# KEY CONCEPTS: Introduction to Real Analysis

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## 1 Sets and Functions

PRELIMINARY NOTE: Many definitions given in these notes are framed in terms specific to the real numbers. This simplifies matters greatly because of the familiar ordering and distance concepts which come as standard features in (finite-dimensional) Euclidean space. However, many of the concepts given below have useful analogs in more exotic spaces (*e.g.*, spaces of functions or spaces of infinite sequences), but those require some knowledge of metric topology, which will be a major emphasis of your Economic Analysis I (06E:200) class. For the purpose of math camp, we will resign ourselves to the quaint shackles of Euclidean space.

### 1.1 Sets

**Definition 1** *A* ***set*** *is a collection of elements.*

- If an element $x$ is in a set $A$ then we denote $x \in A$, if $x$ is not in $A$, then we write $x \notin A$
- If every element of a set $A$ is also in set $B$, then $A$ is a subset of $B$ and we denote this as $A \subseteq B$ or $B \supseteq A$.
- $A$ is a **proper subset** of $B$ if there is at least one element in $B$ that is not in $A$. We denote this by $A \subset B$ or $B \supset A$.
- Two **sets are equal** if they contain all the same elements: $A = B$. That is, $A \subseteq B$ and $B \subseteq A$.

- A set is defined by listing its elements or by specifying the property that determines the elements of the set: $\{x \in A : P(x)\}$, where $P$ is some property.

- Examples:
  - Natural numbers: $\mathbb{N} = \{1, 2, 3, ...\}$
  - Integers: $\mathbb{Z} = \{0, 1, -1, 2, -2, ...\}$
  - Rational numbers: $\mathbb{Q} = \{\frac{m}{n} : m, n \in \mathbb{Z} \text{ and } n \neq 0\}$
  - Real numbers: $\mathbb{R}$
  - $S = \{2k : k \in \mathbb{N}\}$

## 1.2 Set operations

- The **union** of sets $A$ and $B$ is the set

$$A \cup B = \{x : x \in A \text{ or } x \in B\}$$

- The **intersection** of the sets $A$ and $B$ is the set

$$A \cap B = \{x : x \in A \text{ and } x \in B\}$$

- The **complement** of $B$ relative to $A$ (or the **difference** between $A$ and $B$) is the set

$$A \backslash B = \{x : x \in A \text{ and } x \notin B\}$$

- A set with no elements is called the **empty set** and is denoted by $\emptyset$.

- Two sets are **disjoint** if they have no elements in common: $A \cap B = \emptyset$.

- **DeMorgan laws**:

$$\begin{aligned} A \backslash (B \cup C) &= (A \backslash B) \cap (A \backslash C) \\ A \backslash (B \cap C) &= (A \backslash B) \cup (A \backslash C) \end{aligned}$$

- For a countable collection of sets $\{A_1, A_2, ..., A_n, ...\}$ :

- ▶ **union** :

$$\cup_{n=1}^{\infty} A_n = \{x : x \in A_n \text{ for some } n \in \mathbb{N}\}$$

- ▶ **intersection**:

$$\cap_{n=1}^{\infty} A_n = \{x : x \in A_n \text{ for all } n \in \mathbb{N}\}$$

- **Cartesian product** of two sets is a set of all ordered pairs:

$$A \times B = \{(a, b) : a \in A, b \in B\}$$

- A **ball** of radius $\varepsilon$ around $x$ in $\mathbb{R}$ is a set defined by $B_\varepsilon(x) = \{y \in \mathbb{R} : |x - y| < \varepsilon\}$

- For a set $A$, a **limit point** of $A$ is a point $l \in \mathbb{R}$ such that for each $\varepsilon > 0$, there exists some $y \in A$ such that $y \in B_\varepsilon(x)$.

## 1.3 Functions and Mappings

- A **function** $f$ from a set $A$ into a set $B$, denoted $f : A \to B$, is a set $f$ of ordered pairs in $A \times B$ such that for each $x \in A$ there exist a unique $b \in B$ with $(a, b) \in f$.

- The set $A$ of first elements of function $f$ is called the **domain** of $f$ and is denoted $D(f)$. The set $B$ of all second elements in $f$ is called the **range** of $f$ and is denoted by $R(f)$. Note that $D(f) = A$, while $R(f) \subseteq B$.

For the following, consider a function $f$, mapping $A$ into $B$.

