Fluid level in tandem queues with an On/Off source

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Joint work with
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Why fluid queues?

- A simple model for shared resources with high arrival/service rates – e.g. telecommunication networks

- Markov modulated fluid queues can describe correlated input processes – e.g. self-similar traffic

- Often, the only tractable approximation
Tandem fluid queues with On/Off source

PRIOR WORK: Mostly numerical and iterative
• Markov modulated queues – Martingale methods [Kella Whitt 92], Sample path SDEs [Brocket Gong Guo 99]
• General On periods – approximate $X$ by PH distribution and solve for moments of fluid level iteratively [Field Harrison 07]

Q’: How many moments of $X$ do you need? (All? $kn$? $k+n$?)

OUR CONTRIBUTIONS
1. Non-iterative method for Laplace transform of fluid level at each queue
2. Closed-form exact expressions for moments of fluid level

Q: First $k$ moments of fluid level at queue $n$?

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$A'$: # moments of $X$ needed = ??
Analysis Roadmap

STEP 1: Fluid level at queue 1

STEP 2: Busy period analysis

STEP 3:
Making the method non-iterative
Analysis Roadmap

**STEP 1:** Fluid level at queue 1

\[
\lambda \quad X \sim G \quad \text{Exp}(\gamma) \quad \mu_1
\]

**STEP 2:** Busy period analysis

\[
\lambda \quad X \sim G \quad \text{Exp}(\gamma) \quad \mu_1 \quad 0 \quad B \quad \text{Exp}(\gamma)
\]

**STEP 3:** Making the method non-iterative

\[
\lambda \quad \mu_1 \quad 0 \quad \mu_1 \quad \text{Exp}(\gamma)
\]
Fluid level in a queue with On/Off source

\[ L = L_{OFF} 1_{\{\text{source Off}\}} + L_{ON} 1_{\{\text{source On}\}} \]
STEP 1a: Analysis of $L_{OFF}$

$L_{OFF} \overset{d}{=} \text{stationary } M/G/1 \text{ workload with arrival rate } \gamma \text{ and job sizes } \sim (\lambda-1)X$
STEP 1b: Analysis of $L_{ON}$

\[
L_{ON} = L_{OFF} + (\lambda - 1)T \\
L_{ON} = L_{OFF} + (\lambda - 1)X_e
\]

(By PASTA) $L_{OFF}$

stationary excess/age in a renewal process with i.i.d. renewals according to $X$
Finally…

\[ \lambda \xrightarrow{\text{Exp}(\gamma)} X \sim G \]

\[ L = L_{\text{OFF}} \mathbf{1}_{\{\text{source Off}\}} + L_{\text{ON}} \mathbf{1}_{\{\text{source On}\}} \]

\[ d = L_{\text{OFF}} + (\lambda - 1) X_e \cdot \mathbf{1}_{\{\text{source On}\}} \]

\[ d = M(\gamma)/G((\lambda - 1) X)/1 \text{ workload} + (\lambda - 1) X_e \cdot \mathbf{1}_{\{\text{source On}\}} \]

First \( k \) moments of \( L \) completely determined by first \((k+1)\) moments of \( X \)
Analysis Roadmap

**STEP 1: Fluid level at queue 1**

\[ X \sim G \]

\[ \lambda \]

\[ 0 \]

\[ \text{Exp}(\gamma) \]

\[ (k+1) \text{ moments of } X \Rightarrow k \text{ moments of fluid level} \]

**STEP 2: Busy period analysis**

\[ X \sim G \]

\[ \lambda \]

\[ 0 \]

\[ \text{Exp}(\gamma) \]

\[ B \]

**STEP 3: Making the method non-iterative**

\[ \mu_1 \]

\[ \mu_2 \]

\[ \cdots \]

\[ \mu_n \]
Analysis Roadmap

STEP 1: Fluid level at queue 1

STEP 2: Busy period analysis

STEP 3:
Making the method non-iterative

\[ X \sim G \]

\[ \lambda \rightarrow X \rightarrow \mu_1 \rightarrow \mu_2 \rightarrow \cdots \rightarrow \mu_n \]

\[ \lambda \rightarrow \mu_1 \rightarrow \mu_1 \rightarrow 0 \]

\[ \lambda \rightarrow \mu_1 \rightarrow \mu_1 \rightarrow 0 \]

