College Assignment as a Large Contest

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Abstract

We develop a model of college assignment as a large contest wherein students with heterogeneous abilities compete for seats at vertically differentiated colleges through the acquisition of productive human capital. We use a continuum model to approximate the outcomes of a game with large, but finite, sets of colleges and students. By incorporating two common forms of affirmative action in our model, admissions preferences and quotas, we can show that (legal) admissions preference schemes and (illegal) quotas are outcome equivalent. We assess the welfare costs of using human capital accumulation to compete for college admissions. While competition is necessary for (welfare enhancing) assortative match, the welfare losses from the accumulation of human capital solely to compete for a better college seat are also significant.

Keywords: Affirmative action; welfare costs of competition; contests; approximate equilibrium.

JEL subject classification: D44, C72, I20, I28, L53.

1 Introduction

There are many crucially important economic features of the competition between students for admission to colleges. An ideal model would include heterogeneity amongst the colleges in terms of quality, allow for differences amongst the students in terms of ex ante ability, and endogenize the decisions students make to compete for admission. For many policy questions, it is also necessary to allow for asymmetric admissions policies that allow for easier admissions for the children of alumni or students from underrepresented demographic backgrounds. While many models include one or two of these effects, providing a tractable model of the market that includes all three features has proven difficult. A primary reason for the difficulty is the dual role played by human
capital (HC) in college admissions: first, a student’s HC is a productive asset, which yields a *productive channel* of investment incentives, and second, students who acquire more HC gain access to higher quality colleges, which yields a *competitive channel* of incentives.

We model the college admissions market as a contest wherein colleges are rank-ordered and students compete for admission by endogenously choosing the level of HC to accrue prior to the admissions contest. While this model is difficult to solve when it includes a finite number of students and colleges, we show that a more tractable limit model with a continuum of colleges and students closely approximates a model with a large, but finite, number of students and colleges. To understand why a large market setting might simplify things, consider a student (or a student’s parents) that is deciding how much effort to exert in school with an eye toward her senior year when she will apply to various colleges. If the student wants to ascertain whether she is likely to be admitted to a school with a given GPA and SAT score, she does not need to consider the other students who might also be applying (her opponents) or the strategies they are employing. Instead, she simply consults a college guide that describes the qualifications of previously admitted students at her favorite schools. Since the aggregated choices of many market players produces a high degree of predictability in these thresholds, she can have confidence that if she meets them then admission will be reasonably likely.

The combination of model richness and tractability allow us to explore two key aspects of college matching markets that depend on the interplay between the distinct channels of incentives and college/student heterogeneity when admissions thresholds may be demographically conditioned. Our first policy analysis goal is to assess the difference, if any, between affirmative action (AA) schemes implemented through an admissions preference system and those implemented through a quota. We use the term “affirmative action” to refer to any policy that discriminates between individuals based on demographic features. Although this paper focuses on affirmative action in the context of U.S. college admissions, the US government has mandated AA practices in various areas of the economy where it has influence, including education, employment, and procurement. AA has been widely implemented outside the United States as well, in places such as Malaysia, Northern Ireland, India, and South Africa.

An affirmative action system implemented through a quota reserves separate pools of seats for different groups of students, and each student can only compete for the seats allocated to his or her group. In an admissions preference scheme, all applicants compete for the same pool of seats, but the fashion in which the applications are ranked is

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1 Although this example and our main analysis are focused on college admissions, the basic insight regarding the tractability and applicability of limit games as approximations to real-world contests generalizes far outside the context of college admissions and affirmative action.

2 For an in-depth discussion of AA implementations around the world, see Sowell[58].
dependent on the demographics of the applicant. We show that the quota and admissions preference schemes are outcome equivalent. In other words, given any equilibrium of any quota (admissions preference system), one can design an admissions preference system (quota) with an equilibrium that results in both the same school assignments and HC choices.

On a more practical level, the equivalence of quota and admission preference systems raises difficult questions regarding the legality of these schemes in the context of U.S. higher education. Since the 1978 Supreme Court Case Regents of the University of California v. Bakke, constitutional jurisprudence has held that racial quotas violate the equal protection clause of the 14th amendment to the U.S. constitution since some students are prevented from competing for some of the seats. In contrast, admissions preference schemes may be acceptable under constitutional law since any student can compete for any seat, albeit the competition is not on a level playing field.

Our result calls into question the bright-line delineation of quotas and admissions preference systems developed by the 1978 opinion of Justice Powell. The later 2003 judgements of Grutter v. Bollinger et al. and Gratz et al. v. Bollinger et al. turned on whether the University of Michigan had implemented an admissions preference scheme that resulted in a de facto racial quota, which both foreshadows and highlights the legal importance of the perceived differences between quotas and admission preferences. To the extent the constitutionality of an affirmative action scheme hinges on the outcomes of these systems and/or the policy-induced competitive conditions (e.g., incentives) that produced them, then the existing precedents are self-contradictory.

The second policy question we address is the welfare cost of forcing the students to compete for college seats through costly acquisition of HC. Once again, there are two incentives for a student to invest in HC: first, the productive channel—since HC is a productive asset bearing a direct return—and second, the competitive channel—since HC is also used to rank order students at the college matching stage. In our model it would be welfare maximizing to assortatively match students (by innate productivity) and colleges (by institutional quality level) and allow each student to then choose their preferred human capital investment level given his or her college assignment. Unfortunately, since a student’s type is private information, the contest naturally relies on observable HC output to differentiate between higher and lower productivity students. The reliance on HC in the assignment process in turn induces them to choose a wastefully high level of human

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3Both of the authors are in favor of affirmative action schemes. Our argument is purely regarding the consistency (or lack thereof) of prior legal precedent, not an attempt to argue for (or against) the efficiency or ethics of affirmative action.

4Later cases, which we discuss in depth in section 6, have refined the standards that acceptable admissions preference schemes must meet.
capital (i.e., above that resulting from a first-best assignment) to compete for a seat at a good college, and this wasteful over-investment is the cost that society pays to achieve matching assortativity in the presence of private information.

A natural question then is, at the societal level, are the benefits of assortativity in the college match enough to outweigh the costs of wasteful over-investment? There are some signs that the waste caused by the competitive channel may be getting worse with time. Anecdotal evidence of the increasing pressure to compete for college seats includes the “tiger mom” phenomenon brought to popular attention by the 2011 book *Battle Hymn of the Tiger Mother* by Amy Chua as well as popular book *The Overachievers: The Secret Lives of Driven Kids* that documents the competitive pressures placed on high school students.

Combining our model with estimates from Hickman [36] we are able to provide comparisons between a number of alternative systems to provide perspective on the relative merits of different college assignment systems. First we compare the welfare generated by the first-best system and an assortative color-blind contest, which allows us to compute the welfare losses caused by competition. Next we compute the welfare generated by a match that randomly assigns students to schools before the students choose their HC levels. Comparing the outcome of random assignment with the color-blind contest yields a measure of the benefits of assortative assignment. We find that the cost of competition ($1,225 loss per student per year) entirely wipes out the benefits of an assortative match ($822 gain per student per year). In other words student welfare would be improved by randomly assigning students to colleges instead of allowing the students to compete.

The final welfare analysis we conduct is a computation of the second-best contest design. We use our limit model to formalize an optimal control problem that computes the welfare maximizing college assignment contest subject to the incentive compatibility constraints of the contest structure. Since demography is a sign of underlying ability, demographics are taken into account by the contest. We are able to show that only 26% of the losses between the status quo and the first best can be recovered in the second-best contest. We view this as a pessimistic conclusion, since it implies that the losses from competition can not be significantly ameliorated through clever policy intervention.

The remainder of this paper has the following structure: in section 2 we briefly discuss the relation between this work and the previous literature on college admissions and AA. In section 3 we give an overview of the full model of competitive human capital investment and describe the college assignment contest we study. In section 4 we introduce

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5While our model does not include spillovers from the human capital of the college students to broader society, at a minimum our model describes the cost component required for a principled cost-benefit analysis.

6The caveat is that policy intervention might help, but only if more information about the students’ ex ante types is available. However, to the extent that this information is also subject to strategic competition, the benefits of any such policy are ambiguous at best.
the limit model with a continuum of students and prove that equilibria of the limit model are approximate equilibria of the finite model of section 3. In section 6 we prove that quota and admissions preference schemes admit the same set of equilibrium outcomes and discuss the practical ramifications of this result. In section 7 we turn to the welfare analysis of the cost of signaling. Finally, in section 8 we consider a number of extensions of our model and describe what, if anything, changes about our results.

2 Previous Literature

One clear antecedent of our work is the literature on the college admissions problem (Gale and Shapley [31]), which considers the many-to-one assignment of students to colleges. A significant benefit of this literature relative to our approach is that the students are allowed to have heterogeneous preferences over the colleges and vice versa, but these preferences are assumed to be fixed and exogenous. The focus of the literature following Gale and Shapley [31] has largely been on finding efficient mechanisms that give students the incentive to declare their preferences truthfully in a variety of real-world markets. It is unclear how to include an endogenous choice of human capital that influences college preferences and still (relatively) easily compute equilibria in a model of the form used in this literature, which is a crucial feature for studying the welfare cost of the competitive channel.

Our work is also related to the literature on assortative marriage markets, Becker [4] providing an early example. To the best of our knowledge, all of these models assume transferable utility between agents paired in a match as well as complete information regarding the types of the agents. Since the complete information eliminates the welfare losses caused from competition in the college admissions contest, these models cannot address our welfare concerns.

Since our paper is largely theoretical in nature, this literature review focuses on the theory part of the affirmative action literature. There is also rich empirical literature on the effects of affirmative action programs, and we encourage the interested reader to see the summary provided in Hickman [36].

Previous economic theory has studied AA and effort incentives, but existing models either exhibit important limitations or do not facilitate empirical analysis and the development of data-driven counterfactuals. Fain [21] and Fu [30] study models in the spirit of all-pay contests with complete information (i.e., academic ability types are ex ante observable) where two students compete for a college seat. Extrapolating the two-player

\[7\] Schotter and Weigelt [56] study a similar setting in the laboratory. Their results suggest that no equity-achievement tradeoff exists.
insights to real-world settings is difficult because the framework implicitly assumes that all minority students, even the most gifted ones, are at a disadvantage to even the least talented non-minorities. Talented and challenged students exist among all groups, making considerations of performance incentives more complex. Franke [23] extends the contest idea to include more than 2 agents, but at the cost of focusing on a specific form of affirmative action program.

The bilateral matching literature has also touched on incentives under AA, an early example being Coate and Loury [17], which studies both human capital investment and achievement gaps. This paper considers two strategic groups of agents, firms and job applicants. Job applicants make a binary choice to either become qualified at some cost or remain unqualified given a privately known cost of becoming qualified. Firms then observe a noisy signal of the potential employee’s choice and decides to assign the applicant to one of two positions. Coate and Loury [16] provides mild conditions under which discriminatory equilibria exist, analyzes the complex effects of affirmative action on equilibrium outcomes, and demonstrates that the equilibrium beliefs about the typical qualifications of minority applications can be worsened under an affirmative action program.

There are a number of fundamental distinctions between bilateral matching models and a contest framework. Most obviously, the contest framework takes the “prizes” (e.g., jobs, school placements) and how these prizes are assigned as exogenously given. In effect, the colleges in our model are nonstrategic actors. The benefit of the contest approach is that it is easier to incorporate heterogeneity on the part of the applicants (e.g., job seekers, students), allow the applicants to employ a richer action space, and allows us to incorporate heterogeneity in the prizes awarded to the agents (e.g., allow some firms/schools to be more desirable than others). We believe that these features bring us closer to empirical data and real-world counterfactuals, but we acknowledge that firm and college decision-making is an interesting and important component of these markets.

Chan and Eyster [13] focuses on the effect of affirmative action bans on a single school when admission can be conditioned on student traits correlated with race. Epple et al. [20] analyzes a similar question, but considers a set of vertically differentiated colleges. Both papers describe how colleges bias their admissions policy to encourage diversity. These papers assume student quality is fixed and exogenous, and so necessarily can’t say much about the general equilibrium effect of the admission policy changes on student incentives. Fryer et al. [27] partially addresses student incentives by including a binary effort choice in the spirit of Coate and Loury [16], but Fryer et al. [27] simplifies the setting by assuming colleges are homogenous.

Chade, Lewis, and Smith [12] studies a matching model of college admissions with
heterogeneous colleges. However, academic achievement is exogenous, and the analysis focuses on the role of information frictions within the market (e.g., such as when SAT scores are a noisy signal of student ability), as well as the strategic behavior of colleges in setting admissions standards. Our framework is a frictionless matching market, but academic achievement is endogenous. In that sense, our work and Chade, Lewis, and Smith [12] may be considered complementary for understanding the role of broad market forces in college admissions.

Although many of these papers make similar points, our conclusions regarding student welfare would be impossible without incorporating (1) heterogenous colleges, (2) heterogenous and innate student quality, and (3) endogenous human capital accumulation choices by the students. Most of the papers above include flexible specifications for 1 or 2 of these components, but no prior paper includes rich models of all 3.

Fryer [25] touches on our analysis of the equivalence of quota and admissions preference schemes. Fryer [25] studies a model of workplace affirmative action wherein firms that wish to maximize profit are subject to an affirmative action mandate imposed by the government and enforced by an auditor. Fryer [25] finds that firms facing a pool of applicants with few minorities will act as if they are subject to a quota on the number of minorities they must hire, which Fryer [25] argues implies an equilibrium equivalence between quota and hiring preference systems. In our setting the students are the strategic actors, and our equivalence result is in some ways stronger - not only is the racial balance at colleges in the two systems the same, but the endogenous quality of the students is identical. However, the fundamental public policy points are very similar.

Finally, our methodology analyzes approximate equilibria played by a large number of agents, which has been a prominent theme in the industrial organization and microeconomic theory literature. Due to the broad scope of this literature, we provide only a brief survey and a sample of the important papers related to the topic. Early papers focused on conditions under which underlying game-theoretic models could be used as strategic microfoundations for general equilibrium models (e.g., Hildenbrand [37] and [38], Roberts and Postlewaite [52], Otani and Sicilian [50], and Jackson and Manelli [41]). Other early papers focused on conditions under which generic games played by a finite number of agent approach some limit game played by a continuum of agents (e.g., Green [33] and [34], Sabourian [55]). A more recent branch of this literature applies these ideas to simplify the analysis of large markets and mechanisms with an eye to real-world applications (e.g., Swinkels [59]; Cripps and Swinkels [18]; McLean and Postlewaite [47]; Budish [8]; Kojima and Pathak [45]; Fudenberg, Levine and Pesendorfer [28]; Weintrub, Benkard and Van Roy [60]; Krishnamurthy, Johari, and Sundarajan [43]; and Azevedo and
Leshno [21]. Many of the approximation results we use are based on Bodoh-Creed [2]

Of these papers on approximate equilibria and large games, we would like to single
out the contemporaneously developed Olszewski and Siegel [49] as particularly relevant.
Both this paper and Olszewski and Siegel [49] use the equilibria of contests played by a
continuum of agents as an approximation of a contest played by a large, but finite, set
of contenders. Although there are a number of theoretical differences in the projects
(e.g., the definitions of an approximate equilibrium differ slightly), the largest differences
between our works is the applications of our large contest theories. Olszewski and Siegel
[49] does not make any claims regarding the costs of the competition for college seats or
the equivalence of quota and admissions preference systems.

3 The Finite Model

We model the market as a Bayesian game where high school students are characterized
by an observable demographic classification—minority or non-minority—and a privately-
known cost type that governs HC production. Students compete for the right to occupy
seats at colleges of differing quality and the seats are allocated to students according to
a pre-specified rank-order mechanism which is a function of measured HC and demo-
graphic classification. Agents can observe the set of potential match partners (colleges)
and the form of the sorting mechanism before making decisions, but they must make their
invest in HC before entering the market. A student’s ex-post payoff is the utility derived
from placing at a given college with her acquired human capital minus her investment
cost.

In this section we lay out the model when the number of students and colleges are
finite. Although we spend most of the paper working with a limit approximation of this
model, the finite model and the limit approximation share the same underlying primitives
described in this section. A secondary goal is to highlight the difficulty of working with
the finite model directly, which highlights the power of our limit approximation.

3.1 Agents, Actions, and Payoffs

The set of all students is denoted $K = \{1, 2, \ldots, K\}$, but there are two demographic sub-
groups, $M = \{1, 2, \ldots, K_M\}$ (minorities) and $N = \{1, 2, \ldots, K_N\}$ (non-minorities), where

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8There also exists a literature on the relationship between large finite and nonatomic games that focuses
on games of complete information and does not emphasize the use of nonatomic games as a framework for
analyzing equilibria (examples include Housman [39], Khan and Sun [44], Carmona [9], and Carmona and
Podczeck [10], Yang [61] and [62] amongst others). Kalai [42] studies large games show the equilibria of finite
games are robust to modifications of the game form.
\(K_M + K_N = K\) and demographic class is observable. Each agent has a privately-known cost type \(\theta \in [\underline{\theta}, \bar{\theta}] \subset \mathbb{R}^+\), and each student views his or her opponents’ unobservable types as independent random variables \(\Theta\) whose realizations follow group-specific distributions, or \(\Theta \sim F_i(\theta), i = M, N\). For convenience, we denote the unconditional type distribution for the overall population by \(F_K(\theta)\). The reader should assume throughout that high values of \(\theta\) are associated with students that have a high cost of accruing human capital.

We denote the set of colleges \(P_K = \{p_1, p_2, \ldots, p_K\}\), where \(p_k \in [\underline{p}, \bar{p}] \subseteq \mathbb{R}^+\) denotes the quality level of the \(k^{th}\) college. We denote the \(k^{th}\) order statistic by \(p_{(k)}\), so that \(\min_k\{P_K\} = p(1:K)\) and \(\max_k\{P_K\} = p(K:K)\). The colleges are passive in our model, and each college accepts the students assigned at that college through the admissions contest.

Each agent’s strategy space, \(S = [\underline{s}, \infty) \subset \mathbb{R}^+\), is the set of feasible HC levels that can be attained. Each students’ choice of HC is observable to the colleges (e.g., through a standardized examination). Human capital \(\underline{s}\) is the minimum level required to participate in the market; in the current context, this would be a minimum literacy threshold required to attend college. In skilled labor markets \(\underline{s}\) might represent a college degree, and investment above \(s\) could be interpreted as points above the minimum passing college GPA.

Agents value both college quality and HC. The gross match utility derived from being placed at college \(k\) for a student with type \(\theta\) and HC \(s\) is given by \(U(p_k, s, \theta)\). However, in order to acquire HC \(s \in S\), an agent must incur a cost given by \(C(s; \theta)\), which depends on both her unobservable type and her HC investment level. The total utility for a student of type \(\theta\) that choose human capital level \(s\) and is assigned to college \(k\) is

\[
U(p_k, s, \theta) - C(s, \theta)
\]

### 3.2 Allocation Mechanisms

We now describe the contest that allocates students to colleges. Letting \(s_i\) denote student \(i\)’s human capital level and \(s_{-i}\), the vector of all other players’ actions, we let \(P_j(\cdot, s_{-i}) : S \rightarrow P_K, j = M, N\), be an assignment mapping that describes the college to which student \(i\) is assigned given each possible HC choice. Note that we have deliberately

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\(^9\)The assumption that there are two groups is not restrictive. Results for the two-group case extend straightforwardly to the case with \(L > 2\) groups.

\(^{10}\)Costs can arise in various ways: they could come from a labor-leisure tradeoff, where \(\theta\) indexes one’s preference for leisure; they could represent psychic costs, indexed by \(\theta\), involved in exerting effort to learn new concepts, where the more able students learn with the least effort; or costs could be interpreted as monetary investment required for study aides (e.g., computers, tutors, and private education), where \(\theta\) represents the severity of the budget constraint that determines one’s consumption–investment tradeoff.
allowed the assignment mapping to depend on student $i$’s demographic classification to reflect potential discrimination between members of the two groups. With this function in mind, the ex post payoff for student $i$ is then

$$\Pi_r^j(s_i, s_{-i}; \theta) = U(P_r^j(s_i, s_{-i}), s_i, \theta_i) - C(s_i, \theta_i), r \in \{cb, q, ap\} \text{ and } j = M, N$$

The baseline color-blind admission rule (involving no diversity preference) is represented by the following assignment mapping which yields positive assortative matching:

$$P_{cb}^M(s_i, s_{-i}) = P_{cb}^N(s_i, s_{-i}) = P_{cb}(s_i, s_{-i}) = K \sum_{k=1}^{K} p(k : K)1[s_i = s(k : K)] .$$

In the above expression, $1$ is an indicator function equaling 1 if its argument is true and 0 otherwise.