- Let $E \subseteq A$, then the **direct image** of $E$ under $f$ is the subset $f(E) \subseteq B$ given by

$$f(E) = \{f(x) : x \in E\}$$

  Let $H \subseteq B$, then the **inverse image** of $H$ under $f$ is the subset $f^{-1}(H) \subseteq A$ given by

$$f^{-1}(H) = \{x \in A : f(x) \in H\}$$

- The function $f$ is **injective (one-to-one)** if whenever $x_1 \neq x_2$ then $f(x_1) \neq f(x_2)$.

  The function $f$ is **surjective (onto)** if $f(A) = B$. That is, $R(f) = B$.

  If $f$ is both injective and surjective then it is **bijective**.

- If $f : A \to B$ is a bijection of $A$ onto $B$, then

$$f^{-1} = \{(b, a) \in B \times A : (a, b) \in f\}$$

  is a function mapping $B$ into $A$, called the **inverse** of $f$.

  - IMPORTANT NOTE: inverse images always exist; inverse functions exist ONLY for bijections.

- If $f : A \to B$ and $g : B \to C$, and if $R(f) \subseteq D(g) = B$, then the **composite function** $g \circ f$ is the function from $A$ into $C$ defined by

$$(g \circ f)(x) = g(f(x)) \text{ for all } x \in A$$

**Theorem 1** *Let $f : A \to B$ and $g : B \to C$ be functions and let $H \subseteq C$. Then*

$$(g \circ f)^{-1}(H) = f^{-1}(g^{-1}(H))$$

## 1.4 Finite, infinite, and countable sets

- The empty set $\emptyset$ has zero elements

- If $n \in \mathbb{N}$, a set $A$ has $n$ elements if there exist a bijection from $\mathbb{N}_n = \{1, ..., n\}$ onto $A$.

- A set $A$ is **finite** if it either empty or it has $n$ elements for some $n \in \mathbb{N}$

- A set $A$ is **infinite** if it is not finite.

**Theorem 2** *Suppose that $S$ and $T$ are sets and that $T \subseteq S$.*

1. *if $S$ is a finite set, then $T$ is a finite set*

2. *If $T$ is an infinite set, then $S$ is an infinite set*

- A set $S$ is **denumerable (countably infinite)** if there exists a bijection of $\mathbb{N}$ onto $S$.
- A set $S$ is **countable** if it is either finite or denumerable
- A set $S$ is **uncountable** if it is not countable.

**Theorem 3** *Suppose that $S$ and $T$ are sets and that $T \subseteq S$.*

1. *if $S$ is a countable set, then $T$ is a countable set*
2. *If $T$ is an uncountable set, then $S$ is an uncountable set*

**Exercise 1** *Prove that $\mathbb{Z}$ is a countable set.*

**Exercise 2** *Prove that $\mathbb{Q}$ is a countable set.*

**Exercise 3** *Prove that $\mathbb{R} \backslash \mathbb{Q}$ is an uncountable set (HINT: proceed by contradiction and use the fact that the irrationals are expressed by non–terminating, non–repeating decimals, with comparisons being performed in lexicographic fashion).*

**Theorem 4** *The countable union of countable sets is countable is countable.*

- *A set $S \subseteq T$ is said to be **dense** in $T$ if for any two members of $T$, say $x$ and $y$, there exists $s \in S$ such that $x < s < y$.*

**Exercise 4** *Prove that $\mathbb{Q}$ is dense in itself.*

**Exercise 5** *Prove that $\mathbb{Q}$ is dense in $\mathbb{R}$ (HINT: use the familiar lexicographic representation of real numbers).*

**Exercise 6** *Prove that $\mathbb{R} \setminus \mathbb{Q}$ is dense in $\mathbb{R}$ (HINT: use the familiar lexicographic representation of real numbers).*

## 2 The completeness property of $\mathbb{R}$

*Now we consider the set of real numbers.*

- *Let $S \subseteq \mathbb{R}$ and $S \neq \emptyset$. Then*

  1. *The set $S$ is* ***bounded above*** *if there exist a number $u \in \mathbb{R}$ such that $s \leq u$ for all $s \in S$. Then each such $u$ is an* ***upper bound*** *of $S$.*
  2. *The set $S$ is* ***bounded below*** *if there exists a number $w \in \mathbb{R}$ such that $w \leq s$ for all $s \in S$. Each such $w$ is called a* ***lower bound*** *of $S$.*
  3. *A set $S$ is* ***bounded*** *if it is both bounded above and below. A set is* ***unbounded*** *if it is not bounded.*

