^* - \varepsilon$. Passing to that subsequence, for any $\varepsilon' \in (0, \varepsilon)$ there exists K such that for $k \geq K$ we have $|v^* - \beta_k(v^*)| \geq \varepsilon'$, and by monotonicity, for each $v \in [v^* - \frac{\varepsilon'}{2}, v^*]$ we also have $|v - \beta_k(v)| \geq \frac{\varepsilon'}{2} > 0$. Now note that for $k \geq K$ the numerator of equation (10) can be uniformly bounded below on the interval $[v^* - \frac{\varepsilon'}{2}, v^*]$ by

$$\frac{\varepsilon'}{2} \left(\min_{v \in [v^* - \frac{\varepsilon'}{2}, v^*]} g(v) \right) > 0.^{19}$$

Moreover, since $G \left[\beta_k^{-1}(\beta_k(v) - \Delta_k) \right]$ is a monotone function converging pointwise to a continuous limit ($G(v)$), then it converges uniformly on the set $[v^* - \frac{\varepsilon'}{2}, v^*]$. This implies that the denominator of equation (10) is converging uniformly to zero on the interval, or

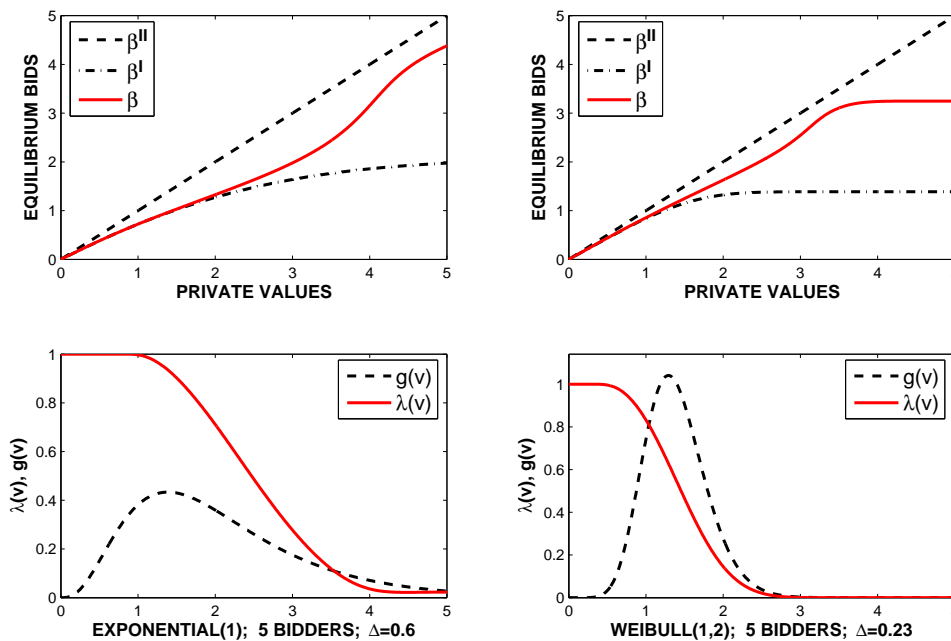
$$\lim_{k \rightarrow \infty} \left[\max_{v \in [v^* - \frac{\varepsilon'}{2}, v^*]} \left\{ G(v) - G \left[\beta_k^{-1}(\beta_k(v) - \Delta_k) \right] \right\} \right] = 0.$$

If this is true, then for any $L > 0$ there exists $K^* \geq K$ such that $\beta'_{K^*}(v) \geq L$ for all $v \in [v^* - \frac{\varepsilon'}{2}, v^*]$. However, this cannot be since for all k and for all $v \in [v^* - \frac{\varepsilon'}{2}, v^*]$ we have $\beta_k(v) < v^*$ by Theorem 3.5. Therefore, $\beta_k(v)$ must converge pointwise to $\beta^{II}(v) = v$, and uniform convergence again follows immediately from monotonicity on a compact set. ■

It is easy to see that first-price and second-price auctions are actually special cases of the EA game. However, Theorems 3.5 and 3.6 indicate something more: bid increments can be manipulated in such a way as to make bidder behavior closely resemble that in *either* format. Knowing this, one might also be interested to know the rate at which β approaches its limiting functions as Δ approaches either limit. As this question is difficult to answer analytically in the general case, I leave it to the next section where I explore the EA equilibrium using numerical methods.

¹⁹One possible exception might occur if v^* happens to be \bar{v} , in which case it could be that $g(\bar{v}) = 0$. If this is so, then one can simply replace the interval $[v^* - \frac{\varepsilon'}{2}, v^*]$ with $[v^* - \frac{\varepsilon'}{2}, v^* - \frac{\varepsilon'}{4}]$ in the proof, and the logic is then identical.