We concentrate on two canonical forms of AA that have received attention due to wide implementation: quotas and admission preferences. A quota is the practice of earmarking seats for each demographic group. Within the current modeling environment, this is equivalent to a set of $K_M$ seats being reserved for minorities and then allowing an assortative match within each group.

Let $P_M = \{P_{M1}, P_{M2}, \ldots, P_{MK_M}\}$ and $P_N = \{P_{N1}, P_{N2}, \ldots, P_{NK_N}\}$ denote the sets of seats earmarked for minorities and non-minorities, respectively, and let $s_M = \{s_{M1}, s_{M2}, \ldots, s_{MK_M}\}$ and $s_N = \{s_{N1}, s_{N2}, \ldots, s_{NK_N}\}$ denote the group-specific human capital profiles. Then a quota assignment mapping is represented by the functions

$$P_q^M(s_i, s_{-i}) = \sum_{m=1}^{K_M} p_M(m : K_M)1[s_i = s_M(m : K_M)], \text{ and}$$

$$P_q^N(s_i, s_{-i}) = \sum_{n=1}^{K_N} p_N(n : K_M)1[s_i = s_N(n : K_N)] .$$

The defining characteristic of a quota is a split of the competition into two separate contests where students compete only within their own race group.

An admission preference allows the students to compete against all of the other students, but the human capital choices of the members of each group are treated differently. In practice, many American admissions committees are thought to treat observed minority applicants’ SAT scores (a commonly used measure of HC) as if they were actually higher when evaluating them against their non-minority competitors.\[11\] More formally, an ad-

\[11\]There has been a fair amount of empirical research estimating a substantial average admission preference for minorities at elite American colleges; e.g., Chung, Espenshade and Walling [15] and Chung and Espenshade [14]. Hickman [36] employs a similar empirical measure for the aggregate US college market and finds
mission preference is a markup function $\tilde{S} : S \rightarrow \mathbb{R}_+$ through which minority output is passed to produce a set of transformed HC levels, $\tilde{s} = \{s_{N1}, \ldots, s_{NK}, \tilde{s}(s_{M1}), \ldots, \tilde{s}(s_{MK})\}$, and allocations are given by the following group-specific functions:

$$
\begin{align*}
    P_{\lambda}^{p}(s_i, s_{-i}) &= \sum_{k=1}^{K} p(k : K) \mathbb{1} \left[ \tilde{S}(s_i) = \tilde{s}(k : K) \right], \quad \text{and} \quad \\
    P_{\lambda}^{s}(s_i, s_{-i}) &= \sum_{k=1}^{K} p(k : K) \mathbb{1} \left[ s_i = \tilde{s}(k : K) \right].
\end{align*}
$$

Regardless of whether admissions are color-blind or follow some form of AA, ties between competitors are assumed to be broken randomly.

### 3.3 Model Assumptions

We now outline a series of assumptions on our model primitives. The assumptions serve three goals:

1. Establish the existence of a monotone equilibrium.
2. Justify the use of a centralized, assortative matching structure.
3. Insure the model is sufficiently well-behaved that our limit approximation is valid.

Although we highlight how the assumptions tie into goals (1) and (2), we defer discussion of (3) until the next section.

**Assumption 1.** For each $(\theta, s) \in [\theta, \tilde{\theta}] \times \mathbb{R}_+$, we have $C_s(s, \theta) > 0$, $C_\theta(s, \theta) > 0$, and $C_{ss}(s, \theta) > 0$

**Assumption 2.** $U_p(p, s, \theta) > 0$, $U_s(p, s, \theta) \geq 0$, $U_\theta(p, s, \theta) \leq 0$, and $U_{ss}(p, s, \theta) \leq 0$

Assumption 1 states that costs are strictly increasing in human capital level $s$ and cost type $\theta$. Moreover, the cost function is (weakly) convex in $s$ for any given $\theta$. Assumption 2 states that utility is differentiable, strictly increasing in college quality $p$, weakly increasing and concave in human capital level $s$ and weakly decreasing in cost type $\theta$. These assumptions imply that the individual decision problems have global maximizers

**Assumption 3.** $U(p, s, \theta) - C(s, \theta) = 0$ and $\arg \max_s U(p, s, \theta) - C(s, \theta) \leq \tilde{s}$

Evidence of a substantial admission preference even at lower-ranked colleges.
Assumption 3 is a boundary condition for our model. First, the assumption requires that the lowest type of student is indifferent between participating in the college admissions contest and not attending college, which normalizes the utility of not attending college to 0. This has the benefit of allowing us to interpret \( U(p,s,\theta) - C(s,\theta) \) as the equilibrium college premium of the students. Second, the assumption demands that the minimally qualified student who chooses to attend college does not have an interest in acquiring more HC. While this is not usually assumed, we find that the assumption is supported in the data we use to calibrate our model in section 7.

**Assumption 4.** There exists \( \tilde{s} \) such that \( U(\bar{p},\tilde{s},\theta) - C(\tilde{s},\theta) < U(\bar{p},\bar{s},\theta) - C(\bar{s},\theta) \)

Assumption 4 requires that there exist a human capital level so large that even the lowest type, the type that gets the largest value from human capital and has the lowest cost for acquiring human capital, would rather not invest in human capital at all and be assigned to the worst school. This implies that we can limit our analysis to human capital levels within \( s \in [\bar{s}, \tilde{s}] \). Unless otherwise stated, we simply let \( S = [\bar{s}, \tilde{s}] \).

**Assumption 5.** \( U_{ps}(p,s,\theta) \geq 0 \) and \( U_{p\theta}(p,s,\theta) \leq 0 \)

Assumption 5 requires (weak) positive complementarity between student HC and college quality and between student and college qualities. These assumptions imply that the efficient, decentralized matching college assignment is characterized by positive assortative matching, which helps justify our use of the centralized rank-order mechanisms to model the market.

**Assumption 6.** \( U_{s\theta}(p,s,\theta) \leq 0 \) and \( C_{s\theta}(s,\theta) > 0 \)

**Assumption 7.** \( F_M(\theta) \) and \( F_N(\theta) \) have continuous and strictly positive densities \( f_M(\theta) \) and \( f_N(\theta) \), on a common support \([\theta, \bar{\theta}]\).

Assumption 6 is key for existence of a monotone pure-strategy equilibrium. Assumption 6 states that marginal benefits of HC are decreasing and marginal costs of HC are increasing in a student’s type. Assumption 7 is a standard regularity condition on the type distributions and are supported by the calibration data.

**Assumption 8.** \( \tilde{S}(s) \) is strictly increasing and differentiable almost everywhere.

Assumption 8 assumes that, holding race fixed, admissions officers always prefer to enroll a student with more HC.

In our model the agents’ types are their private information, and the agents choose their human capital level given knowledge about the number of competitors from each group \( K_M \) and \( K_N \), the distribution of student types in the economy, the set of seats
\( P_K \), and the admission rule \( P_r \), \( r \in \{cb,q,ap\} \) and \( j \in \{M,N\} \). Under the payoff mapping \( \Pi_j^r(\theta, s_i, s_{-i}) \) induced by a particular admission rule, students optimally choose their achievement level based on their type and the types of potential match partners, taking into account opponents’ optimal behavior. The model defined above fits the mold of a contest, which can be thought of as an asymmetric, multi-object, all-pay auction with single-unit bidder demands and bid preferences.

We study the Bayes-Nash equilibria of the finite game. An equilibrium of the game \( \Gamma(K_M, K_N, F_M, F_N, P_K) \) is a set of achievement functions \( \sigma^r_j: [\theta, \bar{\theta}] \to \mathbb{R}_+ \), \( j = M, N \) which generate optimal choices of human capital \( s = \sigma^r_j(\theta) \) given that all other agents follow the equilibrium strategy. It will be convenient at times to denote the inverse equilibrium achievement functions by \( \psi^r_j(s) \equiv \left( \sigma^r_j \right)^{-1}(s) = \theta \).

**Theorem 1.** In the college admissions game \( \Gamma(K_M, F_M, K_N, F_N, P_K) \) with \( r \in \{cb,q,ap\} \), under assumptions 1–8 there exists a monotone pure-strategy (but potentially not group-wise symmetric) equilibrium \( (\sigma^r_M(\theta), \sigma^r_N(\theta)) \). Moreover, any equilibrium of the game must be strictly monotone with almost every type using pure strategies.

## 4 The Limit Game

For large \( K \), the equilibrium of our finite model is analytically and computationally difficult because an agent’s decision problem is a complicated function of the order statistics of opponents’ cost types. However, intuitions based on the law of large numbers would suggest that as the market becomes large (i.e., as \( K \to \infty \)), the distribution of realized types and human capital choices ought to approach some limit measure. If this intuition is true, then the mapping between human capital choices and school assignment in this limit game, which we call the *limit assignment mapping*, ought to provide a good description of the outcomes in games with a sufficiently large, but finite, set of players. This would suggest that an individual agent could come very close to maximizing his or her utility in a large finite game by optimizing against the limit assignment mapping.

To see how this simplifies the student’s problem (and as a result, our analysis), consider the plight of a college applicant in the United States. Do would-be college students go to elaborate lengths to determine who else is applying, where those students are applying, and what the other students’ qualifications are? Of course not - to determine whether an application is likely to be accepted, the would-be college student can simply look at data on the grades and SAT scores of currently enrolled students. In our model, the SAT score of currently enrolled students is akin to knowing the equilibrium mapping between human capital levels and school assignments. Conveniently, metrics such as SAT scores of incoming freshman are stable from year to year.
We repeatedly suggest that the equilibria of the limit game are easy to compute, which can be interpreted in two ways. First (and we think most importantly), the equilibria of the limit game are computed using a decision problem that captures the decision process described in the paragraph above. In other words, a student can discover their optimal action in the limit game through reasoning that we believe is typical of real-life behavior, which we think supports the plausibility of the limit model. Second, even using modern algorithms, it is difficult to numerically compute the equilibrium of the finite model when \( K \) is large. Our limit model can also be used as a powerful analytical tool by researchers. In either case, our challenge is to formally prove that the equilibrium of the limit game approximates the equilibrium of the more realistic finite agent game when \( K \) is large.

Much of the underlying mechanics of the convergence of the finite games to the relevant limit model are the same across the color-blind, admissions preference, or quota games. For each type of college assignment game we consider a sequence of finite games denoted \( \{\Gamma(K_M, F_M, K_N, F_N, P_K)\}^{\infty}_{K_N + K_M = 2} \), and we use the same utility function in the finite games and the limit game. We assume that \( \frac{K_M}{K} \to \mu \in (0, 1) \) as \( K \to \infty \), which implies that \( \mu \) is the asymptotic mass of the minority group.\(^{12}\) We let \( F^K_P(p) \) denote the distribution of college qualities in the \( K \) agent game, and we assume that there is a continuous CDF \( F_P(p) \) such that

\[
\lim_{K \to \infty} \sup_{p \in [p, \bar{p}]} \left\| F^K_P(p) - F_P(p) \right\| = 0
\]  

Alternately, we could assume that college types are drawn in an independent fashion from \( F_P \) in each of the finite games, in which case equation \(^4\) holds almost surely.

In the limit admissions preference game there is a measure \( \mu \) continuum of minority students with types distributed exactly as \( F_M \) and a measure \( 1 - \mu \) continuum of nonminority students with types distributed exactly as \( F_N \). Finally, there is a measure 1 continuum of college seats distributed exactly as \( F_P \). In order to describe the limit equilibria using an ODE, we must make the following regularity assumption on \( F_P \).

**Assumption 9.** \( F_P(p) \) has a continuous and strictly positive density \( f_P(p) \) on support \( [p, \bar{p}] \).

In the following subsections, we address the color-blind, admission preference, and quota games separately. In each subsection we describe the limit assignment mapping and use the mapping to describe the equilibrium strategy in each game. In section 4.4 we provide conditions under which there exists an essentially unique PSNE of the limit game. In the case of color-blind and quota systems, such a unique equilibrium exists

\(^{12}\)Although we assume throughout that the convergence is deterministic, one can think of the agents in the finite games as being created by nature by assigning each agent to group \( M \) with probability \( \mu \in (0, 1) \), after which nature draws a cost type for the student from the corresponding distribution.
for any feasible game. Finally, we close with a description of comparative statics results in section 4.5, which also serves to highlight how our model enriches some common pre-existing models in the contest literature.

**4.1 Color-blind Admissions Game**

Recall that a color-blind outcome yields positive assortative matching of seats and human capital levels. We denote the endogenous distribution of human capital levels chosen by the students as $G_{cb}^K(s)\in \mathcal{M}$ and $r \in \{cb, q, ap\}$. Using this notation, we can describe the match of students to colleges as:

$$P_{cb}^M(s) = P_{cb}^N(s) = P_{cb} = F_{cb}^{-1} \left( G_{cb}^K(s) \right)$$

$$= F_{cb}^{-1} \left( \mu G_{cb}^M(s) + (1 - \mu) G_{cb}^N(s) \right)$$

$$= F_{cb}^{-1} \left( 1 - \mu F_M \left( \psi_{cb}^M(s) \right) - (1 - \mu) F_N \left( \psi_{cb}^N(s) \right) \right) .$$

The intuition is simple: quantiles of the population HC distribution $G_{cb}^K(s)$ are mapped into the corresponding quantiles of the distribution $F_{cb}$. Since limiting payoffs do not depend on race, it follows that $\sigma_{cb}^M(\theta) = \sigma_{cb}^N(\theta) = \sigma_{cb}(\theta)$; hence, the lack of subscripts on the inverse strategies in the third line.

We now turn to describing the equilibrium of the color-blind limit game. Given that the equilibrium will, in the end, be assortative, we can write the endogenous assignment function as a function of agent type as follows

$$P_{cb}^M(\theta) = P_{cb}^N(\theta) = F_{cb}^{-1} \left( 1 - \mu F_M (\theta) - (1 - \mu) F_N (\theta) \right)$$

If we describe the agent’s problem in revelation mechanism form, then the decision problem reduces to choosing a type $\hat{\theta}$ to declare

$$\max_{\hat{\theta}} U \left( P_{cb} (\hat{\theta}) , \sigma_{cb} (\hat{\theta}) , \theta \right) - C \left( \sigma_{cb} (\hat{\theta}) , \theta \right)$$

Manipulating the first order condition for this problem and using the fact that in equilibri-

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13Theorists with experience in asymmetric auctions may find this statement puzzling, but one must keep in mind that it merely applies to limiting payoffs. In a two-player finite game, differing investment behavior arises from the fact that a minority competitor faces a profile of opponents of each type numbering $(K_N, K_M - 1)$, whereas a non-minority competitor faces profile $(K_N - 1, K_M)$, and since costs are asymmetrically distributed across groups, a minority and a non-minority with the same private cost will have differing expectations of their standing in the distribution of realized competition. However, the difference between their expected ranks quickly vanishes as the number of players gets large.
rium we have $\tilde{\theta} = \theta$ yields the following differential equation for investment:

$$\frac{d\sigma_{cb}(\theta)}{d\theta} = -\frac{U_p\left(P_{cb}[\sigma_{cb}(\theta)], \sigma_{cb}(\theta), \theta\right) \cdot f_K(\theta)}{f_p\left(F_p^{-1}(1 - F_K(\theta))\right) \cdot (C_s(\sigma_{cb}(\theta), \theta) - U_s(P_{cb}(\sigma_{cb}(\theta)), \sigma_{cb}(\theta), \theta))},$$

$$\sigma_{cb}(\tilde{\theta}) = \tilde{s} \quad \text{(boundary condition).}$$

The boundary condition comes from the fact that a player of type $\tilde{\theta}$ will always be matched with the lowest seat in a monotone equilibrium, so she cannot do better than to simply choose HC level $\tilde{s}$ to complement $p$. Note that, given the assumptions on the model primitives, the initial value problem defined by (6) results in a strictly decreasing function.

### 4.2 Admissions Preference Game

As in the finite-agent model, an admission preference rule is modeled as a markup function $\tilde{S} : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ and seats are matched assortatively with transformed HC. In other words, minorities are repositioned ahead of non-minority counterparts with investment of $\tilde{S}(s)$ or less. The limiting assignment mapping for group $\mathcal{M}$ is

$$P_{ap\mathcal{M}}(s) = F_p^{-1}\left((1 - \mu)G_N\left(\tilde{S}(s)\right) + \mu G_M(s)\right)$$

$$= F_p^{-1}\left(1 - \left((1 - \mu)G_N\left(\psi_{apN}(\tilde{S}(s))\right) + \mu G_M\left(\psi_{apM}(\tilde{S}(s))\right)\right)\right)$$

and limiting allocations for group $\mathcal{N}$ are given by

$$P_{ap\mathcal{N}}(s) = F_p^{-1}\left((1 - \mu)G_N(s) + \mu G_M\left(\tilde{S}^{-1}(s)\right)\right)$$

$$= F_p^{-1}\left(1 - \left((1 - \mu)G_N\left(\psi_{apN}^{-1}(s)\right) + \mu G_M\left(\psi_{apM}^{-1}(\tilde{S}^{-1}(s))\right)\right)\right).$$

The intuition for the above expressions is similar as before: limiting mechanisms map the quantiles of a distribution into the corresponding college seat quantiles. For non-minorities, it is a mixture of the distributions of non-minority HC and subsidized minority HC. For minorities, it is a mixture of the distributions of minority HC and de-subsidized non-minority HC.

Unlike the color-blind or quota case, it is not possible to describe which kind of student obtains each seat in an admissions preference scheme without first solving for the equilibrium. This feature of the admissions preference mechanism makes it significantly harder to work with than the other two schemes. However, we prove in section 6 that the admissions preference and quota schemes are outcome equivalent - for any admissions preference markup function one can describe a quota scheme that generates the same
school assignment and equilibrium human capital choices. Because of this equivalence, for the majority of the paper we work with color-blind and quota systems. However, for completeness, we now provide the differential equations describing the admissions preference equilibrium when $\tilde{S}$ is differentiable and $\tilde{S}(s) = s$.

$$
\begin{align*}
\left( \psi_{ap}^M \right)'(s) &= -\frac{C_s(s, \psi_{ap}^M(s)) - U_s(p_{ap}^M(s), s, \psi_{ap}^M(s))}{\mu f_M(\psi_{ap}^M(s))} \cdot \frac{fp(p_{ap}^M(s))}{f_P(p_{ap}^N(s))} \cdot \frac{1}{\mu f_M(\psi_{ap}^M(s))} \cdot \frac{1}{\mu f_M(\psi_{ap}^M(s))} \cdot \left( \psi_{ap}^M(s) \right)'(s) \cdot \frac{d\tilde{S}(s)}{ds} \\
\left( \psi_{ap}^N \right)'(s) &= -\frac{C_s(s, \psi_{ap}^N(s)) - U_s(p_{ap}^N(s), s, \psi_{ap}^N(s))}{\mu f_M(\psi_{ap}^N(s))} \cdot \frac{fp(p_{ap}^N(s))}{f_P(p_{ap}^N(s))} \cdot \frac{1}{(1 - \mu) f_N(\psi_{ap}^N(s))} \cdot \left( \psi_{ap}^N(s) \right)'(s) \cdot \frac{d\tilde{S}(s)}{ds}
\end{align*}
$$

These equations are described in terms of the inverse strategy functions for notational ease. If $\tilde{S}$ is not differentiable or $\tilde{S}(s) \neq s$, then the equilibrium may involve jumps. The interested reader is (again) referred to appendix D for a description of how to identify the size and location of these jumps.

### 4.3 Quota Game

We denote the empirical measure of seats allocated to group $j \in \{M, N\}$ in the $K$-agent game be denoted $Q^K_j$. We assume that there exists a distribution of seats $Q_j(p)$ such that

$$
\lim_{K \to \infty} \sup_{p \in [\underline{p}, \bar{p}]} \left\| Q^K_j(p) - Q_j(p) \right\| = 0
$$

In the limit quota game a measure $\mu$ of minority students with types exactly distributed as $F_M$ compete for a pool of seats exactly distributed as $Q_M$. Similarly, in the limit quota game a measure $1 - \mu$ of nonminority students with types exactly distributed as $F_N$ compete for a pool of seats exactly distributed as $Q_N$. The measures $Q_M$ and $Q_N$ are subject to the following feasibility constraint

$$
\text{For all } p \text{ we have } \mu Q_M(p) + (1 - \mu) Q_N(p) = F_p(p)
$$

Finally, we assume that the quota measures admit a density, which will prove useful in describing the equilibrium strategies.
Assumption 10. $Q_j(p), j \in \{M, N\}$, admits a density on $[\underline{p}, \overline{p}]$.