- *Let $S \subseteq \mathbb{R}$ and $S \neq \emptyset$.*

  *If $S$ is bounded above, then a number $u$ is a* ***supremum*** *(or a least upper bound, l.u.b.) of $S$ if:*

  1. *$u$ is an upper bound of $S$, and*
  2. *if $v$ is any upper bound of $S$, then $u \leq v$*

  *If $S$ is bounded below, then a number $w$ is an* ***infimum*** *(or a greatest lower bound, g.l.b.) if*

  1. *$w$ is a lower bound of $S$, and*
  2. *if $r$ is any lower bound of $S$, then $r \leq w$.*

*Note that there is only one supremum (infimum) of a given set $S \subseteq \mathbb{R}$. The set $S$ should have upper (lower) bound in order to have the supremum (infimum). It does not have to be an element of the set $S$. Also, it is unique. Notation:*

$$\sup S \text{ and } \inf S$$

**Lemma 1** *An upper bound $u$ of a nonempty set $S \subseteq \mathbb{R}$ is the supremum of $S$ if and only if for every $\varepsilon > 0$ there exist an $s_\varepsilon \in S$ such that $u - \varepsilon < s_\varepsilon$*

**Theorem 5** ***The Completeness Property of*** $\mathbb{R}$***/Supremum Property of*** $\mathbb{R}$. *Every nonempty set of real numbers that has an upper bound also has a supremum in* $\mathbb{R}$.

*Consider the range of the function* $f : D \to \mathbb{R}$.

- *Then* $f$ *is* ***bounded above*** *if the set* $f(D) = \{f(x) : x \in D\}$ *is bounded above in* $\mathbb{R}$. *That is, there exist* $B \in \mathbb{R}$ *such that* $f(x) \leq B$ *for all* $x \in D$.

- *Then* $f$ *is* ***bounded below*** *if the set* $f(D) = \{f(x) : x \in D\}$ *is bounded below in* $\mathbb{R}$.

- *Then* $f$ *is* ***bounded*** *if it is bounded above and below. That is,* $\exists B \in \mathbb{R}$ *such that* $|f(x)| \leq B$ *for* $\forall x \in D$.

*Some properties:*

- *If* $f(x) \leq g(x)$ *for all* $x \in D$, *then*

$$\sup_{x\in D} f(x) \leq \sup_{x\in D} g(x)$$

- *If* $f(x) \leq g(y)$ *for all* $x, y \in D$, *then*

$$\sup_{x\in D} f(x) \leq \inf_{y\in D} g(y)$$

## 2.1 The Archimedean Property

**Theorem 6** ***Archimedean Property*** *If* $x \in \mathbb{R}$ *then there exist* $n_x \in \mathbb{N}$ *such that* $x < n_x$.

**Corollary 1** *If* $S = \left\{\frac{1}{n} \mid n \in \mathbb{N}\right\}$, *then* $\inf S = 0$

**Corollary 2** *If* $t > 0$, *there exists* $n_t \in \mathbb{N}$ *such that* $0 < \frac{1}{n_t} < t$

# 3 Limits of functions

**Definition 2** *Suppose $f : A \to \mathbb{R}$. Let $x \to p$ and $p$ is a limit point of $A$. Then* ***the limit of $f$ as $x$ approaches*** *$p$ is defined as $\lim_{x \to p} f(x) = q$ if such point $q$ exists, and for every $\varepsilon > 0$ there exist $\delta > 0$ such that if $|p - x| < \delta$ then $|f(x) - q| < \varepsilon$.*

**Theorem 7** *Suppose $f, g : A \to \mathbb{R}$ and $\lim_{x \to p} f(x) = q$ and $\lim_{x \to p} g(x) = r$. Then*

1. $\lim_{x \to p} [f(x) + g(x)] = q + r$
2. $\lim_{x \to p} (fg)(x) = qr$
3. $\lim_{x \to p} \left(\frac{f}{g}\right)(x) = \frac{q}{r}$, *if* $r \neq 0$.