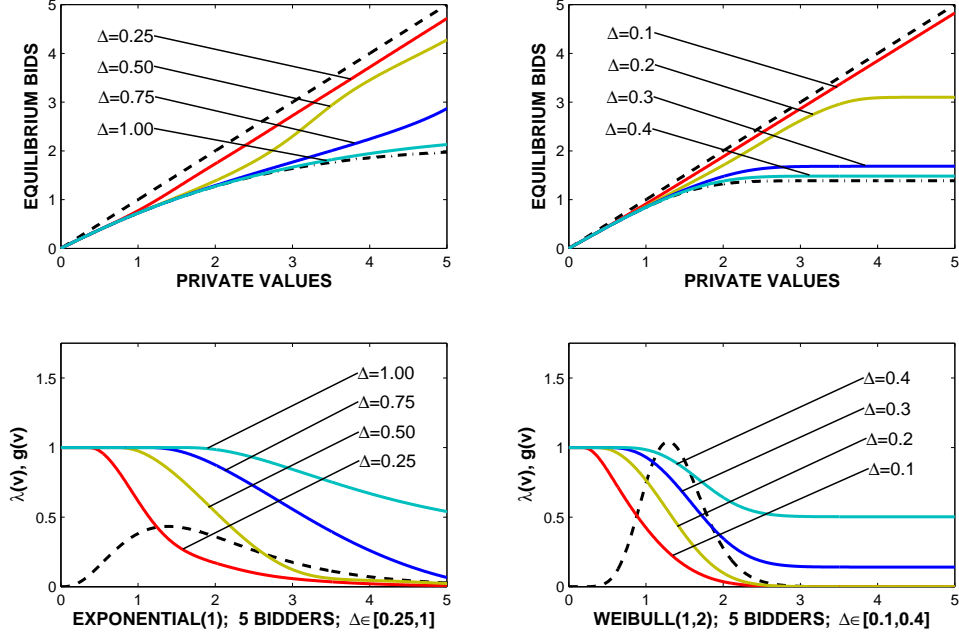
FIGURE 2. EA Bidding: Exponential and Weibull Cases



4. NUMERICAL EXAMPLES: THE EXPONENTIAL AND WEIBULL CASES

Unlike β^I or β^{II} , there is no specification of model primitives which delivers analytic solutions for β . Because it is difficult to obtain closed-form proofs, in this section I present numerical examples to illustrate some EA equilibrium properties graphically for the special case where private values are distributed Weibull, as well as for another nested case in which they are exponential. Interestingly, the shape of β is significantly more flexible than β^I or β^{II} , being both nonlinear and even non-concave. I solved the initial value problem defined by equations (10) and (11) in MATLAB using a fourth-order Runge-Kutta algorithm.

The left two panes of Figure 2 deal with the case where there are five bidders, $\Delta = 0.6$, and private values are distributed as an exponential random variable truncated to the interval $[0, 5]$, having hazard rate one. The right two panes deal with Weibull private values truncated to the interval $[0, 5]$, having cumulative distribution function $F_V(v) = \frac{1-e^{-v^2}}{1-e^{-5^2}}$. The bid increment here is $\Delta = 0.25$. These parameters, along with the others used in this section, were merely chosen to illustrate the flexible shape which $\beta(\cdot)$ may assume. The upper panes compare equilibrium bidding in the EA game (middle, solid line) with that in the second-price (upper, dashed line) and first-price auctions (lower, dash-dot line). The lower panes graph two other functions of interest. One is the density of the highest opponent value (dashed line), and the other function, labeled $\lambda(v)$, gives the probability of paying one's own bid after winning the auction (solid line).

FIGURE 3. Sensitivity to Changes in Δ : Exponential and Weibull Cases


Formally,

$$\lambda(v) \equiv \frac{G(v) - G(\beta^{-1}(\beta(v) - \Delta))}{G(v)}.$$

Some other details of Figure 2 are also worthy of mention. Even though $\lambda(v)$ is very small for values between 3 and 5, $\beta(\cdot)$ remains significantly separated from $\beta^{II}(\cdot)$ in this region. The intuition behind this phenomenon lies in an understanding of the tradeoffs bidders face when selecting a bid. By bidding higher they improve their probability of winning, which produces an upward pressure on bids. I will refer to this upward pressure as the *competitive effect*. However, when they bid higher they also increase the price they pay after winning, which produces a downward pressure. I will refer to this downward pressure the *pricing effect*.

In a second-price auction, the pricing effect is shut down: no tradeoff exists, as the price never depends on the winner's bid. Introducing bid increments changes the way in which both effects work. First, the pricing effect is no longer zero, creating a tradeoff that entices bidders to reduce their bids away from their valuation. Second, the competitive effect is weakened by the presence of a pricing effect. Roughly speaking, when lower bidder types shave their demand, higher bidder types have less incentive to aggressively outbid their lower opponents. This amplifies the incentive to shave demand. The weakening of the competitive effect becomes even more pronounced if more mass of the distribution is concentrated into a tight region, as in the Weibull example.