In the finite game, the students are allocated into two contests in which the students can compete for the seats allocated to their group. The result is an assortative match within each distinct contest. In the limit game the students from group $j \in \{M, N\}$ compete for the measure of seats $Q_j$ allocated to their group. These distinct contests yield group-specific assignment mappings of the form

$$P_j^q(s) = Q_j^{-1}\left(G_j^q(s)\right) = Q_j^{-1}\left(1 - F_j\left(\psi_j^q(s)\right)\right), \ j \in \{M, N\}. \quad (10)$$

As in the color-blind case, the quantiles of the group-specific human capital distributions are mapped into the corresponding quantiles of $Q_j$. We find the following differential equation describing the equilibrium when $Q_j$ have full support.

$$\frac{d\sigma_j^q(\theta)}{d\theta} = -\frac{U_p\left(P_j^q\left(\sigma_j^q(\theta)\right), \sigma_j^q(\theta), \theta\right) \cdot f_j(\theta)}{f_P\left(F_P^{-1}(1 - F_j(\theta))\right) \cdot \left(C_s\left(\sigma_j^q(\theta), \theta\right) - U_s\left(P_j^q\left(\sigma_j^q(\theta)\right), \sigma_j^q(\theta), \theta\right)\right)}, \quad (11)$$

$$\sigma_j^q(\overline{\theta}) = \underline{s} \quad \text{(boundary condition)}.$$ 

When $Q_j$ does not have full support, then there will be jumps in the equilibrium strategies when $P_j^q\left(\sigma_j^q(\theta)\right)$ encounters the left edge of a gap in the support. The interested reader is referred to appendix D for a description of how to identify the location and size of the jumps. Between these jumps, equation (11) describes the equilibrium strategy.

4.4 Equilibrium Existence

Since the limit game is one of complete information, the relevant equilibrium concept is a Nash equilibrium. In a setting with a continuum of agents, this equilibrium concept can be defined as follows:

**Definition 1.** $\sigma$ is a Nash equilibrium of the limit game if for $j \in \{M, N\}$ and all $\theta \in [\underline{\theta}, \overline{\theta}]$

$$\sigma(\theta) \in \arg\max_s \ U\left(P_j^r(s), s, \theta\right) - C(s, \theta)$$

We would like to use the differential equations describing the equilibria under each college assignment model to prove the existence of a unique, pure-strategy Nash equilibrium. While we cannot prove these exact properties, we can prove that they hold almost everywhere. Consider a pair of behavioral strategies $^{14}$$14$$\left(\sigma_M, \sigma_N\right)$. We refer to

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$^{14}$Recall that in a Bayesian game a behavioral startegy is one that maps types into actions and that agents adopt these strategies prior to learning their types.
an equilibrium as essentially group symmetric if for any agent $t$ in group $i = M, N$, the equilibrium strategy adopted by agent $t$ is equal to $\sigma_i$ almost everywhere. Using the theory of strict monotone comparative statics developed by Edlin and Shannon [19], we can show both that any equilibrium must be essentially group-symmetric and that almost all of the agents must adopt pure actions in any equilibrium.

The challenge to proving uniqueness is handling points where the strategy must be discontinuous. For example, in the quota case if $Q_i$ lacks support over an interval of schools $[p_1, p_2]$, then the strategy of the students must exhibit a discontinuity. For quota systems we can identify the size each of these jumps must take in equilibrium, and it is easy to prove that the strategy is continuous almost everywhere. In other words, we can prove the equilibrium is unique for any quota scheme except for the measure 0 set of types where the strategy exhibits a discontinuity, which we refer to as essentially unique. Unfortunately, we do not yet have a direct result for the admissions preference case where $\tilde{S}$ is nondifferentiable or $\tilde{S}(s) = s$. Fortunately, since we work exclusively with quota systems in our later analysis, this result suffices.

**Theorem 2.** In any equilibrium almost all of the agents take pure actions and the strategies are essentially group-symmetric.

Finally, an essentially unique Nash equilibrium exists in the limit model in the following cases

1. The limit game of the admissions preference model with a differential markup function $\tilde{S}$ that satisfies $\tilde{S}(s) = s$ and $F_P$ has full support.

2. The limit game of the quota system with any feasible choice of $Q_M$ and $Q_M$.

### 4.5 Comparative Statics of the Limit Model

Our goal in this section is to identify conditions under which encouragement and discouragement effects occur in our model. Discouragement effects describe situations wherein competition within a contest intensifies, which causes low ability competitors to exert less effort while at the same time high ability competitors work harder. Since affirmative action schemes (most notably admissions preference schemes) in effect make minority students better competitors, one might expect discouragement effects within the nonminority student population. In addition, there will be a symmetric encouragement effect within the minority student population.

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15 At these discontinuities the type could choose the action on either side of the discontinuity or mix arbitrarily over these choices. Since only a measure 0 set of agents face such a discontinuity, the choices of these agents has no effect on the incentives of the other agents.
Our notion of changing competition is captured by examining changes in the distribution of the types of competitors that can be ordered using a single crossing condition applied to the density functions of the relevant distributions.

**Definition 2.** Two functions \( f(\theta) \) and \( g(\theta) \) satisfy the strict single crossing condition (SCC) when for any \( \theta > \tilde{\theta} \) we have \( f(\tilde{\theta}) \geq g(\tilde{\theta}) \) implies \( f(\theta) > g(\theta) \). We denote this relation \( g <_{SCC} f \)

Keeping in mind that students with low values of \( \theta \) choose higher levels of human capital, if \( f_1 \) and \( f_2 \) represent distributions of student types, then \( f_1 <_{SCC} f_2 \) implies that \( f_1 \) represents a stronger student population.

Now we turn to our analysis of the discouragement effect. We first consider a color-blind admission policy and analyze how students react to changes in the distributions of types of the other students. Our first result implies a form of the discouragement effect. When there are fewer high ability students (and hence more low ability competitors), the low ability students make greater investments in human capital.

**Theorem 3.** Consider two population cost distributions, \( F_1(\theta) \) and \( F_2(\theta) \), that have have full support over \([\theta, \tilde{\theta}]\). Assume \( f_1(\theta) <_{SCC} f_2(\theta) \). Under a color-blind admissions policy, there exists a point \( \theta^* \) such that \( \sigma^*_1(\theta) < \sigma^*_2(\theta) \) for each \( \theta \in (\theta^*, \tilde{\theta}) \).

**Proof.** In the color-blind admission scheme, we need not differentiate between the minority and nonminority students. Furthermore, I will drop the “cb” superscript for notational ease. Denote the equilibrium strategies under \( F_1 \) and \( F_2 \) by \( \sigma_1 \) and \( \sigma_2 \), respectively. Note that since the distributions of types are different, the school allocated to each type in equilibrium will also depend on the distributions \( F_1 \) and \( F_2 \), and we denote the associated assignment functions as \( P_1 \) and \( P_2 \).

Using the notation of a revelation mechanism, the FOC is now

\[
U_p(P(\theta), \sigma_i(\theta), \theta)P_i'(\theta) + U_s(P_i(\theta), \sigma_i(\theta), \theta)\sigma_i'(\theta) = C_s(\sigma_i(\theta), \theta)\sigma_i'(\theta)
\]

We can rewrite the first order condition as

\[
\sigma_i'(\theta) = -\frac{U_p(P_i(\theta), \sigma_i(\theta), \theta)f_i(\theta)}{f_p(1 - F_i^{-1}(\theta)) \left[ C_s(\sigma_i(\theta), \theta) - U_s(P(\theta), \sigma_i(\theta), \theta) \right]}
\]

For the lowest ability student, \( \theta = \tilde{\theta} \), we can write

\[
\sigma_i'(\theta) = -\frac{U_p(p, s, \tilde{\theta})f_i(\tilde{\theta})}{f_p(p) \left[ C_s(s, \tilde{\theta}) - U_s(p, s, \tilde{\theta}) \right]}
\]

(12)

where we use the fact that \( P(\tilde{\theta}) = p \), and we have the boundary condition \( \sigma_i(\tilde{\theta}) = s \) for \( i = 1, 2 \). The SCC implies \( f_1(\tilde{\theta}) < f_2(\tilde{\theta}) \), so equation (12) implies \( \sigma_2'(\tilde{\theta}) < \sigma_1'(\tilde{\theta}) < 0 \). This
in turn means that $\sigma_1(\theta) < \sigma_2(\theta)$ within a neighborhood of $\overline{\theta}$ by differentiability (recall also that $\sigma_j$ is strictly decreasing and rises in the leftward direction from the boundary point).

Our second result provides a condition under which there is at most one crossing-point between $\sigma_1^{cb}$ and $\sigma_2^{cb}$. If there is no crossing point, then all of the students increase their human capital choices under $F_1$. If there is a crossing point, then we see both an encouragement and a discouragement effect. In other words, bad students choose higher levels of human capital, good students acquire less human capital, and the gap between the best and worst students contracts. Our proof proceeds by ordering the slope of the equilibrium strategy as function of the underlying primitives.

**Theorem 4.** Assume all of the conditions of theorem 3 hold. In addition we will assume that the distribution of seats is potentially different, and we delineate between the distributions using the notation $F_{p,1}$ and $F_{p,2}$. In addition, assume:

$$\frac{U_p(F_{p,1}^{-1}(1 - F_1(\theta)), s, \theta) f_1(\theta)}{f_{p,1}(F_{p,1}^{-1}(1 - F_1(\theta))) \left[ C_s(s, \theta) - U_s(F_{p,2}^{-1}(1 - F_1(\theta)), s, \theta) \right]} > \frac{U_p(F_{p,2}^{-1}(1 - F_2(\theta)), s, \theta) f_2(\theta)}{f_{p,2}(F_{p,2}^{-1}(1 - F_2(\theta))) \left[ C_s(s, \theta) - U_s(F_{p,2}^{-1}(1 - F_1(\theta)), s, \theta) \right]}$$

in $\theta$

(13)

Under a color-blind admissions policy, there exists at most a single point $\theta^* \in (\theta, \overline{\theta})$ such that $\sigma_1^{cb}(\theta) < \sigma_2^{cb}(\theta)$ for each $\theta \in (\theta^*, \overline{\theta})$ and $\sigma_1^{cb}(\theta) < \sigma_2^{cb}(\theta)$ for each $\theta \in [\theta, \theta^*)$.

**Proof.** The first part of our result follows the proof of theorem 3 and is omitted. What remains is to show that the point $\theta^*$, if it exists, must be unique.

Suppose there are more than two points where $\sigma_1(\theta) = \sigma_2(\theta)$ and let $\theta^*$ denote the largest such point and $\theta^{**}$ the second largest. Let $\sigma_1(\theta^*) = \sigma_2(\theta^*) = s^*$ and $p_i^* = P_i(\theta^*) = F_{p,i}^{-1}(1 - F_i(\theta^*))$. Since $\sigma_1$ must cross $\sigma_2$ from above (recall, $\sigma_1(\theta) < \sigma_2(\theta)$ in $(\theta^*, \overline{\theta})$), the former must have a more negative slope. Combining these facts yields

$$\sigma_1'(\theta^*) = -\frac{U_p(p_{1,i}^*, s^*, \theta^*) f_1(\theta^*)}{f_{p,1}(p_{1,i}^*) \left[ C_s(s^*, \theta^*) - U_s(p_{1,i}^*, s^*, \theta^*) \right]} < -\frac{U_p(p_{2,i}^*, s^*, \theta^*) f_2(\theta^*)}{f_{p,2}(p_{2,i}^*) \left[ C_s(s^*, \theta^*) - U_s(p_{2,i}^*, s^*, \theta^*) \right]} = \sigma_2'(\theta^*)$$

Note that since $U_{ps} = 0$ we have $C_s(s^*, \theta^*) - U_s(p_{1,i}^*, s^*, \theta^*) = C_s(s^*, \theta^*) - U_s(p_{2,i}^*, s^*, \theta^*)$. 

21
Therefore, it must be the case that
\[
\frac{U_p(p_1^*, s^*, \theta^*) f_1(\theta^*)}{f_{P,1}(p_1^*)} > \frac{U_p(p_2^*, s^*, \theta^*) f_2(\theta^*)}{f_{P,2}(p_2^*)}
\]  
(14)

Now we will argue from contradiction that \( \sigma_1 \) and \( \sigma_2 \) can cross at most once. Suppose there were a second point \( \theta^{**} < \theta^* \) such that \( \sigma_1(\theta^{**}) = \sigma_2(\theta^{**}) \). In this case, we would have \( \sigma_1 \) crossing \( \sigma_2 \) from below, which would imply \( \sigma_1 \) is less steep than \( \sigma_2 \), which would require
\[
\sigma'_1(\theta^{**}) = -\frac{U_p(p_1^{**}, s^{**}, \theta^{**}) f_1(\theta^{**})}{f_{P,1}(p_1^{**}) [C_s(s^{**}, \theta^{**}) - U_s(p_1^{**}, s^{**}, \theta^{**})]} > -\frac{U_p(p_2^{**}, s^{**}, \theta^{**}) f_2(\theta^{**})}{f_{P,2}(p_2^{**}) [C_s(s^{**}, \theta^{**}) - U_s(p_2^{**}, s^{**}, \theta^{**})]} = \sigma'_2(\theta^{**})
\]

where \( p_1^{**} \) and \( s^{**} \) are defined analogously. Again, assumption 1 implies \( C_s(s^{**}, \theta^{**}) - U_s(p_1^{**}, s^{**}, \theta^{**}) = C_s(s^{**}, \theta^{**}) - U_s(p_2^{**}, s^{**}, \theta^{**}) \). For \( \sigma'_1(\theta^{**}) > \sigma'_2(\theta^{**}) \) to be true, it must then be the case that
\[
\frac{U_p(p_1^{**}, s^{**}, \theta^{**}) f_1(\theta^{**})}{f_{P,1}(p_1^{**})} < \frac{U_p(p_2^{**}, s^{**}, \theta^{**}) f_2(\theta^{**})}{f_{P,2}(p_2^{**})}
\]

But this is contradicted by assumption 2. Therefore, \( \sigma_1 \) and \( \sigma_2 \) can cross at most once. \( \square \)

The condition required by our theorem is very complex, but much of this is due to the fact that we are allowing more complex interactions between HC choice, school assignment, and type than the typical contest model. For example, suppose we (as in most contest models) assume HC is accrued solely to compete and is not a productive asset. This could be captured by assuming that \( U(p, s, \theta) = p (\overline{\theta} - \theta) \), which implies that good students (i.e., low \( \theta \) values) have a stronger preference for better schools. In this case equation (13) reduces to
\[
\frac{f_1(\theta)}{f_{P,1}(F_{P,1}^{-1}(1 - F_1(\theta)))} >_{SCC} \frac{f_2(\theta)}{f_{P,2}(F_{P,2}^{-1}(1 - F_2(\theta)))} \quad \text{in} \ \theta
\]  
(15)

Equation (15) requires that the population of competitors relative to the population of seats for which they are competing increase with type. For example, this condition would be satisfied if economy 1 (relative to economy 2) had more high cost students and a relatively small proportion of low quality colleges. Such a situation incentives high HC accumulation from high cost students since (1) small increases in HC allows the students to leap in front of a large number of peers and (2) the small number of low quality colleges
means such a leap results in placement in a much better college.

If the condition of theorem 4 holds, then low ability students (i.e., students with high values of $\theta$) will reduce their human capital investment. This is an example of the discouragement effect which is a common feature of the contests literature: when a given cost type $\theta$ falls far enough behind in the sense that the mass of lower-cost competitors $F_2(\theta)$ is large, his incentives for investing in costly effort become weak. If there is a crossing point $\theta^*$, then we also find an encouragement effect that causes low-cost individuals ($\theta$ sufficiently low) to invest more aggressively to keep the larger mass of their like opponents at bay. This means that an increase in the degree of competition will increase the disparity in achievement between top students and the ones at the bottom.

While theorem 4 is useful for identifying encouragement and discouragement effects in generic contest models, it will be difficult to determine whether equation 13 holds in the calibrated model we study below. Fortunately, we can use the underlying data to determine whether equation 13 is satisfied.

5 Approximating the Finite Game with Limit Equilibria

Having defined the equilibria of the limit model using ODEs and proven that the equilibrium thus described exists and is essentially unique, we now argue that the equilibrium of the relevant limit game is a useful approximation of the equilibria in games with a large, but finite, set of players. We use two notions of approximate equilibrium in this paper. Our first definition provides for an approximation in terms of incentives - agents that follow an $\epsilon-$approximate equilibrium can gain at most $\epsilon$ by deviating. Intuitively, students lose little utility if they base their actions on the easy-to-solve limit game equilibrium decision.

**Definition 3.** Given $\epsilon > 0$, an $\epsilon$-approximate equilibrium of the $K$-agent game is a $K$-tuple of strategies $\sigma^\epsilon = (\sigma_1^\epsilon, \ldots, \sigma_K^\epsilon)$ such that for all agents $i, j \in \{M, N\}$, almost all types $\theta$, and all human capital choices $s'$ we have

$$U \left( P^j_i(\sigma^\epsilon_j(\theta)), \sigma^\epsilon_i(\theta), \theta_i \right) - C(\sigma^\epsilon_i(\theta), \theta_i) + \epsilon \geq U \left( P(s'), s', \theta_i \right) - C(s', \theta_i)$$

A $\delta-$approximate equilibrium provides a close approximation of the actual HC choices of each of the agents.

**Definition 4.** Given $\delta > 0$, a $\delta$-approximate equilibrium of the $K$-agent game is a $K$-tuple of strategies $\sigma^\delta = (\sigma_1^\delta, \ldots, \sigma_K^\delta)$ such that for any exact equilibrium of the $K$-agent game $\sigma =
\((\sigma_1, \ldots, \sigma_K)\) we have for all agents \(i, j \in \{M, N\}\), and almost all types \(\theta\)

\[ \left\| \sigma^j_i - \sigma_i \right\| < \delta \text{ in the sup-norm} \]

Our goal in this section is to prove that the equilibria of the limit game are \(\delta\)-approximate equilibria of admissions games with sufficiently many students. Proving this result amounts to proving that the limit game is continuous in the appropriate sense.

We require the following assumption on the markup functions to prove our approximation result for the admissions preference system. The assumption bounds the marginal markup applied to minority student human capital choices, which helps insure that minority student utility functions are continuous in the limit game.

**Assumption 11.** There exists \(\lambda_1, \lambda_2 \in (0, \infty)\) such that for all \(s, s' \in \mathcal{S}\), \(s > s'\), we have

\[ \lambda_2 (s - s') > \tilde{S}(s) - \tilde{S}(s') > \lambda_1 (s - s') \]

We can now state our primary equilibrium approximation result.

**Theorem 5.** Let \(\sigma^j_i, i \in \mathcal{M}, \mathcal{N}\) and \(j \in \{cb, q, ap\}\) denote an equilibrium of the limit game. Assume one of the following cases holds:

1. The limit game of the admissions preference model with a differential markup function \(\tilde{S}\) that satisfies \(\tilde{S}(\underline{s}) = \underline{s}\) and \(F_P\) has full support.

2. The limit game of the quota system with any feasible choice of \(Q_M\) and \(Q_N\) that admit strictly positive PDFs over a connected support.

Under assumptions 7-9, assumption 11, and given \(\varepsilon, \delta > 0\), there exists \(K^* \in \mathbb{N}\) such that for any \(K \geq K^*\) we have that \(\sigma^j_i\) is a \(\varepsilon\)-approximate equilibrium of the \(K\)-agent game and \(\sigma^j_i\) is a \(\delta\)-approximate equilibrium for the \(K\)-agent game.

When many students follow the equilibrium strategy of the limit game, then the realized distribution of human capital and college seats will (with high probability) be approximately the same as the distributions realized in the limit game. If the student utility functions are continuous, then these small differences have a negligible effect on the agent utility for each possible action (and so the maximum utility changes only slightly). This implies that \(\sigma^j_i\) is a \(\varepsilon\)-approximate equilibrium of the \(K\)-agent game.

Theorem 5 also implies that every equilibrium of the finite game must be close to the essentially unique equilibrium of the limit game. First, we show that the continuity of the student utility functions implies that the equilibrium correspondence is upper hemi-continuous. In other words, an exact equilibrium of the admissions game with many
players must be close to some equilibrium of the limit game. However, it could be the
case that there are equilibria of the limit game that are unlike any equilibrium of the finite
game (i.e., lower hemicontinuity of the equilibrium correspondence might fail). To rule
this out, we use the fact that the limit game has a unique equilibrium to prove that the
equilibrium correspondence is in fact continuous, which is equivalent to the claim made
by (ii).