**Theorem 8** ***(L'Hopital's rules)*** *Let $-\infty \leq a < b \leq \infty$, and functions $f$ and $g$ are differentiable on $(a, b)$ such that $g'(x) \neq 0$ for all $x \in (a, b)$.*

*Suppose that $\lim_{x \to a^+} f(x) = 0 = \lim_{x \to a^+} g(x)$*

1. a) *if* $\lim_{x \to a^+} \frac{f'(x)}{g'(x)} = L \in \mathbb{R}$, *then* $\lim_{x \to a^+} \frac{f(x)}{g(x)} = L$

   b) *if* $\lim_{x \to a^+} \frac{f'(x)}{g'(x)} = L \in \{-\infty, +\infty\}$, *then* $\lim_{x \to a^+} \frac{f(x)}{g(x)} = L$

*Suppose that $\lim_{x \to a^+} f(x) = \pm\infty = \lim_{x \to a^+} g(x)$*

1. a) *if* $\lim_{x \to a^+} \frac{f'(x)}{g'(x)} = L \in \mathbb{R}$, *then* $\lim_{x \to a^+} \frac{f(x)}{g(x)} = L$

   b) *if* $\lim_{x \to a^+} \frac{f'(x)}{g'(x)} = L \in \{-\infty, +\infty\}$, *then* $\lim_{x \to a^+} \frac{f(x)}{g(x)} = L$

**Exercise 7** *CRRA Preferences: use L'Hopital's rules to show that preferences given by*

$$\frac{c^{1-\gamma} - \gamma}{1 - \gamma}$$

*approach log preferences as $\gamma \to 1$.*

# 4 Sequences

- *A* ***sequence*** *of real numbers is a function* $f : \mathbb{N} \to \mathbb{R}$*. Therefore, the sequence can be denoted by* $f(1), f(2), \ldots$*. Usually we denote the sequence by* $\{x_n\}_{n=1}^{\infty}$*, where* $x_n = f(n)$*.*

$$\begin{aligned} \left\{1, \frac{1}{2}, \frac{1}{3}, \ldots\right\} &= \left\{\frac{1}{n}\right\}_{n=1}^{\infty} \\ \{-1, 1, -1, 1, -1, \ldots\} &= \{(-1)^n\}_{n=1}^{\infty} \\ b_n &= b \text{ for } \forall n \in \mathbb{N} \end{aligned}$$

## 4.1 The Limit of a Sequence

- *A sequence of real numbers* $\{x_n\}$ *is said to* ***converge*** *to point* $x \in \mathbb{R}$*, or* $x$ *is a* ***limit*** *of* $\{x_n\}$ *if for every* $\varepsilon > 0$ *there exist* $N_\varepsilon \in \mathbb{N}$ *such that for all* $n \geq N_\varepsilon$ *we have* $|x_n - x| < \varepsilon$*.*

  *If sequence has a limit then it is* ***convergent****, if it has no limit then it is* ***divergent****.*

*Notation:*

$$\lim_{n\to\infty} x_n = x \text{ or } x_n \to x$$

**Theorem 9** *Let* $\{x_n\}$ *be a sequence in* $\mathbb{R}$*, and let* $x \in \mathbb{R}$*. Then* $\{x_n\}$ *converges to* $x$ *if and only if for any* $\varepsilon > 0$*,* $B_\varepsilon(x)$ *about* $x$ *excludes at most a finite number of elements of* $\{x_n\}$*.*

**Exercise 8** *Prove that a sequence can converge to at most one point.*

**Example 1** *(a)* $\lim(\frac{1}{n}) = 0$;
*(b)* $\lim(\frac{1}{n^2+1}) = 0$;
*(c)* $\lim(\frac{3n+2}{n+1}) = 3$.

- *In order to show that a sequence* $\{x_n\}$ *does not converge to* $x$*, we have to find one number* $\varepsilon > 0$ *such that for any* $N \in \mathbb{N}$ *one can find a particular* $n \geq N$ *such that* $\left|x_{n_{N_\varepsilon}} - x\right| \geq \varepsilon$*. Alternatively, we need only find one* $\varepsilon > 0$*, such that* $B_\varepsilon(x)$ *excludes infinitely many members of the sequence. Note, however, that showing that* $\{x_n\}$ *diverges is not as simple...*

## 4.2 Limit Theorems

- *A sequence $\{x_n\}$ in $\mathbb{R}$ is **bounded** if there exists $M \in \mathbb{R}$ and $M > 0$ such that $|x_n| \leq M$ for all $n \in N$.*