Figure 3 is an attempt to address the question of the rate at which β approaches its limiting functions as Δ approaches either 0 or $E[V_{(M:M)}]$. The upper and lower panes are similar as in

Figure 2, but now several plots of β are shown for different levels of Δ , ranging between 7% and 50% of $E[V_{(M:M)}]$.²⁰ As Δ increases, the gap between β and β^{II} widens. A comparison of the right- and left-hand panes shows that the rate at which β converges to its limits depends on the distribution of private values. In the exponential case, when $\Delta = 1$ (50.62% of $E[V_{(M:M)}]$) β tracks the first-price equilibrium very closely, and in the Weibull case similar results are produced with only $\Delta = 0.4$ (28.81% of $E[V_{(M:M)}]$).

5. CONCLUSION

I have introduced some new insights into bidder behavior in EAs by showing that, contrary to general understanding, EAs are not equivalent to second-price auctions. Rather, they are a peculiar hybrid of the second-price *and* first-price auction formats. I have also shown that the EA equilibrium lies between the first- and second-price equilibria. Manipulating bid increments can have a dramatic influence on the incentives bidders face, and in turn, on their choice of bids.

I have also provided some empirical evidence for the practical importance of bid increments. In theory, winners of eBay auctions for used laptops will consider the fact that the equilibrium probability of paying their own bid is roughly 23%. The amount of demand reduction resulting from this fact hinges on the underlying distribution of private information, and is therefore an empirical question. However, as the numerical examples demonstrate, the probability of paying one's own bid need not be very high in order for significant demand reduction to occur.

In empirical work, failing to account for the first-price component of EAs may systematically bias estimation of both the distribution of private information, as well as standard errors for parameter estimates. Current structural models of eBay that use a pure second-price assumption have two aspects in common: prices always reflect the second-highest bid, and bids submitted in the closing moments of an auction are generated by the static second-price equilibrium bid function.²¹ However, when increments are present and bidders believe that they may have to pay their own bid, both facts are violated.

Correctly characterizing the processes which generate bids and prices plays a pivotal role in electronic auctions research. Future work must account for their unique strategic environment, and previous work based on the second-price view may need to be re-examined. Clearly, the implications of bid increments should not be ignored.

APPENDIX A

Proof of Proposition 3.2:

Here I prove that in the EA game there exists a symmetric monotonic equilibrium in undominated strategies where bidding ties occur with zero probability. Recall that this result follows from others proven in JS and JSSZ. The main result of JS is their Theorem 6 (p. 114), in which the authors establish (under some assumptions) the existence of an equilibrium in undominated

²⁰In the Exponential example, $E[V_{(M:M)}] = 1.9756$ and in the Weibull example, $E[V_{(M:M)}] = 1.3884$.

²¹Two such models are Nekipelov [10], within the IPV paradigm, and Bajari and Hortaçsu [1], which uses a common values framework.

strategies in which ties among bidders occur with zero probability. One version of their proof invokes Theorem 1 of JSSZ (p. 1720), which states that (under equivalent assumptions) games with indeterminate outcomes may be augmented with a tie-breaking rule, based on reports of private information, in such a way that existence of an equilibrium is guaranteed, and truth-telling induced.²² JSSZ also prove another result (Theorem 2, p. 1722) from which Theorem 1 follows, and which also guarantees that when bidders are symmetric, a symmetric solution exists to the augmented game. Hence, by imposing symmetry and invoking JSSZ Theorem 2 in the proof of JS Theorem 6, we have symmetry as well.

JS Corollary 14 (p. 123) delivers monotonicity in addition to the above properties. Moreover, since imposing symmetry does not alter the proof of JS Corollary 14, I have for the EA game existence of a symmetric, monotonic equilibrium in undominated strategies, so long as the assumptions of JS (hereafter denoted JS1-JS10 and JS8') are satisfied. From here I need only show that the EA game satisfies assumptions JS1-JS10 and JS8', listed below.

Brief Descriptions of JS assumptions 1 – 10 and 8':²³

JS1: (Private values) *Ex-post* utility is determined by private information only.

JS2: Compact type space.

JS3: (Imperfect correlation of player types) The joint distribution of values is absolutely continuous and has a continuous Radon-Nikodym derivative representing the joint density of values.

JS4: The joint distribution of values is atomless.

JS5: Marginal valuations for additional units of the sale object are non-increasing in the number of items held.

JS6: Bidders' preferences over net payoffs are characterized by continuous strictly increasing VonNeumann-Morgenstern utility functions with first derivative that is finite and strictly bounded away from 0.