In order to prove our theorem, we need to rule out discontinuities in our model, which
can arise either through the contest structure or the endogenous equilibrium strategy.
In appendix C we modify our model by assuming that the exact distribution of school
seats uncertain, which reflects the fact that students accrue HC over a long period of
time and they cannot be certain of the exact school qualities that will be available when
they eventually compete for a college seat. With this assumption, our approximation
results hold for any feasible quota. We can also extend the mechanism equivalence result
(theorem C below) as long as we allow the quotas and admissions preference schemes to
depend on the realized distribution of colleges.\textsuperscript{16}

6 Mechanism Equivalence

In this section we make our argument that the quota and admissions preferences systems
are, in fact, equivalent to each other. In any equilibrium of any affirmative action scheme,
the agents in the limit game respond optimally to the assignment mapping, $P(s)$, that
describes how HC choices lead to college assignments. We prove our equivalence result
by showing that if an assignment mapping $P(s)$ is generated by some equilibrium of
an admission preference (quota) system, then there exists an equilibrium of some quota
(admissions preference) system that yields the same $P(s)$. Since the $P(s)$ are the same
under each system, the optimal agent responses must also be the same. Note that the
notion of equivalence we use implies that not only are the same measures of minority
and nonminority students assigned to each school, but the students at each school choose
the same level of human capital under both systems.\textsuperscript{17,18}

\textsuperscript{16}Olszewski and Siegel \textsuperscript{49} does not require continuity to prove their approximation results, but the notion
of approximation they use allows for a small measure of students to take actions that differ significantly from
the predictions of the limit model.

\textsuperscript{17}Note that we have not proven that any choice of $P_i(s)$ can be implemented by either a quota or an
admissions preference scheme. For example, if $P_i(s)$ is strictly decreasing, then it cannot be implemented by
any incentive compatible mechanism.

\textsuperscript{18}We have only conjectured that admissions preference schemes with discontinuous equilibria admit a
unique equilibrium. If this conjecture is false for some admission preference markup function $\tilde{S}$, theorem 6
will be unaffected. However, it will be the case that each of the equilibria induced by $\tilde{S}$ will be equivalent to
different quota systems.
**Theorem 6.** Consider some \( P_i(s) : S \rightarrow \mathcal{P}, i \in \mathcal{M}, \mathcal{N} \). \( P_i(s) : S \rightarrow \mathcal{P}, i \in \mathcal{M}, \mathcal{N} \) is the result of an equilibrium of some quota system if and only if there is an equilibrium of an admissions preference system that also yields these assignment functions and admits the same equilibrium strategies.

Recall that under a quota system the equilibrium assignment of student types to colleges within each group is assortative, which means the only unknown endogenous quantity is the HC accumulation strategy. Given an admission preferences system, both the equilibrium human capital choices and the school assignment need to be computed, which makes the admissions preference schemes more difficult to study. Theorem 6 is useful from a methodological perspective because it shows that there is no loss in generality from focusing solely on outcomes that can be realized using quota schemes.

Our result bears a superficial resemblance to the equivalence of quotas and tariffs in an international trade context (e.g., Bhagwati [5]), although our setting is complicated by the continuum of heterogeneous “goods” (i.e., college seats) being assigned and the continuum of endogenous “prices” (i.e., human capital levels) required to obtain the goods. Even without the insights from the international trade literature, we do not view it as surprising that the same assignment of student types to colleges can be generated using either type of mechanism. We find it much more surprising that the endogenous human capital decisions can also be replicated, and (to the best of our knowledge) there is no analog of this result in the international economics literature. Of course some insights, such as the breakdown of the equivalence in the presence of aggregate uncertainty, are true both in the quota-tariff equivalence and in our college admissions model.

As just hinted, the equivalence of admissions preference and quota schemes relies on the distribution of student types being fixed and known. If there are aggregate shocks to the distribution of student types, the equivalence will no longer hold unless the quota and admissions preference schemes are allowed to be functions of the realized distribution of applicant types. For example, under a quota scheme that is not responsive to the distribution of college students, a fixed number of minority student are enrolled even in the event that the minority student population is significantly better than expected. In contrast, an admissions preference scheme may allow, the additional high quality minority applicants to be enrolled. Large aggregate shocks in the distribution of undergraduate applicants seems unlikely, but it is easy to imagine aggregate shocks such as economic cycles that would affect the pool of applicants to other kinds of educational programs such as MBA programs.

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19Our claim can be easily extended, but the statement is awkward and the value limited. For example, consider an economy where some schools use (potentially different) admissions preference systems and other schools use quotas. We can design either an admissions preference system or a quota system that generates the same equilibrium strategies and assignment of students to colleges.
Another practical issue that may play out is that some aspects of a quota or admissions preference system may be difficult to formalize in terms of the other. For example, if there are quotas for “music students of exceptional promise,” can one precisely specify when a student’s potential is “exceptional,” so he or she is eligible for the reserved seats, rather than merely “good”? As a practical matter, such comparisons might be easier to formalize on a sliding scale that can be accommodated using an admissions preference scheme.

From a legal perspective, theorem 6 throws light on why it has proven so difficult for the court to draw a line between constitutionally permissible and impermissible affirmative action systems. As we discuss in more depth below, the Supreme Court jurisprudence has clearly ruled that quotas violate the U.S. Constitution’s 14th amendment’s guarantee of equal protection because nonminority students cannot compete for the seats reserved for minority applicants. However, the Court has given guidelines under which admissions preference schemes are acceptable, although the members of the court have disagreed as to the extent that admissions preference schemes are functionally different from quotas.

The cornerstone of supreme court jurisprudence regarding affirmative action is the 1978 case University of California Regents v. Bakke [53]. Justice Powell’s opinion established that the government has a compelling interest in encouraging diversity in university admissions founded on principles of academic freedom and a university’s right to take what actions it feels necessary to provide a high quality education to its students. Given this compelling interest, universities are free to implement affirmative action programs, although these programs must be narrowly tailored and are subject to a rigorous “strict scrutiny” standard of review.

Legal challenges to affirmative action schemes (e.g., Gratz et al. v Bollinger [32], Grutter v. Bollinger [35]) turn on whether the affirmative action schemes are narrowly tailored, a term that refers to whether the affirmative action scheme places the smallest possible burden on disadvantaged groups. In an amicus curiae brief to the case of University of California Regents v. Bakke, Harvard College describes their admissions in the following terms:

"In Harvard College admissions, the Committee has not set target quotas for the number of blacks, or of musicians, football players, physicists or Californians to be admitted in a given year. . . . But that awareness [of the necessity of including more than a token number of black students] does not mean that the Committee sets a minimum number of blacks or of people from west of the Mississippi who are to be admitted. It means only that, in choosing among thousands of applicants who are not only ‘admissible’ aca-
demically but have other strong qualities, the Committee, with a number of criteria in mind, pays some attention to distribution among many types and categories of students." (U. of California Regents v. Bakke [53], 438 U.S. 317)

Justice Powell held up the Harvard admissions program as a canonical example of a narrowly tailored, and hence constitutional, affirmative action system. The medical school at the University of California - Davis, the respondent in University of California Regents v. Bakke, used an admissions quota scheme, which Justice Powell ruled was unconstitutional.

Justice Powell’s decision in University of California Regents v. Bakke said little directly about acceptable or unacceptable outcomes of an affirmative scheme, but the opinion provides suggestions about constitutionally acceptable procedures for implementing an affirmative action system. The keys to a constitutionally approved affirmative scheme are:

1. All applicants are in competition for all seats.
2. To the extent that “odious” distinctions between applications, such as race, are a factor in admissions, the consideration of the applicants must be individualized.

The first consideration rules out explicit quotas, and most universities in the United States adapted their admissions policies accordingly in the post-Bakke era. However, Justice Powell’s opinion acknowledges that even admissions preference schemes that treat applicants as individuals can serve as “... a cover for a functionally equivalent quota system” (U. of California Regents v. Bakke [53], 438 U.S. 219).

The 2003 cases Gratz et al. v. Bollinger et al. and Grutter v. Bollinger et al. were the first affirmative action cases addressed by the Supreme Court following the ruling in

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20 As Justice Powell wrote:

“... race or ethnic background may be deemed a “plus” in a particular applicant’s file, yet it does not insulate the individual from comparison with all other candidates for the available seats.” (Page 438 U.S. 318)

21 Returning to Justice Powell’s opinion:

“This kind of program [referring to the Harvard College admissions system] treats each applicant as an individual in the admissions process. The applicant who loses out on the last available seat to another candidate receiving a “plus” on the basis of ethnic background will not have been foreclosed from all consideration for that seat simply because he was not the right color or had the wrong surname. It would mean only that his combined qualifications, which may have included similar nonobjective factors, did not outweigh those of the other applicant. His qualifications would have been weighed fairly and competitively, and he would have no basis to complain of unequal treatment under the Fourteenth Amendment.” (Page 438 U.S. 319 of Bakke)
University of California Regents v. Bakke. These cases turned on whether the University of Michigan admissions preference schemes are narrowly tailored. Interestingly, although Powell’s 1978 opinion appeared to say little about acceptable or unacceptable outcomes of affirmative action, the justices in both of these cases looked to the outcomes to judge the extent to which the systems function as de facto quotas.

Gratz et al. v. Bollinger et al. addressed whether the admissions preference scheme used by the University of Michigan College of Literature, Science, and the Arts (LSA) met the narrow-tailoring criteria. The admissions preference scheme used by the LSA attributed points to applicants based on (for example) academic performance, athletic ability, Michigan residency, and race. Applicants that received more than 100 of the maximum possible 150 points received immediate admissions, while those with fewer points were accepted later in the admissions process or not at all. Minorities received an additional 20 points, while applicants with significant leadership or public service achievements received at most 5 points.

The court ruled the LSA admissions preference scheme unconstitutional for two reasons. First, the across-the-board attribution of 20 points based solely on minority status was not individualized enough to qualify as narrowly tailored. Second, “…virtually all [minority freshman applicants] who are minimally qualified are admitted…” (Gratz v. Bollinger et al. [32], 539 U.S. 278), which means that these students are de facto not competing with nonminority applicants for admission. To summarize, the concurring justices argue in their opinion that although LSA did not formally use a quota, the results were functionally the same.

Grutter v. Bollinger et al. revolved around the admissions process of the University of Michigan Law School (Law School), which the Supreme Court ruled constitutional. The key difference between the LSA’s policy and the Law School’s is that each applicant to the law school is given individualized review without points attributed to particular traits of the applicant. However, if one reads the dissenting opinions, Justices Scalia and Rehnquist made separate arguments that the Law School admissions process was functionally equivalent to a quota:

“I join the opinion of The Chief Justice. As he demonstrates, the University of Michigan Law School’s mystical “critical mass” justification for its discrimination by race challenges even the most gullible mind. The admissions statistics show it to be a sham to cover a scheme of racially proportionate admissions.” (Scalia, p. 346 - 347 of Grutter)

22In her dissent, Justice Ginsburg argued that LSA was simply articulating its policy clearly and unambiguously.

“If honesty is the best policy, surely Michigan’s accurately described, fully disclosed College affirmative action program is preferable to achieving similar numbers through winks, nods, and disguises.” (p. 305)
“... the ostensibly flexible nature of the Law School’s admissions program that the Court finds appealing... appears to be, in practice, a carefully managed program designed to ensure proportionate representation of applicants from selected minority groups.” (Justice Rehnquist in Grutter v. Bollinger \[35\], 539 U.S. 385)

In the end, theorem \[6\] implies that attempts to differentiate between unconstitutional quotas and constitutional admissions preferences on the grounds of the outcomes produced will likely prove futile. For supporters of affirmative action, the equivalence of sophisticated quotas and affirmative action schemes might suggest that all of these systems ought to be constitutional, and universities should be free to use which ever system helps them better achieve their diversity goals. For opponents of affirmative action, this may provide an argument that no admissions scheme that takes race into account ought to be able to pass constitutional muster. At a minimum these ruling seem self-contradictory in light of our equivalence result\[^{23}\]

7 The Welfare Cost of Competition

We now use our model to address the welfare cost of the competition for college seats. We compare four economies to tease apart the relative importance of the assortativity that competition allows relative to the welfare losses caused by the wastefully high level of human capital the students accrue in the process of competing. The total welfare losses from competition amount to roughly $1,153 per student per year, yielding a net present value aggregated over 1996’s newly enrolled college class of over $15.7 billion for the newly entered class of students\[^{24}\]. One might have hoped that the benefits of an assortative match, which can only be accomplished in our incomplete information model via competition, might counterbalance the losses of competition. However, our metrics also suggest that the wasteful competition eliminates all of the benefits of assortative matching. Finally, we argue that the second-best college assignment contest is able to restore only 26% of the welfare losses, which suggests the scope for policy interventions to enhance average welfare are quite limited.

\[^{23}\]We mention two recent cases for completeness. Fisher v. University of Texas at Austin \[^{22}\] addressed issues regarding the application of the strict scrutiny standard without addressing the legality of particular affirmative action schemes. Schuette, Attorney General of Michigan v. Coalition to Defend Affirmative Action, Integration and Immigration Rights and Fight For Equality By Any Means Necessary et al. \[^{57}\] addressed the legal standard under which an amendment to the Michigan state constitution banning affirmative action by state institutions ought to be assessed.

\[^{24}\]In the fall of 1996, the Integrated Postsecondary Education System Dataset dataset studied by Hickman \[^{36}\] recorded 1,056,580 students newly enrolled in 4 year colleges. The net present value was computed using a 5% discount rate.
First, we would like to formally define the source of the welfare loss due to competition. The limiting payoff for agent type $\theta$ with HC $s$ is $\Pi^{cb}(s; \theta) = U(P^{cb}(s), s, \theta) - C(s; \theta)$. Differentiating, we get the following first-order condition (henceforth, FOC):

$$\frac{\partial U}{\partial s} + \frac{\partial U}{\partial P^{cb}(s)} \cdot \frac{dP^{cb}(s)}{ds} = \frac{dC}{ds}$$

The above expression concisely organizes the different aspects of the investment trade-off being made by students. It states that the marginal cost of human capital investment (the right-hand side) must be exactly offset by the marginal benefits (the left-hand side), which can in turn be decomposed into two parts. First, there is the direct value accrued to the student of having human capital level $s$, represented by the term $\partial U/\partial s$; this is the productive channel of investment incentives. There is also the indirect benefit of improving the college to which the student is assigned. An increase in one’s human capital level will improve the quality of one’s match partner by $dP^{cb}(s)/ds$, which increments utility by the margin $\partial U/\partial P^{cb}(s)$; the product of these two terms represents the competitive channel of investment incentives.

For the moment, suppose types were observable to a benevolent planner that could match students to colleges assortatively and allow them to invest ex post, in which case the FOCs would be $U_s(P^{*}_r(\theta), s, \theta) = C_s(s, \theta)$. We refer to the outcome generated by the social planner’s omniscient assignment and the ex post investments of the students as the First-Best Outcome. For concreteness, the following theorem compares the color-blind policy and the first-best outcome, but similar results could be generated for any of our admission schemes. The basic intuition for the result is clear: by shutting down the competitive channel, the first-best outcome reduces the incentive for students to acquire human capital. The net effect is to reduce the human capital obtained for every type of student.

**Theorem 7.** HC investment for all types in the color-blind admissions scheme exceeds that in the first-best outcome.

Given the incomplete information nature of the contest, one must consider two forces when evaluating the welfare effects of the competitive channel. First, color-blind competition between the students for better college seats through the use of human capital accumulation results in an assortative match, which is the necessary assignment in the first-best outcome. Second, the accumulation of human capital merely for the purpose of competing for a better college imposes negative externalities on the other students.

It is not obvious how to maximize surplus. Should the students compete in a color-
blind system that insures an assortative match? Or should society seek to minimize the negative externalities by dampening the competitive channel even though we may end up in a nonassortative match? This is a question that we answer using a model calibrated from real world data - it is unclear whether any parsimonious theoretical answer is possible.

The remaining results we present in this section are numerical solutions of our model conducted using the estimated utility functions and type distributions from Hickman [36] to calibrate our model. Hickman uses the US News and World Report college quality index as his metric for college quality, $p$. A student’s human capital level, $s$, is represented by the student’s SAT score. For the utility functions we used

$$U(p, s, \theta) = \rho(p, s) * u(p)$$
$$\rho(p, s) = -0.176 + 0.000774s - 0.00000049s^2 + 0.00076ps$$
$$u(p) = 40134p^{0.536}$$

where $\rho$ represents the probability of graduating from college and $u$ is the college premium conditional on graduating. The cost function for the model is

$$C(s, \theta) = \theta e^{0.013(s-\bar{s})}, \bar{s} = 520$$

where a student’s $\theta$ is inferred from his or her behavior under the status quo college admissions contest. Note that under this calibration we can interpret the average utility of the students as the ex ante expected college premium per student per year.

The analysis of Hickman [36] is conducted using data from 1995-1996 application year. To the extent that increasing concerns about overinvestment in HC are valid, then one ought to consider the estimates of the welfare cost of competition presented below to be a lower bound on the present cost. On the other hand, unless college quality has changed radically in the past two decades, it is likely that our assessment of the benefits of assortativity are relatively stable.

Our comparative statics are based on the assumption that college quality is exogenous to the HC of the students that are enrolled, which ignores spillovers between students. Our estimate of the welfare costs of competition is unlikely to be affected by allowing spillovers to influence college quality since our computations do not require reallocating many students. While the computation of the welfare of the second-best assignment does

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25 As pointed out by Fryer and Loury [26], many of the theoretical results on the consequences of affirmative actions systems are contingent on assumptions about the underlying economic primitives. Like the previous literature, our results are specific to the functional forms and values we have assumed. Our hope is that the calibration exercise has put the results in close contact with the real world.
require significant reallocation of students, we argue that including these spillovers would, if anything reduce the welfare of the second best, which in turn aggravates our conclusion that policy interventions to mitigate the welfare losses of competition are largely ineffective. However, our estimate of the benefits of assortativity remain sensitive to these spillovers. We address these issues in more depth in section 7.4 and appendix B.

7.1 The Cost of Competition

To estimate the welfare cost of competition we compare the first-best assignment outcome with the welfare generated by a color-blind contest. Note that both of these assignments are perfectly assortative. In the first-best assignment, the student types are known and they are assigned to colleges prior to choosing their HC level. In the color-blind match, the students’ HC level is the tool used to rank and assign the students. The only difference in the outcome is that the students in the color-blind contest are pushed via the competitive channel to accrue more human capital.

The first-best assignment generates $15,415 of social surplus for the students, whereas the color-blind mechanism generates only $14,262. Taking the difference in these values as the cost of competition, we find that the competitive channel results in the loss of $1,153 of welfare per student per year. For a somewhat more granular perspective, we compare the minority and nonminority student outcomes in the table below.

<table>
<thead>
<tr>
<th>Social Surplus</th>
<th>Social Surplus</th>
<th>Social Surplus</th>
</tr>
</thead>
<tbody>
<tr>
<td>Nonminority</td>
<td>Minority</td>
<td>Average</td>
</tr>
<tr>
<td>First Best</td>
<td>$16,584</td>
<td>$10,115</td>
</tr>
<tr>
<td>Color-blind</td>
<td>$15,329</td>
<td>$9,284</td>
</tr>
<tr>
<td>Cost of</td>
<td>$-1,225</td>
<td>$-831</td>
</tr>
<tr>
<td>Competition</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Below we have plotted the welfare costs of competition by student type.

Figure 1 reveals that the costs are highest for students in the top half of the distribution (i.e., those students with $\theta \leq 7.56$), which makes sense given that a student admitted to a college of quality $p$ only imposes externalities on students that are eventually admitted to schools with qualities higher than $p$. Interestingly, the welfare loss begins to decrease for the very lowest cost students since these students are predisposed to accrue high amounts of HC and the number of high quality college seats is relatively abundant relative to the population of low cost students. The students in the middle, who are in relatively high supply relative to the supply of middling quality colleges, have the strongest incentive to compete since small increases in HC would allow them to leapfrog a large mass of
competitors and attain a much better seat.

Aggregating across the population of 1,056,580 students newly enrolled in 4 year colleges in the fall of 1996, the aggregate value of the losses per year to the enrolled students is $1.22 billion. If we use a conservative (for a safe asset) discount rate 5% and assume a 30 year career, we get a total net present value of over $15.7 billion for the newly entered class of students. To the extent that anecdotes about the increasing competition amongst students for college seats, such as Battle Hymn of the Tiger Mother, are accurate, then this large welfare loss ought to be considered a lower bound of the welfare costs facing current college applicants.

7.2 The Benefits of Assortativity

Our second task is to compute the welfare gains from assortativity, which we do by comparing the first-best assignment with a random-assignment. By a random-assignment, we mean that the students are first randomly assigned to colleges and then are allowed to choose their human capital level. Specifically, the random assignment provides each student, regardless of his or her type, a seat at college $p$ with a probability equal to $f_P(p)$. The competitive channel has been shutdown in both the first-best and random assignments - the difference in outcomes is generated by the fact that the first-best is an assortative assignment, whereas the random assignment yields none of the benefits of
assortativity.