**Theorem 10** *A convergent sequence in $\mathbb{R}$ is bounded*

**Theorem 11** *Let $\{x_n\}$ and $\{y_n\}$ be two sequences in $\mathbb{R}$ such that $x_n \to x$ and $y_n \to y$. Then the sequences $\{x_n + y_n\}$, $\{x_n - y_n\}$, $\{x_n y_n\}$, and $\{cx_n\}$ converge to $x+y$, $x-y$, $xy$, $cx$, respectively.*

**Theorem 12** *If $x_n \to x$, $z_n \to z$, $z_n \neq 0$ for all $n \in \mathbb{N}$ and $z \neq 0$, then the quotient sequence $\{x_n/z_n\} \to x/z$.*

**Theorem 13** *If $\{x_n\}$ and $\{y_n\}$ are convergent sequences in $\mathbb{R}$ and if $x_n \leq y_n$ for all $n \in \mathbb{N}$, then $\lim_{n\to\infty} x_n \leq \lim_{n\to\infty} y_n$.*

**Theorem 14** *If $\{x_n\}$ is a convergent sequence in $\mathbb{R}$ and if $a \leq x_n \leq b$ for all $n \in \mathbb{N}$, then $a \leq \lim_{n\to\infty} x_n \leq b$.*

**Theorem 15** *Let $\{x_n\}$,$\{y_n\}$, $\{z_n\}$ be sequences in $\mathbb{R}$. If $x_n \leq y_n \leq z_n$ for all $n \in \mathbb{N}$ and $\lim_{n\to\infty} x_n = \lim_{n\to\infty} z_n$, then $\{y_n\}$ is convergent with $\lim_{n\to\infty} x_n = \lim_{n\to\infty} y_n = \lim_{n\to\infty} z_n$.*

**Exercise 9** *Prove the above theorem*

**Theorem 16** *If $x_n \to x$, then $|x_n| \to |x|$.*

**Exercise 10** *Prove the above theorem*

**Theorem 17** *Let $x_n \to x$ and $x_n \geq 0$. Then sequence $\left\{\sqrt{x_n}\right\} \to \sqrt{x}$*

**Exercise 11** *Prove the above theorem*

**Theorem 18** *Let $x_n$ be a sequence of positive real numbers such that $L = \lim_{n\to\infty} \frac{x_{n+1}}{x_n}$ exists. If $L < 1$ then $\{x_n\}$ converges and $x_n \to 0$.*

**Theorem 19** *Let $x_n$ be a sequence of real numbers converging to $x$. Then any rearrangement of the members of the sequence will also converge to $x$.*

**Exercise 12** *Prove the above theorem*

## 4.3 Monotone Sequences

- *A sequence $\{x_n\}$ is **non–decreasing** if $x_1 \leq x_2 \leq ... \leq x_n \leq x_{n+1} \leq ....$ If the inequalities are strict, we say it is **increasing**. A sequence $\{x_n\}$ is **non–increasing** if $x_1 \geq x_2 \geq ... \geq x_n \geq x_{n+1} \geq .....$ If the inequalities are strict, we say it is **decreasing**. Sequence $\{x_n\}$ is **monotone** if it is either non–increasing or non–decreasing.*

**Theorem 20** ***Monotone Convergence Theorem.*** *A monotone sequence of real numbers is convergent if and only if it is bounded.*

- *Let $\{x_n\}$ be a sequence of real numbers and let $n_1 < n_2 < ... < n_k < ...$ be a strictly increasing sequence of natural numbers. Then the sequence $\{x_{n_k}\} = \{x_{n_1}, x_{n_2}, ..., x_{n_k}, ...\}$ is called a **subsequence** of $\{x_n\}$.*

**Theorem 21** *If $x_n \to \infty$ then every subsequence $x_{n_k} \to x$ as well.*

**Theorem 22** ***Monotone Subsequence Theorem.*** *If $\{x_n\}$ is a sequence in $\mathbb{R}$, then it has a monotone subsequence.*

**Theorem 23** ***Bolzano-Weierstrass Theorem.*** *Every bounded sequence in $\mathbb{R}$ has a convergent subsequence.*

**Theorem 24** *If every subsequence of $\{x_n\}$ converges to $x \in \mathbb{R}$ then $\{x_n\}$ converges to $x$.*

- *A sequence $\{x_n\}$ in $\mathbb{R}$ is a **Cauchy sequence** if for every $\varepsilon > 0$ there exist $N_\varepsilon \in \mathbb{N}$ such that for all $n, m \geq N_\varepsilon$, $|x_n - x_m| < \varepsilon$.*