JS7: (Conditional on winning, payments are non-decreasing in bids) If a player is a net buyer of h units, his net payment is non-decreasing in the bids for the 1st through h^{th} additional units, and constant in all others; If he is a net seller of h units, the reverse is true.²⁴

JS8: Payments in tie situations are dependent only upon a player's bid.

JS9: Marginal payments are non-decreasing in the number of units won.

²²Here, I use the term truth-telling in the mechanism design sense. The idea is that there is a principal who explains the bidding function and tie-breaking rules to the agents who then truthfully report their private information to the principal. The principal then implements the equilibrium strategy on the players' behalf. This is not the same sense in which truth-telling was meant in the body of this paper. There, truth-telling referred to a particular equilibrium in which bids and private information are the same.

²³EXPLANATORY NOTE: JS includes assumptions JS5, JS9 and JS8' to cover the case of auctions for multiple units of an object. These assumptions are trivially satisfied in the single unit case.

²⁴JS study a general double auction setting in which bidders enter the auction market with an endowment of a certain number of units of a good for sale, and a vector of marginal valuations for additional units beyond those that they already possess. The single object auction is a special case in which all bidders are endowed with zero units, and their marginal valuation vector is one-dimensional.

JS10: For each i we can define a measurable map $M : [0, \bar{v}] \times \mathbb{R}_+ \rightarrow [0, \bar{v}] \times \mathbb{R}_+$ that sends any pair (v_i, b) into another pair (v_i, b') where b' is in the closure of the set of all undominated strategies, given value v_i .

JS8': JS8 is satisfied and holding bids of the other players constant, the effect of a change in i 's bid on his payments is a function only of what the change in that bid is.

First note that JS5, JS7, JS9 and JS8' are meant to cover the general case of auctions for multiple objects, and are trivially satisfied for the special case of a single object. This leaves only seven conditions to satisfy. JS1-JS4 and JS6 are true by assumption in the present model (see section 2), and the tie breaking rule deals only with allocation of the object, so JS8 is also satisfied. Finally, since any bid $b \in (0, v_i)$ is undominated, we can define the mapping M by

$$M(v_i, b) = \begin{cases} (v_i, b) & \text{if } b \leq v_i \\ (v_i, v_i) & \text{otherwise.} \end{cases}$$

This produces a measurable mapping, so JS10 is satisfied, which completes the proof. ■

APPENDIX B

Proof of Proposition 3.3:

Before proving that a symmetric monotonic EA equilibrium is differentiable, it will be necessary to establish a few intermediate results.

Lemma 3.3.1: *Any symmetric, monotonic equilibrium of the EA game is continuous everywhere.*

Proof: I will first establish left-continuity. Suppose on the contrary that there exists $\bar{v} \in (0, \bar{v}]$ and $\varepsilon > 0$ such that for each $v < \bar{v}$ we have

$$\beta(\bar{v}) - \beta(v) \geq \varepsilon.^{25}$$

It will suffice to consider only the case where $\varepsilon < \Delta$. Consider the bidder with value \bar{v} . Conditional on the event

$$\beta(\bar{v}) - \Delta \leq \beta(V_{(M:M)}) \leq \beta(\bar{v}),$$

he will pay exactly his bid. Since the equilibrium is increasing, he could reduce his bid by the amount of ε without changing the probability of being outbid by an opponent. Moreover, with positive probability he will have decreased the price he pays if he wins, thus improving his expected payoff, a contradiction. Therefore, β must be left-continuous.

Next, I will establish right continuity. Suppose on the contrary that there exists $\bar{v} \in [0, \bar{v})$ and $\varepsilon > 0$ such that for each $v > \bar{v}$ we have

$$\beta(v) - \beta(\bar{v}) \geq \varepsilon.$$

Consider a bidder with value $\bar{v} + \delta$. Similarly as in the argument above, this player can gain by reducing his bid by the amount ε , as this will decrease the price he pays upon winning. The following expression provides a lower bound on the gain to be had:

$$(B.1) \quad (\beta(\bar{v} + \delta) - \varepsilon) \Pr \left[\beta(\bar{v} + \delta) - \Delta \leq \beta(V_{(M:M)}) \leq \beta(\bar{v} + \delta) - \varepsilon \right].$$

²⁵This type of jump discontinuity is the only one possible, as a removable discontinuity would violate monotonicity.