As noted above, the first-best assignment generates $15,415 of social surplus for the students. The random assignment generates a social surplus of $14,593. Taking the difference of these quantities to represent the gains from assortativity, we find a welfare gain of an assortative match equal to $822 per year. Comparing this with the $1,153 welfare loss from competition, we see that the competitive channel more than wipes out the gains from assortativity. In other words, social surplus would be improved if we could eliminate competition by randomly assigning students to colleges before HC decisions were made! Total yearly value of the benefit of assortativity are $868 million per year for a total net present value of $13.6 billion for the newly entered class of students.

If we break down the outcome by demographic group we find

<table>
<thead>
<tr>
<th></th>
<th>Social Surplus Nonminority</th>
<th>Social Surplus Minority</th>
<th>Social Surplus Average</th>
</tr>
</thead>
<tbody>
<tr>
<td>First Best</td>
<td>$16,584</td>
<td>$10,115</td>
<td>$15,415</td>
</tr>
<tr>
<td>Random</td>
<td>$14,886</td>
<td>$13,226</td>
<td>$14,593</td>
</tr>
<tr>
<td>Benefits of Assortativity</td>
<td>$1,698</td>
<td>-$3,331</td>
<td>$822</td>
</tr>
</tbody>
</table>

We see that minority students suffer a significant loss when moving from a random to an assortative match, but this loss is overwhelmed by the gains made by the more numerous nonminority students. The fact that minority students do better under a random assignment may be initially surprising, but this outcome is driven by the fact that minority students have significantly higher costs to accrue human capital and are assigned a disproportionate fraction of seats at lower quality schools under an assortative outcome.

7.3 Computing the Second Best

Our third and final comparative static is to compute the second-best college assignment contest to discover what fraction of the welfare gains of the first-best can be recovered by either admissions policy changes or a clever intervention on the part of the government. We take as given that colleges all wish to enroll the highest human capital students possible. Any move outside of a contest model would require that colleges either abandon these goals or coordinate on some alternative mechanism, both of which we view as implausible in the near-term. Similarly, contests with stochastic outcomes (i.e., wherein colleges deliberately randomize their admissions decisions) would mean the randomizing college is choosing not to admit the best applicants, which strikes us as implausible given
our underlying beliefs about the motivations of colleges.

Let us now describe the optimal control problem we solve to derive the second-best contest. The control, \( u(p) \in [0, 1] \), represents the fraction of seats at school \( p \) that are allocated to minority students. Because the index variable is the college, \( p \), all of the variables in our problem must be written as functions of \( p \). The state variables of our control problem are the equilibrium strategies of the students, \( \sigma_M(p) \) and \( \sigma_N(p) \), and the type of the student from each group that is assigned a seat at college \( p \), \( \theta_M(p) \) and \( \theta_N(p) \). Our optimal control problem can be written

\[
\max_u \int_p u(p)[U(p, \sigma_M, \theta_M) - C(\sigma_M, \theta_M)] + (1 - u)[U(p, \sigma_N, \theta_N) - C(\sigma_N, \theta_N)] f_P(p)dp
\]

such that

\[
\begin{align*}
\dot{\theta}_M(p) &= -\frac{u * f_P(p)}{\mu f_N(\theta_M(p))} \\
\dot{\theta}_N(p) &= -\frac{(1 - u) * f_P(p)}{(1 - \mu) f_N(\theta_N(p))} \\
\dot{\sigma}_M(p) &= \frac{U_p(p, \sigma_M, \theta_M)}{C_s(\sigma_M, \theta_M) - U_s(p, \sigma_M, \theta_M)} \\
\dot{\sigma}_N(p) &= \frac{U_p(p, \sigma_N, \theta_N)}{C_s(\sigma_N, \theta_N) - U_s(p, \sigma_N, \theta_N)} \\
\int_p u(p)f_P(p)dp &= \mu \\
\sigma_M(p) &= \sigma_N(p) = 520, \ \theta_M(p) = \theta_N(p) = 1517
\end{align*}
\]

The objective of the problem is simply a rewriting of the average social surplus using the index variable of the control problem, \( p \). Equations 16-19 denote the laws of motion for the state variables, and equation 20 insures that enough seats are allocated to minorities that the entire measure \( \mu \) of minorities obtains a college seat. Equation 21 provides boundary conditions for our state variables that are derived from Hickman [36]. We impose the boundary condition \( \sigma_j(p) = \overline{\xi} = 520 \) regardless of whether or not both groups are assigned seats at the worst school, \( p \). We are, in effect, assuming that both groups are assigned at least a vanishingly small fraction of a seat at every college.

Since the objective function and the equations of motion are linear in \( u \), we know im-

---

\(^{26}\)In section 8, we consider situations that generalize our contest model. These extensions include an extension wherein colleges get noisy observations of the students’ human capital choice, but the randomness this noise generates has a very different interpretation than deliberate randomization by colleges.

\(^{27}\)We do this primarily so that the optimal control problems are tractable. An alternative to our approach would be to allow the boundary conditions for each group to be defined by indifference conditions.
mediately that the solution will have a bang-bang structure. In other words, the social surplus maximizing affirmative action scheme will generically involve complete segregation - all of the seats in each school will be allocated to one of the two groups. However, it could be that the two groups are allocated essentially identical schools. For example, it could be that each time a school \( p \) is allocated to nonminority students, minority students are allocated a school with a quality very close to \( p \).

![Figure 2](image.jpg)

However, the second-best outcome assigns all of the low quality college seats to minority students and reserves the best colleges for nonminority students, a result depicted in figure 2. Figure 2 includes the allocation under the color-blind scheme for comparison. There are two stages to the logic underlying the second-best. First, the incentive to compete is driven by the difference in the quality of the prizes. To reduce this incentive, we must make the difference between the best and worst prize available to each group as small as possible, which can be done by breaking the prize space into two intervals. Second, we ought to assign the better prizes to the group that will reap larger complementarities from being matched to a high quality school. Since membership in a nonminority group is a signal that human capital accumulation costs are low, the second-best gives the majority of the seats at high quality colleges to nonminority students.

We compare the welfare from the first best, second best, color-blind, status quo and random assignment schemes in the following table. By status quo, we simply mean the social surplus estimated by Hickman given the affirmative action policies in place in
the 1990s.

<table>
<thead>
<tr>
<th></th>
<th>Social Surplus</th>
<th>Social Surplus Relative to Status Quo</th>
</tr>
</thead>
<tbody>
<tr>
<td>First Best</td>
<td>$15,415</td>
<td>$1,266</td>
</tr>
<tr>
<td>Second Best</td>
<td>$14,472</td>
<td>$323</td>
</tr>
<tr>
<td>Color-Blind</td>
<td>$14,262</td>
<td>$113</td>
</tr>
<tr>
<td>Random Assignment</td>
<td>$14,593</td>
<td>$444</td>
</tr>
<tr>
<td>Status-Quo</td>
<td>$14,149</td>
<td>$0</td>
</tr>
</tbody>
</table>

The primary takeaway from these results is that most of the losses caused by the competition in the status quo contest, represented by the difference between the status quo and the first best, cannot be recovered by a policy intervention. Of the $1,266 of welfare losses per student per year due to competition, only $323 (26%) can be recovered in the second-best. Given the extreme unfairness of the second-best assignment, it cannot be taken seriously as a policy recommendation - surely social planners care about many concerns other than average welfare (e.g., diversity). However, our result suggests that since market interventions cannot seriously address the significant losses resulting from competition, policy-makers ought to focus their attention on other issues.

### 7.4 Caveats

At this point we would like to discuss our model’s assumption that school quality is fixed and exogenous. Although it is outside of the scope of this paper to provide a model that endogenizes school quality, it is well known that there are spillovers between students in many contexts that make school quality a function of which students enroll. The spillovers may be a function of student characteristics (e.g., Hoxby [40]) or student effort choices (e.g., Fruewirth [24]). In addition, if we take at face value the briefs filed by the universities in Regent of the University of California v. Bakke [53], Gratz v. Bollinger [32], and Grutter v. Bollinger [35], universities believe that student welfare is directly enhanced by diversity amongst the student body.

Since the first best outcome as well as the color-blind and status quo contests all result in nearly assortative matches without radical differences in the human capital choices of the agents, these concerns are less pressing for these cases. However, the second best match and the random assignment counterfactuals both imply significantly nonassortative outcomes, and these welfare computations should be reviewed in light of the concerns presented above. Without a well calibrated model of these spillovers, it is hard
to say much about the magnitude of these effects. At one extreme, if one believed that assigning a few great students to a school would create a high quality school, then the benefits of assortativity would be even lower. At the other extreme, if a few bad students destroys a school’s quality, then assortativity is crucial for student welfare.

To address these concerns more concretely, appendix B repeats the analysis of this section using an ad hoc model of endogenous school quality that assumes school quality is generated by the median student quality. We believe this exercise helps to identify key features of the problem. However, since the model of school quality is one of many possible choices, we do not view the numbers provided in appendix B as necessarily realistic estimates of the cost of competition or the benefits of assortativity when school quality is endogenous.

8 Extensions

Our goal in this section is to discuss some of our modeling choices, potential alternative assumptions, and to what extent the alternatives make a substantive difference in our model. Our first extension is to allow the matching to be imperfectly assortative by including a noise term in the colleges’ observations of a student’s human capital choice, which slightly alters the ODEs that characterize the equilibria, but does not substantively change our results. Second we consider the effect of allowing the colleges to have multi-dimensional types or the students to make multidimensional investment decisions, which we find do not make a significant difference in our results.

Finally we consider whether we can incorporate heterogeneous preferences on either side of the market. The key to incorporating a diversity of preferences is to construct a well-defined notion of an assortative outcome that pertains to the new model. If the notion of assortativity either is ill-defined or fails to hold, then modeling the market as a contest becomes implausible. When student preferences are allowed to be heterogeneous, we find that the notion of an assortative outcome fails to be well defined since the colleges cannot be completely ordered. However, we can include these feature in our model by generalizing our contest mechanism to a serial dictatorship. When we allow for heterogeneity amongst the preferences of the colleges, assortativity continues to be well-defined and hold in equilibrium since students with lower $\theta$ endogenously choose higher human capital levels along all dimensions.

8.1 Noisy observations of HC

In the real world, the matching between colleges and students is not perfectly assortative on SAT scores. Presumably this is because the colleges view other attributes of a student’s
application as informative about the applicant. One way to model this is to assume that
the colleges observe a multidimensional investment decision on the part of the students,
which we discuss in section 8.2. An alternative route is to assume that the investment
decision of the student is randomly shocked, which yields a noisy signal of the student’s
investment decision \( t = s + \epsilon \), where \( \epsilon \) refers to the error in the observation. We assume
that \( \epsilon \) is common across colleges and that students are matched to schools assortatively
based on \( t \) and \( p \).

Given the random shock, the decision problem of the students in the limit game
becomes

\[
\max_s E [U(P(s + \epsilon), s, \theta))] - C(s; \theta) \]

where the expectation is over the random shock. All of the assumptions (e.g., super-
modularity, monotonicity) placed on the function \( U(P(s), s, \theta) \) continue to hold for the
expectation \( E[U(P(s + \epsilon), s, \theta)] \). It is straightforward to show that all of our results
regarding the existence and uniqueness of equilibria and the approximation of the finite
equilibria with the limit game equilibrium continue to hold.

8.2 Multidimensional investment decisions and college types

In our main model we assume that student human capital is represented by a single
number and that colleges are characterized by a unidimensional quality index. Obvi-
ously students choose to build many different kinds of human capital (e.g., performance
in class, SAT scores, extracurricular activities), and colleges evaluate them along many
different dimensions. Similarly, colleges differ along many dimensions (e.g., quality of
different degree programs, appeal of the location) that affect how students evaluate them.
As it turns out, allowing for complex action and type spaces does not change our results
so long as we retain the assumption of a common student preference over the college
types and a common college preference over the student investments.

Suppose the colleges have a multidimensional type \( p \in \mathbb{R}^N_+ \). If we assume the
(mild) conditions required for the students’ preferences to be represented by a utility
function, then we can use the student utility associated with a college as the college’s
unidimensional “type” in our model and all of our results carry through immediately.

Now consider the colleges faced with evaluating a pool of applicants that have each
made a multidimensional investment choice \( s \in \mathbb{R}^N_+ \). In order for our results to hold, we
need to assume some structure on the human capital decision. Namely, we assume that
the colleges’ preference ranking is strictly monotone increasing in \( s \) and that the decision
problem of the agent

\[
s(\theta) \in \arg\max_{s \in \mathbb{R}^N_+} U(P(s), s, \theta) - C(s, \theta)\]
is pairwise supermodular in \((s_i, s_j), i \neq j\), and \((s_i, \theta)\) and differentiable in each component of \(s\). This insures that \(s(\theta)\) is strictly increasing in \(\theta\), which in turn means that \(P(s(\theta))\) is strictly increasing in \(\theta\).

With this structure in place, it is possible to characterize an equilibrium of the limit game using a system of partial differential equations. Establishing the continuity of the utility functions in the limit game is straightforward, which allows us to prove that any equilibrium of the limit game is an \(\varepsilon\)-approximate equilibrium of a sufficiently large finite game. Under stricter conditions than assumed in theorem 2 (e.g., under a quota scheme wherein \(Q_M\) and \(Q_M^\prime\) have full support), we can prove that the limit equilibrium is unique and is a \(\delta\)-approximate equilibrium of a sufficiently large finite game.

8.3 Diverse preferences amongst students

Once we allow colleges to be characterized by a multidimensional type \(p \in \mathbb{R}_+^N\), it is natural to consider a setting wherein the students have diverse preferences over \(p\). For example, it may be that colleges are characterized by the quality of the education provided and their tuition fees, and students have diverse preferences over these attributes either because of differences in family wealth or because some students pay a reduced tuition (e.g., in-state v. out-of-state tuition rates at many public universities). Because the contestants differ in their rankings of the prizes that can be won in the college admissions contest, this setting does not neatly fit into the contest model we have studied above.

We can incorporate these features into our analysis by recasting our model as a serial dictatorship. In this recasting, the students sequentially choose their preferred school in order from highest to lowest human capital choice. The outcome remains assortative in the sense that students with higher human capital investments are allowed to choose their preferred school earlier in the queue. However, without some regularity assumptions on the preferences of the students, it is difficult to apply any of our limit game approximation results since significant discontinuities may be present.

Let us assume that a student’s preference are described by a multidimensional variable \(t \in \mathbb{R}^n\) with the utility of the student as follows

\[ U(p, s, \theta, t) - C(s, \theta) \]

Assume \(U\) is continuous in \(t\) and increasing in each dimension of \(p\). We can still use a function \(P(s)\) to describe the colleges available to a student with human capital level \(s\),

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\(^{28}\)When the investment decision is multidimensional, our method for proving the uniqueness of the discontinuities in the equilibrium strategy described by the differential equations no longer works. However, standard results on the uniqueness of solutions of partial differential equations apply if there is no such discontinuity.
but this will be a set valued function. First note that in this setting, the colleges available to a student with human capital level \( s \) depends not only on how many students have higher levels of human capital but on the preferences of those individuals. Second, in any equilibrium we can describe the possible colleges available to a student with human capital level \( s \) as an \( n - 1 \) dimension surface of \( p \in \mathbb{R}_+^N \). This second insight also implies that we can describe the assortative match of a quota system without explicitly computing the strategies, so this model retains some of the tractability of the unidimensional model studied above.

Using our redefined function \( P(s) \), we try to write-down an ODE describing the equilibrium human capital investments. However, without some structure on how \( t \) varies with \( \theta \), we cannot insure that our ODE has a solution (much less a unique one). For example, if \( t \) is a deterministic and almost everywhere continuous function of \( \theta \), then the ODE will have a solution. Since the utility each type of student receives varies continuously with \( s \) in the limit game, one can show that any equilibrium of the limit game is an \( \varepsilon \)-approximate equilibrium of sufficiently large finite games. If we have a unique limit equilibrium, then the equilibrium of the limit game is a \( \delta \)-approximate equilibrium of sufficiently large finite games. In other words, our approximation results continue to hold.

### 8.4 Diverse preferences amongst colleges

Finally, consider the case where students make multidimensional investment decisions \( s \in \mathbb{R}^n \) and colleges have heterogeneous preferences over these decisions. We assume that all colleges have monotone increasing preferences over \( s \). We maintain the benchmark assumption that student preferences over colleges are characterized by a unidimensional college quality type. If we make regularity assumptions as per those of section 8.2, then we obtain an assortative match in the sense that \( s(\theta) \) is increasing in each dimension, which means that all of the colleges agree on the endogenous ranking of the students since \( \{s(\theta) : \theta \in [\theta, \bar{\theta}]\} \) is totally ordered in equilibrium. In other words, better students (low \( \theta \)) make larger human capital investments and are matched to better colleges. Once we have this insight regarding the complete ordering of the equilibrium investments and the assortativity of the match, the discussion of section 8.2 pertains and we see that most of our results carry through to this setting.

This extension is significantly easier to handle than the case where colleges have multidimensional types and students have diverse preferences (section 8.3). Section 8.3 allowed for models wherein the college types were only partially ordered, which allowed for student preferences to resolve this partial order in different ways. In this section the different possible investment decisions are only partially ordered, but the endogenous
decisions are completely ordered. If the equilibrium investment decision were only partially ordered, we would have to deviate from the contest structure to a serial dictatorship or some other description. The failure of the equilibrium investment decisions to be fully ordered might result if (for example) the utility of the agents was not supermodular in \( s_i, s_j \) for some pair \( i \neq j \), which would imply \( s(\theta) \) might not be monotone in all dimensions. Any such violation of monotonicity could cause the set of investment decisions observed in equilibrium to be partially ordered, which would require an analysis in the spirit of section 8.3.

9 Conclusion

The purpose of this paper has been to introduce a new model of college admissions and use it to enrich the debate around the differences between quota and admissions preference systems as well as throwing light on the welfare costs of the competition for admission, which anecdotal evidence suggests has ratcheted up in intensity in recent years. Modeling college admissions and affirmative action programs is challenging since one must consider the human capital investment decisions of students, heterogeneity in underlying quality on both sides of the market, and the strategic decisions of universities given the information they are presented. Prior papers in the literature gain tractability by simplifying various components of the problem (e.g., assuming all college are homogenous or student quality is innate), we instead consider a market with a continuum of agents, and the continuum approximation greatly simplifies the analysis and allows us to produce a large number of novel results.

Our first application of the model is to study the difference between quota-based and admissions preference-based affirmative action systems, and we find that in fact there is no difference in the equilibrium outcomes produced. It is not particularly surprising that one can achieve the same diversity levels under each scheme - for example, one might imagine duplicating the effect of a quota in an admissions preference system by giving minority applicants a small boost when evaluating their applications. However, we believe it is surprising that the equivalence holds over both the diversity at individual colleges and the human capital accumulation decisions of those admitted.

Moreover, our equivalence formalizes comments made in the jurisprudence regarding the difference between quotas and affirmative action schemes. The legality of affirmative action turns on what was perceived by Justice Powell as a sharp difference between quotas and well designed admissions preference systems. Later opinions about the legality of different admissions preference schemes hinged on how closely the respective justice thought the admissions preference scheme mimicked a quota. Our analysis suggests
that drawing a sharp line between quotas and admissions preference schemes based on outcomes will be futile. Not being legal scholars, we have little to say about the eventual legal ramifications for universities that employ color-sighted affirmative action programs.

Our last contribution is to analyze the welfare effects of competition for college admission using a calibration of our model drawn from Hickman [36] that used admissions data from 1996. In an incomplete information world, competition is necessary to achieve an assortative match between students and colleges. Unfortunately, our analysis suggests that competition wipes out most (and possibly more than all) of the benefits of assortativity. We are able to compute the welfare in the second-best contest, and we find that we can only recover roughly 20% of the welfare lost due to competition. This suggests that policy or market interventions will have little effect on blunting the disutility of competition unless a radical departure from the current contest-like admissions process becomes possible.

Three other interesting directions for further research exist. First, if we could incorporate a model of how student assignment endogenously influences college quality, we could discuss with more confidence the general equilibrium implications of massive changes in the assignment of students to schools. Unfortunately, we are unaware of any structural models we could use as the basis for this research agenda.

Second, the current model focuses on student behavior, conditional on participation in the college market, but there is another interesting group of individuals to consider as well: those whose college/work-force decisions may be affected by a given policy. This question could be addressed by formalizing the “supply-side” comprised of potential colleges and firms who may enter the market and supply post-secondary education services or unskilled jobs. Such a model might illustrate how the marginal agent (i.e., the individual indifferent between attending college and entering the workforce) is affected by a given college admission policy. This would help to characterize the effect of AA on the total mass of minorities enrolling in college.