**Theorem 25** *If $\{x_n\}$ is convergent then $\{x_n\}$ is Cauchy.*

**Theorem 26** *A Cauchy sequence of real numbers is bounded*

**Theorem 27** ***Cauchy Convergence Criterion.*** *A sequence $\{x_n\}$ in $\mathbb{R}$ is convergent if and only if it is a Cauchy sequence.*

**Example 2** ***Claim:*** *$\{x_n\}$ is a Cauchy sequence if $x_n = \frac{1}{n} \forall n \in \mathbb{N}$.*
***Proof:*** *Given $\varepsilon > 0$, let $N_\varepsilon > \frac{1}{\varepsilon}$. Let $n, m$ be greater than $N_\varepsilon$ and WLOG, let $n < m$. Thus, we have the following inequalities, which prove the result:*

$$\left|\frac{1}{x_n} - \frac{1}{x_m}\right| = \left|\frac{1}{n} - \frac{1}{m}\right| < \left|\frac{1}{n}\right| < \varepsilon,$$

*where the first inequality follows from the fact that $n < m$.* $\square$

## 5 Series

- *Consider an infinite series with the general term denoted $x_n$; the series then has the form*

$$\sum_{n=1}^{\infty} x_n$$

- *Let $s_n$ be the n-th partial sum given by $\sum_{i=1}^{n} x_i$. If $\lim_{k\to\infty} s_k$ exists then series is said to* ***converge****. Otherwise, we say that it diverges.*

- *The following condition is necessary for the convergence of a series $\sum_{n=1}^{\infty} x_n$:*

$$\lim_{n\to\infty} x_n = 0.$$

  *Otherwise, the sequence of partial sums will diverge. Note, however, that it is not a sufficient condition. For example:*

  - *Consider the series $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$*

$$s_n = 1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + ... + \frac{1}{\sqrt{n}} \geq \frac{n}{\sqrt{n}} = \sqrt{n}$$

    *The n-th partial sum increases without bound as n increases. Hence the the series diverges.*

- *If $\{x_n\}$ is a sequence in $\mathbb{R}$, then the series $\sum x_n$ is* ***absolutely convergent*** *if the series $\sum |x_n|$ is convergent in $\mathbb{R}$. The series is* ***conditionally convergent*** *if it is convergent but not absolutely convergent.*

**Theorem 28** *If a series in $\mathbb{R}$ is absolutely convergent, then it is convergent.*

*Some convergence tests:*

**Theorem 29** ***The $n-$th Term Test.*** *If the series $\sum x_n$ converges, then* $\lim x_n = 0$

**Theorem 30** ***Cauchy Criterion for series.*** *The series $\sum x_n$ converges if and only if for every $\varepsilon > 0$ there exists $N_\varepsilon \in \mathbb{N}$ such that if $m > n \geq N_\varepsilon$, then*

$$|s_m - s_n| = |x_{n+1} + x_{n+2} + ... + x_m| < \varepsilon$$

**Theorem 31** ***Comparison Test.*** *Let $\{x_n\}$ and $\{y_n\}$ be real sequences such that for some $K \in \mathbb{N}$*

$$0 \leq x_n \leq y_n \text{ for all } n \geq K$$

1. *Then the convergence of $\sum y_n$ implies the convergence of $\sum x_n$*

2. *The divergence of $\sum x_n$ implies the divergence of $\sum y_n$*

**Theorem 32** ***Root Test.*** *Given $\sum x_n$, let $\alpha = \lim_{n\to\infty} \sqrt[n]{|x_n|}$. Then if $\alpha < 1$, $\sum x_n$ converges, if $\alpha > 1$ then $\sum x_n$ diverges, and if $\alpha = 1$, then the test provides no information.*

**Theorem 33** ***Ratio Test.*** *The series $\sum x_n$ converges if $\left|\frac{x_{n+1}}{x_n}\right| < 1$, and diverges if $\left|\frac{x_{n+1}}{x_n}\right| \geq 1$.*

- *Given a sequence $\{c_n\}$ in $\mathbb{R}$, the series $\sum c_n x^n$ is called the **power series**. The numbers $c_n$ are called the **coefficients** of the series. Let $\alpha = \lim_{n\to\infty} \sqrt[n]{|c_n|}$. The **radius of convergence** is given by $R = \frac{1}{\alpha}$. Then $\sum c_n x^n$ are convergent if $|x| < R$, and diverges if $|x| > R$.*