However, this gain is now offset by the fact that he might lose to one of his competitors whose type is in the interval $(\tilde{v}, \tilde{v} + \delta)$, whom he would have outbid otherwise. The associated cost is given by

$$(B.2) \quad (\tilde{v} + \delta - \beta(\tilde{v} + \delta)) \cdot \Pr \left[V_{(M:M)} \in (\tilde{v}, \tilde{v} + \delta) \right].$$

Note that expression (B.2) approaches zero as δ gets small, whereas expression (B.1) remains positive in the limit. Thus, there exists some δ small enough so that all player types in the interval $(\tilde{v}, \tilde{v} + \delta)$ could profit by decreasing their bid, another contradiction. This establishes right-continuity, which completes the proof of general continuity. ■

Lemma 3.3.2: *For any symmetric, monotonic, continuous equilibrium of the EA game, we have $v - \beta(v) > 0$ for each $v \in (0, \bar{v}]$.*

Proof: Suppose not. I already know that $\beta(v) \leq v$ since β is an equilibrium in undominated strategies. So, the supposition implies that $\beta(\tilde{v}) = \tilde{v}$ for some $\tilde{v} \in (0, \bar{v}]$. Since $f_V(v)$ is positive on the entire interval, there is positive probability that a player with type \tilde{v} will pay his bid if he wins. Consider the effect of a reduction in that player's bid by some margin $\varepsilon > 0$. For $\varepsilon < \Delta$, the benefit comes from an expected price decrease associated with the reduction. This expected price decrease is bounded below by

$$\varepsilon \cdot \left[G(\beta^{-1}(\beta(\tilde{v}) - \varepsilon)) - G(\beta^{-1}(\beta(\tilde{v}) - \Delta)) \right] > 0.$$

The cost of the reduction arises from the fact that this player type will now be outbid by an additional interval of opponents with $v \in (\beta^{-1}(\beta(\tilde{v}) - \varepsilon), \tilde{v})$. However, winning the auction against any such opponents means that player type \tilde{v} would pay exactly $\beta(\tilde{v}) = \tilde{v}$, so the cost is zero. Thus, player type \tilde{v} could do strictly better by reducing his bid and by continuity there is an interval of player types close to \tilde{v} that could also do better, a contradiction. ■

I will now argue that the slope of the equilibrium, if it exists, must be positive and finite. The following arguments apply to the entire interval $(0, \bar{v}]$, and they make repeated use of the following expression: $G(\beta^{-1}(\beta(v) - \Delta))$. Although this expression is only defined on the interval $[v_\Delta, \bar{v}]$, for simplicity I will define it to be zero on the interval $(0, v_\Delta)$. This will eliminate the need to consider additional cases where the logic is the same, but with different notation.

Lemma 3.3.3: *For each $v \in (0, \bar{v}]$ and each sequence $\{v_k\}_{k=1}^\infty \rightarrow v$, the following quantity is finite:*

$$\limsup_{k \rightarrow \infty} \frac{\beta(v) - \beta(v_k)}{v - v_k}.$$

Proof: Suppose not. That is, suppose that there is some type v and some sequence $\{v_k\}_{k=1}^\infty \rightarrow v$ such that the above lim sup is $+\infty$.²⁶ Passing to a monotone subsequence, suppose that $\{v_k\}_{k=1}^\infty \uparrow v$. Consider the effect of a bid reduction for player type v from $\beta(v)$ to $\beta(v_k)$ for some k . For

²⁶Since β is increasing, the lim sup could not be $-\infty$.

large k , the benefit of this bid reduction is bounded below by

$$(\beta(v) - \beta(v_k)) \left[G(v_k) - G(\beta^{-1}(\beta(v) - \Delta)) \right].^{27}$$

On the other hand, the cost is given by

$$(v - \beta(v)) [G(v) - G(v_k)].$$

Since G is C^1 , one can approximate the second term in the above product within a neighborhood of v by $g(v)(v - v_k)$. Thus, the lower bound of the benefit-cost ratio behaves similarly as the following ratio for large k :

$$(B.3) \quad \frac{G(v_k) - G(\beta^{-1}(\beta(v) - \Delta))}{(v - \beta(v)) g(v)} \cdot \frac{\beta(v) - \beta(v_k)}{(v - v_k)}.$$

Since β is an equilibrium, ratio (B.3) cannot rise above 1, as that would imply that the net benefit of the bid reduction was positive. However, by Lemma 3.3.2 the first term is positive and finite, and by our supposition the second term tends toward $+\infty$ as k gets large. Thus, there exists some large K such that for each $k \geq K$, ratio (B.3) is greater than one, a contradiction. For the case where the monotone subsequence is decreasing, a similar contradiction is produced when player type v_k reduces his bid to $\beta(v)$. Thus, the result of the lemma follows. ■