Finally, the eventual goal for this line of research should be to answer the question of how AA helps or hinders the objective of erasing the residual effects of institutionalized racism. This will require a dynamic model in which the policy-maker is not only concerned with short-term outcomes for students whose types are fixed, but also with the long-run evolution of the type distributions. Empirical evidence suggests that academic competitiveness is determined by factors such as affluence, as well as parental education. If AA affects performance and outcomes for current minority students, then the next question is what effect it might have on their children’s competitiveness when the next generation enters high school? If a given policy produces the effect of better minority enrollment and higher achievement in the short-run, then one might conjecture
that a positive long-run effect will be produced. However, given the mixed picture on the various policies considered in this paper, it seems evident that a long-run model is needed in order to give meaningful direction to forward-looking policy-makers. It is our hope that the theory developed here will serve as a basis for answering these important questions in the future.

References


A Proofs

A.1 Proving Theorem 5

Since the equilibrium strategies are strictly decreasing (proposition 1), we know immediately that the equilibrium strategy must be almost everywhere differentiable. We now prove that there is a lower bound on the derivative of the equilibrium strategy, which implies that the distribution of human capital in any equilibrium must be nonatomic. Moreover, it implies if we look at sequences of equilibrium strategies, the resulting limit strategy generates a nonatomic distribution of human capital.

Lemma 1. There exists $\omega < 0$ such that for any equilibrium strategy, $\sigma(\theta)$, of either a finite game or the limit game, we have $\frac{d}{d\theta} \sigma(\theta) < \omega$ at points where the strategy is differentiable.
Proof. We prove our lemma for the color-blind game, but the proof extends directly to the admissions games with quotas by treating each group separately. Finally, the proof technique easily extends to the score function game, although the notation becomes cumbersome since one must accommodate both groups of students and account for the variation in $\tilde{S}$.

Suppose there is no such upper bound on the derivative, which means that for any $\omega < 0$ there exists $\theta$ such that $\sigma'(\theta) > \omega$. From the a.e. differentiability of $\sigma$, there exists an interval $[\theta_L, \theta_U]$ such that $\sigma'(\theta) > \omega$ for all $\theta \in [\theta_L, \theta_U]$ where $\sigma'(\theta)$ exists. Without loss of generality, we assume $\sigma'(\theta)$ exists at $\theta_L$. Let $s_L = \sigma(\theta_L)$ and $s_U = \sigma(\theta_U)$, and note that $0 < s_L - s_U < \omega(\theta_L - \theta_U)$. Since $\sigma$ must be decreasing, we have

$$\Pr\{\sigma(\theta) \in [s_U, s_L]\} = F(\theta_U) - F(\theta_L)$$

Rearranging this we find

$$\frac{\Pr\{\sigma(\theta) \in [s_U, s_L]\}}{s_L - s_U} = \frac{F(\theta_U) - F(\theta_L)}{\sigma(\theta_U) - \sigma(\theta_L)} > \frac{-1 F(\theta_U) - F(\theta_L)}{\omega \theta_U - \theta_L}$$

Let $\eta_\theta = \inf_{\theta} f(\theta) > 0$. Taking limits we find

$$\lim_{\theta_U \to \theta_L} \frac{\Pr\{\sigma(\theta) \in [\sigma(\theta_U), \sigma(\theta_L)]\}}{\sigma(\theta_U) - \sigma(\theta_L)} > \frac{-1 \lim_{\theta_U \to \theta_L} F(\theta_U) - F(\theta_L)}{\omega \theta_U - \theta_L} = \frac{-1 \omega f(\theta_L)}{\omega \eta_\theta} > 0$$

This means that in intervals where $\sigma'(\theta)$ is close to 0, the “density” of individuals making the associated human capital choices is arbitrarily large. We call such a point of high density a pseudo-atom.

Let $\delta_p = \sup p f_p(p) < \infty$. Increasing the human capital choice from $s_L$ to $s_U$ yields a minimal benefit of increasing the rank of one’s school in the limit game by

$$-\frac{1}{\omega \delta_p} (s_L - s_U)$$

If we let $\eta_U = \min_{s,p,\theta} U_p(p,s,\theta) > 0$, then the utility benefit must be at least

$$-\frac{1}{\omega \delta_p} \eta_U (s_L - s_U)$$

Let the maximum marginal cost of human capital that we can observe in any equilibrium be denoted

$$\delta_C = \max_{s \in \tilde{S}, \theta \in \Theta} C_s(s, \theta) < \infty$$
This means the cost of deviating from \( s_U \) to \( s_L \) is bounded from above by

\[
(s_L - s_U) \delta_C
\]

For such a deviation to be suboptimal, we must have

\[
(s_L - s_U) \delta_C \geq -\frac{\eta_U \eta_\theta}{\omega \delta_p} (s_L - s_U)
\]

which requires

\[
\sigma'(\theta) \leq \omega \leq -\frac{\eta_\theta \eta_U}{\delta_p \delta_C}
\]

In the game with \( K \) students, the formation of pseudoatoms is probabilistic. Consider an agent with type \( \theta_U \) who in equilibrium chooses \( s_U = \sigma^K(\theta) \). Suppose such a student considers increasing her human capital choice to \( s_L \). For each student she leap frogs, her school placement improves by at least \( (\delta_p)^{-1} \frac{1}{K} \), which generates a utility benefit of at least

\[
\frac{\eta_U}{\delta_p} \frac{1}{K}
\]

For each student, there is a probability of at least \(-\frac{\eta_\theta}{\omega} (s_L - s_U)\) of observing a human capital choice in \([s_U, s_L]\), which yields a lower bound on the expected benefit of the deviation equal to

\[
\frac{\eta_U}{\delta_p} \frac{1}{K} E[i]
\]

where \( i \) is distributed binomially with \( K \) draws using a parameter equal to \(-\frac{\eta_\theta}{\omega} (s_L - s_U)\). Given the distribution of \( i \), we can write

\[
\frac{\eta_U}{\delta_p} \frac{1}{K} E[i] = \frac{\eta_U}{\delta_p} \frac{1}{K} \left[-\frac{\eta_\theta}{\omega} (s_L - s_U) K \right] = -\frac{\eta_U}{\delta_p} \frac{\eta_\theta}{\omega} (s_L - s_U)
\]

The remainder of the argument proceeds as above.

Before proving our approximation results, we provide a few background results from the theory of the weak convergence of empirical processes.

**Theorem 8.** Consider a random variable \( X : \Omega \rightarrow \mathbb{R}^d, d < \infty \), with measure \( \pi_0 \) and associated CDF \( F(y) = \int_c 1\{x \leq y\} * \pi_0(dx) \). For \( N \) i.i.d. realizations, \( \{X_1, ..., X_N\} \), drawn from \( \pi_0 \), denote the \( N \) realization empirical CDF as \( F_N(y) \). Then we have

\[
\sup_{y \in \mathbb{R}^d} \|F_N(y) - F(y)\| \rightarrow 0 \text{ almost surely as } N \rightarrow \infty
\]
Proof. Follows from van der Vaart et al. [?], p. 135 and noting that sets of the form 
\{x : x \leq y\} for \(y \in \mathbb{R}^d\) are lower contours and form a VC Class. 

The topology over the space of measures generated by the sup-norm over the space of
CDFs is referred to as the Kolmogorov topology. It is straightforward to show that the Kolmogorov topology and the weak-* topology\(^{29}\) are identical over any space of nonatomic measures.

**Corollary 9.** Define the empirical measure generated by the counting measure over \{\(X_1, ..., X_N\}\) as \(\pi_N\). Then \(\pi_N \rightarrow \pi_0\) almost surely in the weak-* topology over \(\Delta(\mathbb{R}^d)\)

Proof. From Billingsley (p. 18, [?]) we have that \(F_N(y) \rightarrow F(y)\) at continuity points of \(F\) impies \(\pi_N \rightarrow \pi_0\) almost surely in the weak-* topology. Since we have uniform convergence \(F_N(y) \rightarrow F(y)\) for all \(y\) almost surely, we have \(\pi_N \rightarrow \pi_0\) in the weak-* topology. \(\square\)

**Corollary 10.** Consider a random variable \(X : \Omega \rightarrow \mathbb{R}^d, d < \infty\), and associated CDF \(F(y)\). Denote the \(N\) realization empirical CDF as \(F_N(y)\). Then

\[
\Pr\{\sqrt{N}\sup_{y \in \mathbb{R}^d} \|F_N(y) - F(y)\| > t\} \leq C * e^{-2t^2}
\]

where the constant \(C > 0\) depends only on the dimension \(d\).

Proof. This result follows directly from Theorems 2.6.7 and 2.14.9 of van der Vaart and Wellner [?]. \(\square\)

With this background in hand, we can now proceed with the proof of our approximation results. We prove our results in two steps. First we provide a weaker version of theorem\(^{[\text{[2]}]}\) that assumes that the measures defining the quota schemes have strictly positive PDFs. This assumption allows us to prove continuity results that imply our claim. We then prove our original result by arguing that even when we allow \(Q_M\) and \(Q_N\) to have disconnected supports, the equilibrium conditions of our model insure that equilibria of the limit game remain equilibria of the finite game with sufficiently many players.

**Theorem 11.** Let \(\sigma_i^j, i \in M, N\) and \(j \in \{cb, q, ap\}\) denote an equilibrium of the limit game. Assume one of the following cases holds:

1. The limit game of the admissions preference model with a differential markup function \(\tilde{S}\) that satisfies \(\tilde{S}(\tilde{s}) = \tilde{s}\) and \(F_P\) has full support.

\(^{29}\)The sup-norm and the Prokhorov-Levy metric that metricizes the weak-* topology are identical when one of the measures is nonatomic.
2. The limit game of the quota system with any feasible choice of \(Q_M\) and \(Q_N\) that admit strictly positive PDFs over a connected support.

Under assumptions 2-9, assumption 11, and given \(\epsilon, \delta > 0\), there exists \(K^* \in \mathbb{N}\) such that for any \(K \geq K^*\) we have that \(\sigma^*_i\) is a \(\epsilon\)-approximate equilibrium of the \(K\)-agent game and \(\sigma^i_j\) is a \(\delta\)-approximate equilibrium for the \(K\)-agent game.

Proof. We prove our theorem through a series of lemmas steps. These lemmas are primarily necessary for the application of the theorems in Bodoh-Creed [6].

Since our proofs rely on results on the convergence of empirical processes to their true distributions, we will need to define the spaces in which these measures live. The true distributions of the student types are \(F_M(\theta)\) or \(F_N(\theta)\). We have assume these distributions have full support, which means that their respective PDFs are bounded from below. This means that we can define a compact set \(\Delta(\Theta)\) of measures over \(\Theta\) such that \(F_M(\theta), F_N(\theta) \in \Delta(\Theta)\) and all of the measures in \(\Delta(\Theta)\) have PDFs that are uniformly and strictly bounded above 0. We endow the space \(\Delta(\Theta)\) with the Kolmogorov topology.

First we establish some initial properties of the objects we are working with. Lemma 8 establishes that the equilibrium strategies have an upper bound on their derivative, which implies that pseudo-atoms cannot exist. Let \(\Sigma^R\) be the set of strategies that adhere to the bound prescribed by lemma 8 and we endow this space with the sup-norm. Let \(\Delta^R(S)\) denote the space of pushforward measures generated by a strategy \(\sigma \in \Sigma^R\) and a distribution of student types, \(F_i(\theta) \in \Delta(\Theta)\). The set \(\Delta^R(S)\) is a tight family of measures and is compact as a result (Theorem 15.22 of Aliprantis and Border [3]). We endow the space \(\Delta^R(S)\) with the weak-* topology. Since the measures in \(\Delta^R(S)\) are all nonatomic, the weak-* topology is equivalent to the Kolmogorov topology generated by the sup-norm applies to the space of CDFs.

Let \(\Delta_K(X)\) denote the set of empirical measures generated \(K\) draws from the set \(X\). We will let \(G^{K_X}_N\) denote the CDF of the empirical measure of human capital choices for the nonminority students when there are \(K_N\) such students in the economy, and we use the notation \(\Delta_{K_X}(S)\) to refer to the set of such CDFs. \(G^{K_M}_M\) denotes the empirical measure of human capital choices for the minority students when there are \(K_M\) such students in the economy, and we use \(\Delta_{K_M}(S)\) to refer to the set of such CDFs. We endow \(\Delta_{K_X}(S)\) and \(\Delta_{K_M}(S)\) with the weak-* topology. Note that since \(\sigma \in \Sigma^R\) are strictly monotone, then all of the elements of \(\Delta_{K_X}(S)\) and \(\Delta_{K_M}(S)\) must be nonatomic, which implies that the weak-* topology and the Kolmogorov topology are equivalent.

We now prove two useful lemmas about inverse functions that, for some reason, we cannot find in the existing literature.
Lemma 2. Let $G_i, i = \mathcal{M}, \mathcal{N}$ be the CDF of the pushforward measure generated by $\sigma_i \in \Sigma^R$ and the type distribution $F_i \in \mathcal{D}$. Then $G_i$ is uniformly continuous in $\sigma_i$ when the space of CDFs is endowed with the sup-norm.

Proof. Consider $\tilde{\sigma}_i$ with the associated pushforward measure $\tilde{G}_i$. Then

$$\tilde{G}_i = F_i \left( \tilde{\sigma}_i^{-1}(s) \right)$$

Since $\sigma_i \in \Sigma^R$ has a slope that is bounded from below, we know that $\sigma_i^{-1}$ is uniformly continuous in $\sigma_i$ under the sup-norm.\footnote{A quick proof of this claim starts by noting that for any $\theta$}

Since $F_i$ is nonatomic, this implies that $G_i$ is uniformly continuous in $\sigma_i$. \hfill \Box

Lemma 3. Let $K$ be any increasing function where for some $\gamma > 0$ and any $t > t'$ we have $K(t) - K(t') \geq \gamma(t - t')$. Then $K^{-1}(\cdot)$ is uniformly continuous.

Proof. Consider $k, k'$ where $K(t) = k$ and $K(t') = k'$. Then we have

$$\|k - k'\| > \gamma \|t - t'\|$$
$$\|t - t'\| < \frac{1}{\gamma} \|k - k'\|$$

which implies that $K^{-1}(\cdot)$ is uniformly continuous. \hfill \Box

Our next proves that the limit assignment mapping must be continuous in both $s$ and the distributions of agent actions, $G_i, i = \mathcal{M}, \mathcal{N}$. To this end, let $P'_i (\cdot; G_N, G_M), i = \{\mathcal{M}, \mathcal{N}\}$ and $r \in \{cb, q, ap\}$, denote the assignment mapping generated if the agent actions are distributed as per $(G_N, G_M)$.

Lemma 4. $P'_i (s; G_N, G_M), i = \{\mathcal{M}, \mathcal{N}\}$ and $r \in \{cb, q, ap\}$, is uniformly continuous in $(s, G_N, G_M)$.

Proof. We provide a proof for the admissions preference game since the other systems are special cases of an admissions preference scheme. In the admissions preference game, we have $P_{M}^{ap}(s) = F_p^{-1} \left( (1 - \mu)G_N \left( \tilde{S}(s) \right) + \mu G_M (s) \right)$. Let

$$q(s) = (1 - \mu)G_N \left( \tilde{S}(s) \right) + \mu G_M (s)$$
$$\tilde{q}(s') = (1 - \mu)\tilde{G}_N \left( \tilde{S}(s') \right) + \mu \tilde{G}_M (s')$$
\(G_N\) and \(G_M\) must be uniformly continuous with respect to \(s\) since the slopes of \(\sigma_N\) and \(\sigma_M\) are strictly bounded away from 0, and \(\tilde{S}\) is uniformly continuous from assumption 11. Therefore, for any \(\gamma > 0\) we can choose \(\delta > 0\) sufficiently small that for any \((s, G_N, G_M)\) and \((s', \tilde{G}_N, \tilde{G}_M)\) that satisfy

\[
\|G_N - \tilde{G}_N\| + \|G_M - \tilde{G}_M\| + \|s - s'\| < \delta
\]

we have

\[
\|\tilde{q}(s') - q(s)\| < \|q(s) - (1 - \mu)G_N(\tilde{S}(s')) + \mu G_M(s')\| + \delta
\]

\[
= \|q(s) - (1 - \mu)G_N(\tilde{S}(s)) + \mu G_M(s)\| + \delta + \frac{\gamma}{2} < \gamma
\]

Since \(f_P\) has a lower bound, then \(F_P\) has a lower bound on its slope, so from lemma 3 \(F_P^{-1}\) is uniformly continuous. Therefore for any \(\epsilon > 0\) we can choose \(\delta > 0\) sufficiently small that for any \((s, G_N, G_M)\) and \((s', \tilde{G}_N, \tilde{G}_M)\) that satisfy

\[
\|G_N - \tilde{G}_N\| + \|G_M - \tilde{G}_M\| + \|s - s'\| < \delta
\]

we have

\[
\|P_r'(s; G_N, G_M) - P_r'(s'; \tilde{G}_N, \tilde{G}_M)\| < \epsilon
\]

which establishes our claim.

We use the notation \(\Pi_j'(s, \theta; G_N^{K_N}, G_M^{K_M}, K)\) to refer to the expected utility in the \(K\)-agent game of an agent of type \(\theta\) in demographic group \(j = M, N\) that chooses human capital level \(s\) given \(G_N^{K_N}\) and \(G_M^{K_M}\) and admissions system \(r = cb, q, ap\). Let \(\Pi_j'(s, \theta; G_N^{K_N}, G_M^{K_M})\) refer to the utility received by an agent of type \(\theta\) that chooses human capital level \(s\) given \(G_N^{K_N}\) and \(G_M^{K_M}\) in the limit game. Finally we use our lemmas to prove that the agent utility is continuous, which is the linchpin of our \(\epsilon\)-approximate equilibrium result.

**Lemma 5.** \(\Pi_j'(s, \theta; G_N, G_M)\) is uniformly continuous in \((s, \theta, G_N, G_M) \in S \times \Theta \times \Delta(S) \times \Delta(S)\).

**Proof.** The continuity result follows from lemma 4 and the continuity of \(U\) and \(C\) with respect \((p, s, \theta)\). The uniform equicontinuity comes from the compactness of \(S \times \Theta \times \Delta(S) \times \Delta(S)\). 

We now employ a bit of sleight of hand to prove the final result. In our model we
assume that all ties are broken randomly. For the next result we define an assignment mapping \( \hat{\Pi}_i^f (s; G_N, G_M) \) in the limit game and \( \hat{\Pi}_i^f (s_i, s_{-i}; K) \) that instead assign each of the students in the tie the best school available in the random tie-break. We let the associated payoff functions be

\[
\hat{\Pi}_i^f (s, \theta; G_N^{K'}, G_M^{K'}) = E \left[ U \left( \hat{\Pi}_i^f (s_i, s_{-i}; K), s, \theta \right) - C(s, \theta) \right]
\]

\[
\hat{\Pi}_i^f (s, \theta; G_N^{K'}, G_M^{K'}) = E \left[ U \left( \hat{\Pi}_i^f (s; G_N, G_M, s, \theta) - C(s, \theta) \right) \right]
\]

Note that in cases when \( G_N^{K'}, G_M^{K'} \) are nearly nonatomic, these modified payoff functions will be almost the same as \( \Pi_i^f (s, \theta; G_N^{K'}, G_M^{K'}, K) \) and \( \Pi_i^f (s, \theta; G_N^{K'}, G_M^{K'}) \). We refer to any game that uses \( \hat{\Pi}_i^f (s; G_N, G_M) \) or \( \hat{\Pi}_i^f (s_i, s_{-i}; K) \) as a modified game.

**Lemma 6.** For all \( \varepsilon > 0 \) there exists \( K^* \) such that for all \( K > K^* \) and all \( (s, \theta; G_N^{K'}, G_M^{K'}) \in S \times \Theta \times \Delta_{K'}(S) \times \Delta_{K'}(S) \) we have

\[
\left\| \hat{\Pi}_i^f (s, \theta; G_N^{K'}, G_M^{K'}, K) - \hat{\Pi}_i^f (s, \theta; G_N^{K'}, G_M^{K'}) \right\| < \varepsilon
\]

**Proof.** The minority student assignment mapping in the limit game is

\[
\hat{\Pi}_M^{up} (s; G_N, G_M) = F_P^{-1} \left( (1 - \mu)G_N \left( \tilde{S}(s) \right) + \mu G_M (s) \right)
\]

It will be useful to define \( \hat{G}_N \) and \( \hat{G}_M \) to be the empirical CDFs of the other agents’ actions in the \( K \) agent game, which allows us to write the assignment mapping of the finite game as

\[
\hat{\Pi}_M^{up} (s; G_N, G_M, K) = F_P^{-1} \left( (1 - \mu)G_N \left( \tilde{S}(s) \right) + \mu \frac{K_M - 1}{K_M} G_M (s) + \frac{1}{K} \right)
\]

As \( K \to \infty \) we have

\[
(1 - \mu)G_N \left( \tilde{S}(s) \right) + \mu \frac{K_M - 1}{K_M} G_M (s) + \frac{1}{K} \to (1 - \mu)G_N \left( \tilde{S}(s) \right) + \mu G_M (s)
\]

uniformly over \((s, \theta; G_N^{K'}, G_M^{K'})\). From lemma 3 and the lower bound on the slope of \( F_P \), we know that \( F_P^{-1} \) is uniformly continuous. Together these imply that for any \( \gamma > 0 \) we can choose \( K \) sufficiently large that for all \( s \)

\[
\left\| \hat{\Pi}_M^{up} (s; G_N, G_M) - \hat{\Pi}_M^{up} (s; G_N, G_M, K) \right\| < \gamma
\]

Since \( U \) is uniformly continuous in \( p \), this immediately implies for for any \( \varepsilon > 0 \) we can
choose $K$ sufficiently large that

$$\left\| \tilde{\Pi}_j(s, \theta; G^K_M, G^K_M, K) - \tilde{\Pi}_j(s, \theta; G^K_N, G^K_M) \right\| < \epsilon$$

Finally, we establish the link between the original and modified game.