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Busy period analysis [Boxma,Dumas 98]

Fluid level

\[ B = (U_1 + U_2 + \ldots + U_n) + (V_1 + V_2 + \ldots + V_n) \]
\[ = \left[ \frac{1}{(\lambda - 1)} + 1 \right] (V_1 + V_2 + \ldots + V_n) \]
\[ = \left[ \frac{1}{(\lambda - 1)} + 1 \right] M(\gamma)/G((\lambda - 1)X)/1 \text{ busy period} \]

First \( k \) moments of \( B \) completely determined by first \( k \) moments of \( X \)
Analysis Roadmap

STEP 1: Fluid level at queue 1

STEP 2: Busy period analysis

STEP 3:
Making the method non-iterative
Analysis Roadmap

**STEP 1: Fluid level at queue 1**

\[ \lambda \text{ Exp}(\gamma) \]

\[ X \sim G \]

\[ \rightarrow \mu_1 \]

\[ \lambda \text{ Exp}(\gamma) \]

\[ 0 \]

(k+1) moments of \( X \) \( \Rightarrow \) \( k \) moments of fluid level

**STEP 2: Busy period analysis**

\[ \lambda \text{ Exp}(\gamma) \]

\[ X \sim G \]

\[ \rightarrow \mu_1 \]

\[ \mu_1 \text{ Exp}(\gamma) \]

\[ 0 \]

(k+1) moments of \( X \) \( \Rightarrow \) (k+1) moments of busy period

**STEP 3:**
Making the method non-iterative
Putting it together

\[ \lambda \xrightarrow{\text{Exp}(\gamma)} X \sim G \]

First \((k+1)\) moments of \(X\) completely determine first \(k\) moments of fluid level at each queue.
Analysis Roadmap

STEP 1: Fluid level at queue 1

\( \lambda \)

\( 0 \)

\( \text{Exp}(\gamma) \)

\( X \sim G \)

\( \mu_1 \)

(k+1) moments of \( X \) \( \Rightarrow \) k moments of fluid level

STEP 2: Busy period analysis

\( \lambda \)

\( 0 \)

\( \text{Exp}(\gamma) \)

\( X \sim G \)

\( \mu_1 \)

\( \mu_1 \)

\( B \)

\( \text{Exp}(\gamma) \)

(k+1) moments of \( X \) \( \Rightarrow \) (k+1) moments of busy period

STEP 3:
Making the method non-iterative
Analysis Roadmap

**STEP 1:** Fluid level at queue 1

\[ X \sim G \]

\[ \lambda \]

\[ 0 \]

\[ \text{Exp}(\gamma) \]

\[(k+1) \text{ moments of } X \Rightarrow k \text{ moments of fluid level} \]

**STEP 2:** Busy period analysis

\[ X \sim G \]

\[ \lambda \]

\[ 0 \]

\[ \text{Exp}(\gamma) \]

\[(k+1) \text{ moments of } X \Rightarrow (k+1) \text{ moments of busy period} \]

**STEP 3:**

Making the method non-iterative
Getting rid of busy period iteration

\[ \lambda \xrightarrow{\text{Exp}(\gamma)} X \sim G \xrightarrow{} \mu_1 \xrightarrow{} \cdots \xrightarrow{} \mu_{n-1} \xrightarrow{\text{Exp}(\gamma)} \mu_n \]

\[ B_{n-1} = \text{function of } B_{n-2} = \ldots \]

\[ \mu_1 > \mu_2 > \ldots > \mu_{n-1} \Rightarrow B_{n-1} \text{ is identical to:} \]

\[ \lambda \xrightarrow{\text{Exp}(\gamma)} X \sim G \xrightarrow{} \mu_{n-1} \xrightarrow{\text{Exp}(\gamma)} \mu_n \]

Can obtain \( B_{n-1} \) in one step (no need to iterate)
Analysis Roadmap

STEP 1: Fluid level at queue 1

STEP 2: Busy period analysis

STEP 3:
Making the method non-iterative
Contributions/Conclusion

• Non-iterative method to obtain fluid level transform and moments in a tandem network

• Show that first $(k+1)$ moments of On period determine first $k$ moments of fluid level at each queue

• Method generalizes to a wider class of input processes