Lemma 3.3.4: *For each $v \in (0, \bar{v}]$ and each sequence $\{v_k\}_{k=1}^{\infty} \rightarrow v$, the following quantity is strictly positive:*

$$\liminf_{k \rightarrow \infty} \frac{\beta(v) - \beta(v_k)}{v - v_k}.$$

Proof: Suppose not. That is, suppose that there is some type v and some sequence $\{v_k\}_{k=1}^{\infty} \rightarrow v$ such that the above lim inf is equal to zero. Passing to a monotone subsequence, suppose that $\{v_k\}_{k=1}^{\infty} \uparrow v$. Consider the effect of a bid increase for player type v_k from $\beta(v_k)$ to $\beta(v)$ for large k . The benefit of such an increase is given by

$$(B.4) \quad (v_k - \beta(v)) [G(v) - G(v_k)].$$

Since G is C^1 , one can approximate the second term in the above product within a neighborhood of v by $g(v)(v - v_k)$. On the other hand, the cost is given by

$$(B.5) \quad \begin{aligned} & (\beta(v) - \beta(v_k)) \left[G(v_k) - G(\beta^{-1}(\beta(v) - \Delta)) \right] \\ & + \left[G(\beta^{-1}(\beta(v) - \Delta)) - G(\beta^{-1}(\beta(v_k) - \Delta)) \right] \\ & \times \int_{\beta^{-1}(\beta(v_k) - \Delta)}^{\beta^{-1}(\beta(v) - \Delta)} g(V_{(M:M)}) \cdot \left(\beta(V_{(M:M)}) + \Delta - \beta(v_k) \right) dV_{(M:M)}. \end{aligned}$$

In expression (B.5) above, the first term captures the uniform price increase over an interval where the price rose from $\beta(v_k)$ to $\beta(v)$. The second term captures the non-uniform expected price

²⁷For the intuition behind this lower bound, see equation (B.5) and its explanation below, which describes the exact effect. Note that the benefit of a bid reduction is the same as the cost of a bid increase.

increase over an adjoining interval of realizations of $V_{(M:M)}$ such that $\beta(v_k) < \beta(V_{(M:M)}) + \Delta < \beta(v)$, where the price rose from $\beta(v_k)$ to $\beta(V_{(M:M)}) + \Delta$.

I claim that for large k this second term may safely be ignored, implying that the benefit-cost ratio behaves similarly as the following ratio:

$$(B.6) \quad \frac{(v_k - \beta(v)) g(v)}{G(v_k) - G(\beta^{-1}(\beta(v) - \Delta))} \cdot \frac{(v - v_k)}{\beta(v) - \beta(v_k)}.$$

To justify this claim, I will show that the second term in expression (B.5) vanishes at a faster rate than both the first term (*i.e.*, the denominator of (B.6)) and expression (B.4) (*i.e.*, the numerator of (B.6)). Again, since G is C^1 , for large k one can rewrite

$$\begin{aligned} & G(\beta^{-1}(\beta(v) - \Delta)) - G(\beta^{-1}(\beta(v_k) - \Delta)) \\ & \approx g(\beta^{-1}(\beta(v) - \Delta)) \cdot \left(\beta^{-1}(\beta(v) - \Delta) - \beta^{-1}(\beta(v_k) - \Delta) \right). \end{aligned}$$

Also by the C^1 property of G , $g(V_{(M:M)})$ can be bounded on the appropriate interval by some constant, call it ρ . Moreover, since $\beta(v)$ is monotonic, I have the following inequality:

$$\begin{aligned} & \int_{\beta^{-1}(\beta(v_k) - \Delta)}^{\beta^{-1}(\beta(v) - \Delta)} g(V_{(M:M)}) \cdot \left(\beta(V_{(M:M)}) + \Delta - \beta(v_k) \right) dV_{(M:M)} \\ & \leq \rho \cdot \left[\beta(\beta^{-1}(\beta(v) - \Delta)) + \Delta - \beta(v_k) \right] \cdot \int_{\beta^{-1}(\beta(v_k) - \Delta)}^{\beta^{-1}(\beta(v) - \Delta)} dV_{(M:M)} \\ & = \rho \cdot (\beta(v) - \beta(v_k)) \cdot \left(\beta^{-1}(\beta(v) - \Delta) - \beta^{-1}(\beta(v_k) - \Delta) \right). \end{aligned}$$

Thus, the second term in (B.5) vanishes no slower than

$$(\beta(v) - \beta(v_k)) \cdot \left[\beta^{-1}(\beta(v) - \Delta) - \beta^{-1}(\beta(v_k) - \Delta) \right]^2.$$

This is clearly faster than the rate at which the first term in (B.5) vanishes, which by our supposition is faster than the rate at which (B.4) vanishes. Thus, the second term in (B.5) may be safely ignored, and ratio (B.6) adequately represents the behavior of the benefit-cost ratio for large k .