**Lemma 7.** $(\sigma_{M}^{ap}, \sigma_{N}^{ap})$ is an equilibrium of the modified $K$–agent (limit) game if and only if it is an equilibrium of the original $K$-agent (limit) game.

**Proof.** Since $\tilde{\Pi}_j(s, \theta; G^K_N, G^K_M) = \Pi_j(s, \theta; G^K_N, G^K_M)$, the result is immediate for the limit game. The payoffs in the original and modified game only differ in the event that two agents choose the same HC level. Since this is a measure 0 event, the payoffs are almost surely the same for any choice in the modified and original $K$-agent game. Therefore, $(\sigma_{M}^{ap}, \sigma_{N}^{ap})$ is an equilibrium of the modified $K$–agent game if and only if it is an equilibrium of the original game. □

We now establish our two approximation results in separate lemmas.

**Lemma 8.** If $(\sigma_{M}^{ap}, \sigma_{N}^{ap})$ is an exact equilibrium of the limit game, then for any $\epsilon > 0$ we can choose $K^*$ such that for any $K > K^*$ $(\sigma_{M}^{ap}, \sigma_{N}^{ap})$ is an $\epsilon$–approximate equilibrium of the $K$-agent game.

**Proof.** Given our previous results on uniform convergence and uniform continuity of the modified game, theorem 8 of Bodoh-Creed [6] implies that if $(\sigma_{M}^{ap}, \sigma_{N}^{ap})$ is an exact equilibrium of the modified limit game, then for any $\epsilon > 0$ we can choose $K^*$ such that for any $K > K^*$ $(\sigma_{M}^{ap}, \sigma_{N}^{ap})$ is an $\epsilon$–approximate equilibrium of the modified $K$-agent game. Combined with lemma 7 we have our result. □

**Proof.** Our next result deals with sequences of equilibrium strategies. We use the notation $(\sigma_{M}^{ap}(K), \sigma_{N}^{ap}(K))$ to denote an exact equilibrium of the limit game.

**Lemma 9.** Consider any sequence $\{(\sigma_{M}^{ap}(K), \sigma_{N}^{ap}(K))\}_{K=2}^{\infty}$ such that $(\sigma_{M}^{ap}(K), \sigma_{N}^{ap}(K)) \rightarrow (\sigma_{M}^{ap}, \sigma_{N}^{ap})$. Then $(\sigma_{M}^{ap}, \sigma_{N}^{ap})$ is an exact equilibrium of the limit game.

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31One might have thought theorem 6 of Bodoh-Creed [6] would provide this result. Unfortunately, the proof of that theorem required the strategy space to be compact, which is not be the case in our application.
Proof. Let \((G^K_N, G^K_M)\) refer to the pushforward measure generated by \((\sigma^p_M(K), \sigma^p_N(K))\) and the student type distributions. From lemma 2, we have \((G^K_N, G^K_M) \rightarrow (G_N, G_M)\) in the sup-norm. Let \((\tilde{G}^K_N, \tilde{G}^K_M)\) denote an empirical measure defined by random draws from \((G^K_N, G^K_M)\) in the \(K\)-agent game. From theorem 8 for any \(\rho, \delta > 0\) we can choose \(K\) sufficiently large that with probability at least \(1 - \rho\)

\[
\|G^K_N - \tilde{G}^K_N\| + \|G^K_M - \tilde{G}^K_M\| < \delta
\]

Suppose \((\sigma^p_M, \sigma^p_N)\) is not an equilibrium of the modified limit game. For concreteness, let us assume that minority students of some type \(\theta\) where \(\sigma^p_M(\theta) = s^*\) have a profitable deviation. Formally that means there exists \(s'\) and \(\epsilon > 0\) such that

\[
\Pi_M(s^*, \theta; G_N, G_M) + \epsilon < \Pi_M(s', \theta; G_N, G_M)
\]

We now translate this into a statement regarding payoffs in the \(K\)-agent game. For sufficiently large \(K\) we have with probability at least \(1 - \rho\)

\[
\begin{align*}
\Pi_M(s^*, \theta; \tilde{G}^K_N, \tilde{G}^K_M) + \epsilon &< \Pi_M(s', \theta; \tilde{G}^K_N, \tilde{G}^K_M) \\
\Pi_M(\sigma^p_M(K), \theta; \tilde{G}^K_N, \tilde{G}^K_M) + \frac{\epsilon}{2} &< \Pi_M(s', \theta; \tilde{G}^K_N, \tilde{G}^K_M) \\
\Pi_M(\sigma^p_M(K), \theta; G^K_N, G^K_M, K) &< \Pi_M(s', \theta; \tilde{G}^K_N, \tilde{G}^K_M)
\end{align*}
\]

The first line follows from our continuity result (lemma 5). The second line follows from our continuity result (with respect to action) and the fact that \(\sigma^p_M(K) \rightarrow s^*\) as \(K \rightarrow \infty\). The third line follows from our uniform convergence result (lemma 6). The third line implies that \(\sigma^p_M(K)\) is in the \(K\)-agent game with probability at least \(1 - \rho\). Since the utility function is bounded, this implies that for \(\rho\) sufficiently small \(s'\) represents a profitable deviation for an agent of type \(\theta\) in the \(K\)-agent game. From this contradiction we conclude that \((\sigma^p_M, \sigma^p_N)\) is an exact equilibrium of the limit game.

Since lemma 9 holds for all such sequences of strategies, we have from Theorem 17.16 of Aliprantis and Border [3] that the equilibrium correspondence is upper hemicontinuous.

Our claim regarding \(\delta\)-approximate equilibria could still fail if there were equilibria of the limit game that were not close to any equilibrium of arbitrarily large finite game. In other words, the argument might fail if the equilibrium correspondence were not lower hemicontinuous in \(K\). Since there is a unique equilibrium of the limit game, it must be the case that the sequence \(\{\sigma^K : \Theta \rightarrow \Delta(S)\}_{K=1}^\infty\) converges in the sup-norm to the unique equilibrium of the limit game.

\[ \Box \]
A.2 Remaining Proofs

The remaining results in the paper are produced roughly in the order they appear in the main body.

Theorem 1. In the college admissions game $\Gamma(K_M, F_M, K_N, F_N, P, r, U, C, S)$ with $r \in \{cb, q, ap\}$, under assumptions there exists a monotone pure-strategy (but potentially not group-wise symmetric) equilibrium $(\sigma^r_M(\theta), \sigma^r_N(\theta))$. Moreover, any equilibrium of the game must be strictly monotone with almost every type using pure strategies.

Proof. Existence and weak monotonicity is a straightforward application of Athey [1, Theorem 3] who establishes these conditions in a general class of auction-related games to which our model belongs. Strict monotonicity follows from lemma.

Now we prove that all of the equilibria must be increasing and that almost all types must use pure strategies. First recall that $E[P(s, s_{-i})]$, although endogenous, must be increasing in $s_i$. Let the maximizers of the problem faced by an agent of type $\theta_i$ be

$$ C(\theta_i) = \arg\max_s E[U(P_j(s, s_{-i}), s, \theta_i)] - C(s, \theta_i), j \in \{M, N\} $$

The super modularity of the objective function in $(s, -\theta_i)$ and theorem 1 of Edlin and Shannon [19] implies that for $\theta > \theta'$, $s \in C(\theta)$, and $s' \in C(\theta')$ we have $s > s'$. Since in equilibrium $\sigma_j(\theta_i) \in C(\theta)$ we must have $\sigma_j(\theta) > \sigma_j(\theta')$. Since the strategy is strictly increasing and constrained to lie within the compact set $S$, then for each $\sigma_j, j \in \{M, N\}$, there can only be a countable set of points $\Theta_D = \{ \theta : \lim_{t \to \theta^+} \sigma_j(t) < \lim_{t \to \theta^+} \sigma_j(t) \}$. Since such a discontinuity is required for a type to mix in equilibrium and the set $\Theta_D$ is countable (and hence measure 0 with respect to $F_K(\theta)$), we have that almost all types must use a pure strategy in any equilibrium.

Theorem 2. HC investment for all types in the color-blind admissions scheme exceeds that in the first-best outcome.

Proof. Let $\sigma^{cb}$ be the equilibrium strategy under color-blind admissions and $\sigma^{FB}$ be the ex post investment strategy in the first-best benchmark. The boundary condition for both problems is the same:

$$ \sigma^{cb}(\theta) = \sigma^{FB}(\theta) = \bar{s} $$

Since the first order condition must hold at $\theta = \bar{\theta}$ in the color-blind scheme, we have

$$ \frac{\partial U}{\partial p} (P(\bar{s}, \bar{s}, \bar{\theta}), \bar{s}, \bar{\theta}) \cdot \frac{dP(\bar{s})}{ds} + \frac{\partial U}{\partial s} (P(\bar{s}, \bar{s}, \bar{\theta}), \bar{s}, \bar{\theta}) \frac{dC(s; \bar{\theta})}{ds} \geq \frac{\partial U}{\partial s} (P(\bar{s}, \bar{s}, \bar{\theta}), \bar{s}, \bar{\theta}) \frac{ds}{ds} $$
This can only hold if \( \frac{dP(s)}{ds} = 0 \), which requires \( \frac{\partial \sigma_{cb}(\theta)}{\partial \theta} \to -\infty \) as \( \theta \to \overline{\theta} \), or if \( \frac{\partial \sigma_{FB}(\theta)}{\partial \theta} = 0 \).

In either case, we know for an interval of \( \theta \) in the neighborhood of \( \overline{\theta} \) that \( \sigma_{cb}(\theta) \geq \sigma_{FB}(\theta) \) where the inequality is strict for \( \theta \neq \overline{\theta} \) in that neighborhood.

Now assume that \( \sigma_{cb}(\theta^*) = \sigma_{FB}(\theta^*) \) for some \( \theta < \overline{\theta} \). In this case, we have, as per the argument above, that either \( \frac{dP(s)}{ds} = 0 \) or \( \frac{\partial \sigma_{FB}(\theta)}{\partial \theta} = 0 \), which implies \( \sigma_{cb}(\theta) > \sigma_{FL}(\theta) \) for \( \theta \) sufficiently close to, but less than, \( \theta^* \). Therefore, it cannot be the case that \( \sigma_{cb} \) and \( \sigma_{FB} \) ever cross - \( \sigma_{cb} \) is always weakly greater than \( \sigma_{FB} \).

**Theorem 3.** In any equilibrium almost all of the agents take pure actions and the strategies are essentially group-symmetric.

Finally, an essentially unique Nash equilibrium exists in the limit model in the following cases:

1. The limit game of the admissions preference model with a differential markup function \( \tilde{S} \) that satisfied \( \tilde{S}(\tilde{s}) = \tilde{s} \) and \( F_P \) has full support.

2. The limit game of the quota system with any feasible choice of \( Q_M \) and \( Q_M \) that admit strictly positive PDFs over a connected support.

**Proof.** We begin our analysis with two lemmas that imply we can focus on the group symmetric equilibria in pure strategies defined by the ODEs.

**Lemma 10.** All but a measure 0 set of agents have a unique optimal action in any equilibrium.

**Proof.** First define the following equilibrium value function

\[
V(s, \theta) = U(P^j_i(s), s, \theta) - C(s, \theta), \quad j = q, ap \text{ and } i = M, N
\]

From assumptions 5 and 6 we have that \( V_{s\theta}(s, \theta) > 0 \). From theorem 1 of Edlin and Shannon [19] we have that if \( \theta > \tilde{\theta} \), \( s^* \in \arg \max_{s} V(s, \theta) \), and \( \tilde{s}^* \in \arg \max_{s} V(s, \tilde{\theta}) \) then it must be the case that \( s^* > \tilde{s} \). Since all agents within the same group have the same decision problem, this implies that the actions of the different agent types within a demographic must be ordered in this way.

Note that if an agent is indifferent between two actions \( s, s' \) where \( s > s' \), then it must be that no other type of agent within that demographic group chooses an action in the interval \([s', s]\). Since at most a countable number of such “jumps” can occur within the bounded space \( S \), it must be that only a countable number of types of agent in each group has such an indifference. Finally, since the type distributions are nonatomic, this countable set must have measure 0.

Lemma 10 has two crucial implications. First, for a mixed action to be optimal, it must be that the agent is indifferent between the pure actions over which he or she mixes.
Lemma 10 implies that only a measure 0 set of agents can mix in this fashion. Second, for two agents to use different strategies \( \sigma, \sigma' \) in equilibrium, it must be that there exists a type \( \theta \) where the agent is indifferent between \( \sigma(\theta) \) and \( \sigma'(\theta) \). Since this can happen only for a measure 0 set of agents, then the strategies used by the agents in each demographic group must be the same almost everywhere.

First consider the admissions preference case and let \( \sigma_i^{ap} \) be defined by equation 9 and is therefore continuous. Standard results on differential equations imply that \( \sigma_i^{ap} \) is uniquely defined. First note that equation 9 can be treated a direct mechanism wherein the agent’s problem is to choose a declared type \( \widehat{\theta} \) where \( i = \mathcal{M}, \mathcal{N} \)

\[
\max_{\widehat{\theta}} U_i(p_i^{ap}(\widehat{\theta}), \sigma_i^{ap}(\widehat{\theta}), \theta) - C(\sigma_i^{ap}(\widehat{\theta}), \theta)
\]

If one computes the first order conditions with respect to \( \widehat{\theta} \) and recognizes that in equilibrium \( p_i^{ap}(\widehat{\theta}) = p_i^{ap}(\sigma_i^{ap}(\widehat{\theta})) \), we find that the local incentive compatibility conditions are already “built in” to the differential equations. It remains to prove that local incentive compatibility implies global incentive compatibility. Suppose \( \overline{\theta} < \theta \) and let \( p_i^{ap}(\overline{\theta}) = \overline{p} > p = p_i^{ap}(\theta) \) and \( \sigma_i^{ap}(\overline{\theta}) = \overline{s} > s = \sigma_i^{ap}(\theta) \). We then have from local incentive compatibility that

\[
U_i(p, s, \theta) \frac{d p_i^{ap}(\overline{\theta})}{d \overline{\theta}} \bigg|_{\overline{\theta} = \theta} + U_s(p, s, \theta) \frac{d \sigma_i^{ap}(\overline{\theta})}{d \overline{\theta}} \bigg|_{\overline{\theta} = \theta} = C_s(s, \theta) \frac{d \sigma_i^{ap}(\overline{\theta})}{d \overline{\theta}} \bigg|_{\overline{\theta} = \theta}
\]

Since \( \overline{\theta} < \theta \), we have from assumption 5 that \( U_i(p, s, \overline{\theta}) > U_i(p, s, \theta) \), and from assumption 6 that \( U_s(p, s, \overline{\theta}) > U_s(p, s, \theta) \) and \( C_s(s, \theta) < C_s(s, \overline{\theta}) \). The value of the first order condition that results if type \( \overline{\theta} \) deviated from truthfulness upwards by declaring \( \widehat{\theta} = \theta \) is

\[
U_i(p, s, \overline{\theta}) \frac{d p_i^{ap}(\overline{\theta})}{d \overline{\theta}} \bigg|_{\overline{\theta} = \theta} + U_s(p, s, \overline{\theta}) \frac{d \sigma_i^{ap}(\overline{\theta})}{d \overline{\theta}} \bigg|_{\overline{\theta} = \theta} > C_s(s, \overline{\theta}) \frac{d \sigma_i^{ap}(\overline{\theta})}{d \overline{\theta}} \bigg|_{\overline{\theta} = \theta}
\]

which, since there is an inequality, implies declaring \( \widehat{\theta} = \theta \) cannot be optimal if the true type is \( \overline{\theta} \). Similar arguments imply that deviating downward from truthfulness also cannot be optimal.

As a final step we must rule out cases where it might be optimal for the agent to choose a human capital level outside of the range of \( \sigma_i^{ap} \). Since \( \sigma_i^{ap} \) is continuous, this implies that the deviation satisfies \( \overline{s} > s = \sigma_i^{ap}(\overline{\theta}) \). Let \( p = p_i^{ap}(\overline{\theta}) \). We know from our
first order conditions that

\[ U_p(p, s, \theta) \left. \frac{dP_{ii}(\theta)}{d\theta} \right|_{\hat{\theta} = \theta} + U_s(p, s, \theta) \left. \frac{d\sigma_{ii}(\hat{\theta})}{d\theta} \right|_{\hat{\theta} = \theta} = C_s(s, \theta) \left. \frac{d\sigma_{ii}(\hat{\theta})}{d\theta} \right|_{\hat{\theta} = \theta} \]

Increasing from \( s \) to \( \tilde{s} \) does not change the school \( p \), and so it can only be optimal if

\[ U_s(p, \tilde{s}, \theta) = C_s(\tilde{s}, \theta) \]

(22)

Since \( U_{ss} < 0 \), \( C_{ss} > 0 \), and \( U_s(p, s, \theta) \leq C_s(s, \theta) \), equation (22) cannot hold, which means deviating to \( \tilde{s} \) cannot be optimal.

Now consider case 2 and assume that \( \sigma_i^q \) is defined by equation (11) over the intervals where \( Q_i \) has support. Again, by standard results in differential equations, if \([p, p']\) is an interval where \( Q_i \) has support and type \( \theta \) is assigned to college \( p \) with human capital level \( \sigma_i^q(\theta) = s \), then the strategy is uniquely defined by equation (11) for all types that are in equilibrium assigned a seat at a college in \([p, p']\). The only way there could exist two equilibria \( \sigma_i^q \) and \( \tilde{\sigma}_i^q(\theta) \) is if there is some discontinuity at \( \theta \) shared by both strategies such that

\[ \text{Lim}_{\epsilon \to 0^+} \sigma_i^q(\theta + \epsilon) \neq \text{Lim}_{\epsilon \to 0^+} \tilde{\sigma}_i^q(\theta + \epsilon) \]

In other words, the jump over an interval in which \( Q_i \) lacks support is not uniquely defined. We prove that the first such jump in the strategy must be uniquely defined. An (omitted) induction step using essentially the same argument can be used to prove that all of the jumps must be uniquely defined.

Suppose that \( Q_i \) lacks support over the interval \( [p, p'] \) and \( Q_i \) has full support over both \([p, p] \) and \([p', p' + \delta] \) for some \( \delta > 0 \). Let \( \theta \) satisfy \( P_i^q(\sigma_i^q(\theta)) = p \) - in other words, \( \theta \) is where the first jump must occur. In equilibrium it must be the case that \( \theta \) is indifferent about whether to make this jump, so

\[ U(p, s, \theta) - C(\sigma_i^q(\theta), \theta) = U(p', s', \theta) - C(s', \theta) \]

where \( s = \sigma_i^q(\theta) \) and \( s' = \text{Lim}_{\epsilon \to 0^+} \sigma_i^q(\theta + \epsilon) \). Suppose there was a second equilibrium \( \tilde{\sigma}_i^q \) starting at type \( \theta \) such that \( s'' = \text{Lim}_{\epsilon \to 0^+} \tilde{\sigma}_i^q(\theta + \epsilon) > s' \). Then it must be the case that

\[ U(p, s, \theta) - C(\sigma_i^q(\theta), \theta) = U(p', s', \theta) - C(s', \theta) = U(p', s'', \theta) - C(s'', \theta) \]

But since \( U_{ss} \) is strictly concave and \( C_{ss} \) is strictly convex, this cannot be true. Therefore

\[ \text{We have written the first order condition with the derivatives defined using limits from the right (i.e., using sequences contained in } \Theta). \]

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Theorem 5. Consider some $P_i(s) : S \rightarrow P, i \in \mathcal{M}, \mathcal{N}$. $P_i(s) : S \rightarrow P, i \in \mathcal{M}, \mathcal{N}$ is the result of an equilibrium of some quota system if and only if there is an equilibrium of an admissions preference system that also yields these assignment functions and admits the same equilibrium strategies.