Since β is an equilibrium, ratio (B.6) cannot rise above 1, as that would imply that the net benefit of the bid increase was positive. However, by Lemma 3.3.2 the first term is positive and non-vanishing, and by the supposition, the second term tends toward $+\infty$ as k gets large. Thus, there exists some large K such that for each $k \geq K$, ratio (B.6) is greater than one, a contradiction. For the case where the monotone subsequence is decreasing, a similar contradiction is produced when player type v increases his bid to $\beta(v_k)$. Thus, the result of the lemma follows. ■

Finally, the following result follows immediately from Lemmas 3.3.3 and 3.3.4 and from the properties of the lim inf and lim sup operators:

Lemma 3.3.5: *Let $\phi(\cdot)$ denote the inverse bidding function. For each $v \in (0, \bar{v}]$ and each sequence $\{v_k\}_{k=1}^{\infty} \rightarrow v$, and defining $b \equiv \beta(v)$ and $b_k \equiv \beta(v_k)$ for each k , then the following inequalities hold:*

$$0 < \liminf_{k \rightarrow \infty} \frac{\phi(b) - \phi(b_k)}{b - b_k} \leq \limsup_{k \rightarrow \infty} \frac{\phi(b) - \phi(b_k)}{b - b_k} < \infty.$$

Now, armed with lemma 3.3.5, I am ready to prove Proposition 3.3. From here, the general form of the proof follows a strategy used by Lizzeri and Persico [7, Appendix A.4, Lemma 7] when they proved differentiability of equilibrium bidding in a two-player auction game. Recall that $\pi(v_i, b_i, b_{(M:M)})$ denotes the payoff to player i in the event of a win against $b_{(M:M)}$ and it is defined by

$$\pi(v_i, b_i, b_{(M:M)}) = \begin{cases} v_i - b_{(M:M)} - \Delta, & \text{if } b_i > b_{(M:M)} + \Delta; \\ v_i - b_i, & \text{if } b_i \leq b_{(M:M)} + \Delta. \end{cases}$$

Note that π is continuous and differentiable, except along the line $b_i - b_{(M:M)} = \Delta$, where the partial derivative with respect to the second argument does not exist. Also, for any v_i , $\pi(v_i, \cdot, \cdot)$ is differentiable within a neighborhood of (b_i, b_i) . As in Lemma 3.3.5, let $\phi(b)$ denote the inverse bid function.

Fix any $v \in (0, \bar{v}]$, and consider an increasing sequence $\{v_k\}_{k=1}^{\infty} \uparrow v$. Let $b_k \equiv \beta(v_k)$, $b \equiv \beta(v)$ and note that continuity implies $\{b_k\}_{k=1}^{\infty} \uparrow b$. Since players of type v_k always prefer to bid b_k , I have the following inequality:

$$\begin{aligned} & \int_0^{\phi(b_k)} \pi(v_k, b_k, \beta(V_{(M:M)})) g(V_{(M:M)}) dV_{(M:M)} \\ & \geq \int_0^{\phi(b)} \pi(v_k, b, \beta(V_{(M:M)})) g(V_{(M:M)}) dV_{(M:M)}. \end{aligned}$$

Now, subtracting from both sides the quantity

$$\int_0^{\phi(b_k)} \pi(v_k, b, \beta(V_{(M:M)})) g(V_{(M:M)}) dV_{(M:M)},$$

I am left with

$$\begin{aligned} & \int_0^{\phi(b_k)} \left[\pi(v_k, b_k, \beta(V_{(M:M)})) - \pi(v_k, b, \beta(V_{(M:M)})) \right] \\ & \quad \times g(V_{(M:M)}) dV_{(M:M)} \\ & \geq \int_{\phi(b_k)}^{\phi(b)} \pi(b, \beta(V_{(M:M)}), v_k) g(V_{(M:M)}) dV_{(M:M)}. \end{aligned}$$

Note that the LHS can be broken up into intervals on which the partial derivative of π with respect to its second argument is known to exist. This yields the following:

$$\begin{aligned} & \int_0^{\phi(b_k - \Delta)} \left[\pi(v_k, b_k, \beta(V_{(M:M)})) - \pi(v_k, b, \beta(V_{(M:M)})) \right] \\ & \quad \times g(V_{(M:M)}) dV_{(M:M)} \\ & + \int_{\phi(b_k - \Delta)}^{\phi(b_k)} \left[\pi(v_k, b_k, \beta(V_{(M:M)})) - \pi(v_k, b, \beta(V_{(M:M)})) \right] \\ & \quad \times g(V_{(M:M)}) dV_{(M:M)} \\ & \geq \int_{\phi(b_k)}^{\phi(b)} \pi(v_k, b, \beta(V_{(M:M)})) g(V_{(M:M)}) dV_{(M:M)}. \end{aligned}$$

Dividing both sides by $b - b_k$ and taking lim sup yields

$$(B.7) \quad \int_{\phi(b-\Delta)}^{\phi(b)} \frac{\partial}{\partial b} \pi(v, b, \beta(V_{(M:M)})) g(V_{(M:M)}) dV_{(M:M)} \\ \geq \limsup_{n \rightarrow \infty} \frac{1}{b - b_k} \int_{\phi(b_k)}^{\phi(b)} \pi(v_k, b, \beta(V_{(M:M)})) g(V_{(M:M)}) dV_{(M:M)},$$

where the first integral on the LHS has been dropped because $\frac{\partial \pi(v, b, \beta(V_{(M:M)}))}{\partial b} = 0$ for $V_{(M:M)}$ in the interval $[0, \phi(b - \Delta)]$.

By way of a digression, consider separately the expression on the RHS of inequality (B.7) and define a sequence of functions $\{g_k\}_{k=1}^{\infty}$ by the following

$$g_k : [\phi(b_k), \phi(b)] \rightarrow \mathbb{R}_+ \quad k = 1, 2, 3, \dots \quad \text{where} \\ g_k(V_{(M:M)}) \equiv \pi(v_k, b, \beta(V_{(M:M)})) g(V_{(M:M)}).$$

Also, define $\theta \equiv \pi(v, b, \beta(v)) g(\phi(b))$ and note that by continuity of π , g and ϕ , it follows that for each k there exists ε_k such that

$$\theta - \varepsilon_k \leq g_k \leq \theta + \varepsilon_k \quad \text{and} \\ \{\varepsilon_k\}_{k=1}^{\infty} \rightarrow 0.$$

Thus, I have the following

$$\limsup_{k \rightarrow \infty} \int_{\phi(b_k)}^{\phi(b)} \frac{(\theta - \varepsilon_k)}{b - b_k} dV_{(M:M)} \\ \leq \limsup_{k \rightarrow \infty} \int_{\phi(b_k)}^{\phi(b)} \frac{g_k(V_{(M:M)})}{b - b_k} dV_{(M:M)} \\ \leq \limsup_{k \rightarrow \infty} \int_{\phi(b_k)}^{\phi(b)} \frac{(\theta + \varepsilon_k)}{b - b_k} dV_{(M:M)}.$$

The LHS and RHS above both involve constants that can be pulled out of the integral, allowing me to rewrite the inequalities as

$$\limsup_{k \rightarrow \infty} (\theta - \varepsilon_k) \frac{\phi(b) - \phi(b_k)}{b - b_k} \\ \leq \limsup_{k \rightarrow \infty} \int_{\phi(b_k)}^{\phi(b)} \frac{g_k(V_{(M:M)})}{b - b_k} dV_{(M:M)} \\ \leq \limsup_{k \rightarrow \infty} (\theta + \varepsilon_k) \frac{\phi(b) - \phi(b_k)}{b - b_k}.$$

Finally, Lemma 3.3.5 makes it possible to rewrite these as

$$\begin{aligned}
 (B.8) \quad \theta \limsup_{k \rightarrow \infty} \frac{\phi(b) - \phi(b_k)}{b - b_k} & \\
 & \leq \limsup_{k \rightarrow \infty} \int_{\phi(b_k)}^{\phi(b)} g_k(V_{(M:M)}) \frac{dV_{(M:M)}}{b - b_k} \\
 & \leq \theta \limsup_{k \rightarrow \infty} \frac{\phi(b) - \phi(b_k)}{b - b_k}.
 \end{aligned}$$

Thus, having reached the end of the digression, I now substitute (B.8) into the RHS of inequality (B.7) and rearrange to get

$$\begin{aligned}
 \limsup_{k \rightarrow \infty} \frac{\phi(b) - \phi(b_k)}{b - b_k} & \leq \\
 & \frac{\int_{\phi(b-\Delta)}^{\phi(b)} \frac{\partial}{\partial b} \pi(v, b, \beta(V_{(M:M)})) g(V_{(M:M)}) dV_{(M:M)}}{\pi(v_k, b, \beta(v)) g(\phi(b))}.
 \end{aligned}$$

Similar reasoning also leads to

$$\begin{aligned}
 \liminf_{k \rightarrow \infty} \frac{\phi(b) - \phi(b_k)}{b - b_k} & \geq \\
 & \frac{\int_{\phi(b-\Delta)}^{\phi(b)} \frac{\partial}{\partial b} \pi(v, b, \beta(V_{(M:M)})) g(V_{(M:M)}) dV_{(M:M)}}{\pi(v_k, b, \beta(v)) g(\phi(b))},
 \end{aligned}$$

and since the same exercise could also be done with a decreasing sequence $\{v_k\}_{k=1}^{\infty} \downarrow v$, it follows that $\frac{d\phi(b)}{db}$ exists. Last of all, since $\frac{d\phi(b)}{db} > 0$ by Lemma 3.3.5, it follows that $\frac{d\beta(v)}{dv}$ exists as well. ■

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