Proof. Suppose $P_i^q : S \rightarrow P, i \in \mathcal{M}, \mathcal{N}$, is the result of an equilibrium of some quota system and denote the equilibrium strategies $\sigma_i^q : \Theta \rightarrow S$. Since $P_i^q$ and $\sigma_i^q$ are strictly monotone, the functions are invertible. Let $P_{ap}(s)$, the assignment function under admissions preferences, be $P_{ap}(s) = P_i^q \big|_{\mathcal{N}}$. Since the assignment functions are the same for the nonminority students, $P_{ap}$ and $P_i^q \big|_{\mathcal{N}}$ generate identical decision problems for the nonminorities. Therefore, if $\sigma_i^q \big|_{\mathcal{N}}$ was an equilibrium for nonminority students under $P_i^q \big|_{\mathcal{N}}$, then $\sigma_{ap} \big|_{\mathcal{N}}(\theta) = \sigma_i^q \big|_{\mathcal{N}}(\theta)$ will be an equilibrium for the nonminority students under $P_{ap}(s)$.

To construct the outcome equivalent score function, let

$$
\tilde{S}(s) = (P_i^q \big|_{\mathcal{N}})^{-1}(P_M(s))
$$

A minority student who chooses human capital level $s$ will then be assigned to college

$$
P_{ap}(\tilde{S}(s)) = P_i^q \big|_{\mathcal{N}}(\tilde{S}(s)) = P_i^q \big|_{\mathcal{N}}((P_i^q \big|_{\mathcal{N}})^{-1}(P_M(s))) = P_M(s)
$$

Since the assignment functions are the same for the minority students, $P_{ap}$ and $P_i^q \big|_{\mathcal{N}}$ generate identical decision problems for the minorities. Therefore, if $\sigma_{ap} \big|_{\mathcal{N}}$ was an equilibrium for minority students under $P_i^q \big|_{\mathcal{N}}$, then $\sigma_{ap} \big|_{\mathcal{N}}(\theta) = \sigma_{ap} \big|_{\mathcal{N}}(\theta)$ will be an equilibrium for the nonminority students under $P_{ap}(s)$.

Now suppose $P_{ap} : S \rightarrow P$ with score function $\tilde{S}$ is the result of an equilibrium of some admissions preference system and denote the equilibrium strategies $\sigma_{ap} : \Theta \rightarrow S, i \in \mathcal{M}, \mathcal{N}$. To define the equivalent quota system, we need to define allocations of seats to each group. Let these distributions be denoted $Q_i, i \in \mathcal{M}, \mathcal{N}$, and define them as

For all $p$ let $Q_M(p) = 1 - F_i \left[ \psi_i^{ap} \left( \tilde{S}^{-1} \left( (P_{ap})^{-1}(p) \right) \right) \right]$

For all $p$ let $Q_N(p) = 1 - F_i \left[ \psi_i^{ap} \left( (P_{ap})^{-1}(p) \right) \right]$

Note that the total measure of nonminority students choosing $s$ and minority students choosing $\tilde{S}^{-1}(s)$ under $P_{ap}$ (in equilibrium is)

$$
1 - \mu F_M \left[ \psi_M^{ap} \left( \tilde{S}^{-1} \left( (P_{ap})^{-1}(p) \right) \right) \right] - (1 - \mu) F_N \left[ \psi_i^{ap} \left( (P_{ap})^{-1}(p) \right) \right] = 
\mu Q_M(p) + (1 - \mu) Q_N(p) = F_P(p)
$$

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which implies $Q_M$ and $Q_N$ are feasible quotas. Written more formally, $P_N(s) = P^{ap}(s)$ and $P_M(s) = P^{ap}(S(s))$. As argued above, since the decision problems for the agents are the same, the equilibrium strategies in the original admissions preference scheme and the constructed quota are the same.

B Endogenous School Quality

In this section we provide an analysis of a version of our model that includes a stylized model of endogenous school quality, and our goal is to repeat the calculations of section 7. The precise numbers we provide in this section are probably not comparable to real-world outcomes since the model of school quality we have chosen is ad hoc. However, the differences between the results presented in this section and those provided in section 7 help illustrate important features of the analysis that may change when endogenous school quality is taken into account.

The model we employ is identical to the calibrated model used in section 7, except that instead of assuming that the distribution of college qualities is exogenous and equal to the distribution estimated by Hickman [36], we assume that the “effective” school quality is a function of the types of agents assigned to the school. Suppose in equilibrium that school $p$ has a fraction $u(p)$ of it’s student body drawn from the minority student population and that the type of minority and nonminority student assigned to school $p$ are $\theta_M(p)$ and $\theta_N(p)$. The effective school quality, $p_E$, is then equal to

$$p_E = F^{-1}_P \left[ \mu (1 - F_K(\theta_M(p))) + (1 - \mu) (1 - F_K(\theta_N(p))) \right]$$

(23)

where $F_P$ refers to the estimate of the distribution of school qualities provided by Hickman [36]. To decompose this, first note that

$$\mu (1 - F_K(\theta_M(p))) + (1 - \mu) (1 - F_K(\theta_N(p)))$$

represents the average cost-type distribution quantile of the students assigned to school $p$, which we take to represent the average ability of the students assigned to school $p$. We then assign school $p$ the effective quality, $p_E$, of the school of the same quantile in distribution $F_P$.

33Since agent HC level is a strictly monotone function of the agent’s type in equilibrium, we could also treat this as a model of spillovers in agent HC. One might wonder whether the externalities imposed on other agents through the effect of HC choice on school quality introduce new strategic factors to consider. In the limit game these effects are not present. We conjecture that as long as the school quality is continuous function of agent HC, the effects will vanish as the economy grows.
quality. If the assignment of students to schools is perfectly assortative, as is the case for the first-best and color-blind assignments, then equation 23 predicts an effective school quality distribution equal to $F_p$. Since the assignments of students to schools in the real-world (as estimated by Hickman [36]) had only small deviations from assortativity, our model would predict almost the same distribution of school qualities as we observe in reality. This suggests that our model is consistent with the real-world. Unfortunately, without observing how significant variations in the assignment influence school quality, we cannot say much about how well our model captures the effect of large deviations from the status-quo assignment observed in the real-world.

Using our model of spill-overs we recompute the assignments used in section 7 to compute the benefits of assortativity, the cost of competition, and the second-best outcome. Note that, as expected, the perfectly assortative first-best and color-blind assignments yield the same welfare since the distribution of effective school qualities is identical to $F_p$.

<table>
<thead>
<tr>
<th></th>
<th>Social Surplus</th>
</tr>
</thead>
<tbody>
<tr>
<td>First Best</td>
<td>$15,415</td>
</tr>
<tr>
<td>Second Best</td>
<td>$14,556</td>
</tr>
<tr>
<td>Color-Blind</td>
<td>$14,262</td>
</tr>
<tr>
<td>Random Assignment</td>
<td>$13,959</td>
</tr>
</tbody>
</table>

When we use our comparative statics to compute the benefits of assortativity and the cost of competition, we find

<table>
<thead>
<tr>
<th></th>
<th>Social Surplus</th>
</tr>
</thead>
<tbody>
<tr>
<td>Benefits of Assortativity</td>
<td>$1,456</td>
</tr>
<tr>
<td>Cost of Competition</td>
<td>-$1,153</td>
</tr>
</tbody>
</table>

The cost of competition is unchanged since the assignment and school quality in the first-best and color-blind assignment are the same in both the endogenous and exogenous school quality models. However, the benefits to assortativity are 77% higher when school quality is endogenous. In summary, the benefits of assortativity are about 26% higher than the costs of competition when school quality is endogenous, whereas in the case of exogenous school quality we found that the cost of competition eliminated all of the benefits of assortativity.

To see why the benefits of assortativity are higher when school quality is endogenous, let us decompose the source of the assortativity benefits into two pieces. First, for assortativity to matter there must be increasing returns to assigning high HC students (i.e., low cost students) to high quality schools. Second, there must be some variation
in school quality so that there are higher (lower) quality schools to which high (low) HC
students can be assigned. When school quality is exogenous (as in section [7]), we only
lose the first component of the benefit - some low cost students still get assigned to high
quality schools. However, when school quality is endogenous, we lose all of the benefits
of assortativity - since the colleges all have the same median quality ($p_{MED} = 0.5630$), we
reap no complementarities between student type/HC level and school quality.

C Smoothing the Model

In this section we present a modification of the model that smoothes the discontinuities in
the equilibrium strategies and insures that the equilibria of the limit quota model always
approximate the equilibria of the finite game (assuming the quota is feasible). The basic
intuition is that our approximation can fail when small changes in the students’s HC can
cause large changes in his or her assignment. This can occur because of gaps in the
support of the college seats the students are competing for. When there is a gap in the
support, the limit game equilibrium will involve discontinuities. As the HC starts to
exceed the HC choice of the type assigned to the school on the left of the gap, marginal
increases in HC do not change the assignment. Once the HC level finally surpasses the
level chosen by the type at the right of the gap, the assigned discretely jumps in quality
to the school at the right of the gap.

The random distribution of colleges is denoted with the CDF $F(p; \omega)$ where $\omega$ is an
element of the probability space $(\Omega, B(\Omega))$ with underlying measure $\nu$. This distribution
represents the final quality of the colleges to which a student can be assigned, which we
assume is unknown at the time that the HC is being accumulated. This lack of knowledge
could reflect actual changes in the quality of the schools or the innate uncertainty of the
students that is only resolved at the time of college application, the point at which most
students begin relatively detailed research into potential colleges.

We let $Q_j(\rho; \omega), j \in \{M, N\}$, denote the quota given state $\omega$, while $Q_j(\rho)$ denotes the
random CDF of the quota. If we let the empirical distribution of seats in the $K$-agent
game allocated to group $j$ in stated of the world $\omega$ be denoted $Q^K_j(\rho; \omega)$, then we require
that the measure of seats assigned to each group converge uniformly over $p$ and $\omega$.

**Assumption 12.** $\lim_{K \to \infty} \sup_{p, \omega} \left\| Q_j(p; \omega) - Q^K_j(p; \omega) \right\| \to 0$

In addition to the convergence assumption, we require that (random) assignment of
students to schools is continuous.

**Assumption 13.** $Q_j^{-1}(r), j \in \{M, N\}$, is continuous in $r$ under the weak-* topology.
This assumption is (for example) fully compatible with the measures $Q_j(c; \omega)$ admitting disjoint supports, but the edges of the support cannot vary too much and the distributions over each interval of the support cannot radically change at many points. As we show below, the continuity of the assignment mapping implies that there are no discontinuities in the equilibrium strategy.

The ODE describing the equilibrium strategies becomes

$$\frac{d\sigma_j^q(\theta)}{d\theta} = \frac{\frac{\partial}{\partial \theta} E \left[ U \left( Q_j^{-1}(1 - F_j(\bar{\theta})), \sigma_j^q(\theta), \theta \right) \right]_{\bar{\theta} = \theta}}{
\left. \frac{\partial}{\partial \theta} E \left[ U_s \left( Q_j^{-1}(1 - F_j(\theta)), \sigma_j^q(\theta), \theta \right) \right] \right|_{\theta = \theta} - E \left[ U_p \left( Q_j^{-1}(1 - F_j(\theta)), \sigma_j^q(\theta), \theta \right) \right]} \tag{24}$$

$$\sigma_j^q(\theta) = s \quad (\text{boundary condition}).$$

Although we have stated the ODE governing the equilibrium relatively abstractly, there are a number of conditions under which a solution exists. For example, if $Q_j(p; \circ)$ is almost everywhere continuous for all $p$, then assumption 13 holds. If we further assume that $Q_j(c; \omega)$ admits a PDF $f_{Q_j}(c; \omega)$ and there exists $\zeta > 0$ such that $f_{Q_j}(c; \omega) > \zeta$ and $f_{Q_j}(c; \omega)$ is continuous over the support of $f_{Q_j}(c; \omega)$ for all $\omega$, then

$$\frac{\partial}{\partial \theta} E \left[ U \left( Q_j^{-1}(1 - F_j(\bar{\theta})), \sigma_j^q(\theta), \theta \right) \right]_{\bar{\theta} = \theta} = -E \left[ U_p \left( Q_j^{-1}(1 - F_j(\theta)), \sigma_j^q(\theta), \theta \right) \right]$$

will be continuous, which implies that the solution to equation 24 is well-defined and unique.

The following proposition is proven by arguing that the agent utility is continuous as a function of $s$ in the limit game, which then allows us to use the bulk of the argument in the proof of theorem 5 to prove our claims.

**Proposition 1.** Assume that a solution to equation 24 exists and assumption 13 holds. Then for any $\epsilon, \delta > 0$ there exists $K^*$ such that $\sigma_j^q$ is an $\epsilon-$approximate equilibrium of the $K$-agent game and $\sigma_j^q$ is a $\delta-$approximate equilibrium of the $K$-agent game.

**Proof.** First we argue that there are no jumps in $\sigma_j^q$. By contradiction, suppose there exists $\theta$ such that $\lim_{\gamma \to 0^+} \sigma_j^q(\theta - \gamma) = s < \lim_{\gamma \to 0^+} \sigma_j^q(\theta + \gamma) = s'$. From the convexity of $C$ with respect to $s$ and concavity of $U$ with respect to $s$, we have $U(Q_j^{-1}(1 - F_j(\theta)), s, \theta) - C(s, \theta) > U(Q_j^{-1}(1 - F_j(\theta)), s', \theta) - C(s', \theta)$. But for the discontinuity in $\sigma_j^q$ to be an equilibrium, the type $\theta$ must be indifferent between $s$ and $s'$, which provides our contradiction.

Given that the strategy is strictly increasing and continuous, we immediately have that
the assignment mapping $P^q_j(s)$ is uniformly continuous in the weak-* topology, which implies that $E \left[ U(P^q_j(s), s, \theta) \right] - C(s, \theta)$ is uniformly continuous in $s$. It is a matter of slight changes in notation (e.g., using expected utility as opposed to utility) to modify lemmas 6, 7, 8, and 9 to the setting with uncertainty, which yields the $\varepsilon-$approximate equilibrium claim. The uniqueness of the equilibria of the limit game then implies our $\delta-$approximate equilibrium.

D Identifying Discontinuities in the Equilibrium

In this appendix we briefly describe how to identify discontinuities in the equilibrium of the limit game. This section will be of practical interest primarily to practitioners who wish to use equations 11 and 9 to compute equilibria numerically.

First let us consider quota schemes, where jumps are caused by gaps in the support of $Q_j$. The size of the jump must make the types on the edge of the gap indifferent about making the jump. Formally written, suppose $(p_L, p_U)$ is an interval such that $Q_j([p_L, p_U]) = 0$ and for all $\varepsilon > 0$ we have $Q_j([p_L - \varepsilon, p_U]) > 0$ and $Q_j([p_L, p_U + \varepsilon]) > 0$. Let $\theta$ be such that $P(\sigma^q_j(\theta)) = p_L$. Then it must be that for $s = \lim_{\varepsilon \to 0^+} \sigma^q_j(\theta + \varepsilon)$ (i.e., the human capital choice on the other side of the jump) we have

$$U(p_L, \sigma^q_j(\theta), \theta) - C(\sigma^q_j(\theta), \theta) = U(p_U, s, \theta) - C(s, \theta)$$

which identifies $s$.

Now we discuss how to identify gaps in an admissions preference scheme, which are caused by kinks or discontinuities in $\tilde{S}$. When these issues arise, the marginal incentives for both groups will change. We handle each possible issue in turn.

First consider a kink in $\tilde{S}$ at $s$ such that $\frac{d}{ds} \tilde{S}$ jumps at $s$. Without loss of generality, assume that the strategies are lower semicontinuous and let $\theta_N = \psi^ap_N(s)$ and $\theta_M = \psi^ap_M(\tilde{S}^{-1}(s))$, and consider the first order conditions that would have to hold at $s$ if the strategies are continuous

For $i = \mathcal{N}$,

$$\mathcal{N}, U_p(P^{ap}(s'), s', \theta_N) \frac{d P^{ap}(s)}{ds} \bigg|_{s = \tilde{S}(s')} + U_s(P^{ap}(s'), s', \theta) = C_s(s', \theta)$$

For $i = \mathcal{M}$,

$$\mathcal{M}, U_p(P^{ap}(\tilde{S}(s')), s', \theta) \frac{d P^{ap}(s)}{ds} \bigg|_{s = \tilde{S}(s')} + U_s(P^{ap}(\tilde{S}(s')), s, \theta) = C_s(s', \theta)$$

\[34\text{See the proof of theorem 5 for the details of the argument.}\]
\[35\text{Since the edges of these jumps are described by an indifference condition, we could just as easily construct an upper semicontinuous equilibrium.}\]
For both group’s strategies to be continuous, we would need the first order conditions across the discontinuity in \( \frac{d\overline{S}}{ds} \) to be continuous. In other words, we would require

\[
\frac{dP^p(s)}{ds} = \frac{dP^p(s)\, d\overline{S}(s)}{ds}
\]

which is clearly impossible at the discontinuity in \( \frac{d\overline{S}(s)}{ds} \).

To resolve this problem, one of the groups must jump. We will construct an equilibrium where the minority students jump, but the construction and the test for the validity of the construction is symmetric in the case where the nonminority students jump. If the minority student strategy exhibits a jump, then it must be the case that the first order condition for the nonminority students is smooth across the discontinuity, so \( \frac{dP^p(s)}{ds} \) must be continuous. Although convoluted, we write the equation for \( \frac{dP^p(s)}{ds} \) below for clarity:

\[
\frac{dP^p(s)}{ds} = -\frac{\Phi}{f_p(1-\Phi)} \text{ where } \Phi = \frac{(1-\mu)f_N(\psi^p_N(s))}{(\sigma_{\text{ap}}^N)'(s)} + \frac{hf_M(\psi^p_M(\overline{S}^{-1}(s)))}{(\sigma_{\text{ap}}^M)'(s)\overline{S}'(s)}
\]

and we let \((\sigma_{i}^p)'\) be infinity if group \(i\) has jumped across that level of human capital. In other words, if the minority students jump, it must be that \((\sigma_{N}^p)'(s)\) drops discontinuously to keep \( \frac{dP^p(s)}{ds} \) constant at \(s\). For the duration of the minority student jump, we can use the differential equations \ref{eq:nonminority} to describe the nonminority student strategy. This construction is successful if in the gap we have

\[
U_p(P^p(\overline{S}(s)), s, \theta) \frac{dP^p(s)}{ds} \frac{d\overline{S}(s)}{ds} + U_s(P^p(\overline{S}(s)), s, \theta) \geq C_s(s, \theta)
\]

If the inequality is reversed, then it must be the case that nonminority student jump and minority students do not. Finally, we need to define the size of the jump in the minority student strategy. Suppose the minority strategy is lower semicontinuous and we have \(\sigma_{\text{ap}}^{M}(\theta) = s\) (i.e., \(\theta\) is the type of minority student that jumps). To define the jump we need to compute \(s'\) such that

\[
U(P^p(\overline{S}(s)), s, \theta) - C(s, \theta) = U(P^p(\overline{S}(s')), s', \theta) - C(s', \theta)
\]

and let the minority strategy jump to \(s'\) at \(\theta\). Again, it will often be the case that the first order conditions for the two groups will not align and

\[
\frac{dP^p(s')}{ds} \neq \frac{dP^p(s')}{{ds}} \frac{d\overline{S}(s')}{{ds}}
\]
If this occurs, it is treated as noted above. It is easy to construct examples where the groups repeatedly jump and never compete at the same college. For example, if $\tilde{S}(s) = s + \Delta, \Delta > 0$, the equilibrium has this structure.

Second, assume that there is a discontinuity in $\tilde{S}$ at human capital level $s$. Since we have assumed $\tilde{S}$ is increasing, $\tilde{S}$ must jump upwards. Let $\sigma_j^q(\theta) = s$ and assume $\sigma_j^q$ is lower semicontinuous at $\theta$. In this case, there must be a jump in the equilibrium strategy of the minority students that is defined by

$$U(P^{ap}(\tilde{S}(s)), s, \theta) - C(s, \theta) = U(P^{ap}(\tilde{S}(s')), s', \theta) - C(s', \theta)$$

and we let the value of the minority student strategy jump to $s'$ at $\theta$. Again, if the first order conditions cannot both line up, then we will have to allow one of the groups’ strategies to jump again, which requires the construction techniques outlined above.

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36 The construction would be essentially the same if we chose to let $\sigma_j^q$ be upper semicontinuous at $\theta$. 70