Theory of Income, Fernando Alvarez

UofC, Fall of 2008

<u>Final Exam</u>

The exam consists of one long problem. It is a particular example of the neoclassical growth model with endogenous labor supply. We consider conditions under which labor converges to its steady state from below or from above when the starting capital stock is below its steady state. This question is interesting because hours worked are procyclical, high in booms and low in recessions. The exam has a total of 240 points.

Instructions: All the answers should be in the space provided, just below each question. Your preliminary calculations [which will NOT be graded] should be done elsewhere. Do not use the back of the exam pages, except, perhaps, for intermediate calculations [that will NOT be graded].

Consider the following version of the (Gary) Hansen economy:

$$\int_0^\infty e^{-\rho t} u\left(c\left(t\right), 1 - n\left(t\right)\right) dt$$

subject to

$$\dot{k}\left(t\right) + \delta k\left(t\right) + c\left(t\right) = F\left(k\left(t\right), n\left(t\right)\right)$$

Throughout, use u_c for the derivative of u w.r.t. c, and u_l for the derivative of u w.r.t. ℓ, where $\ell = 1 - n$.

1) [10 points] Write the Hamiltonian for this problem. Use λ for the co-state and use c and n as controls.

Answer:

$$H(c,n,k) = u(c,1-n) + \lambda\left[F(k,n) - c - \delta k\right]$$

2) [10 points] Write the first order conditions for this problem (this are 3 equations, 2 of them ode's)

Answer:

$$\begin{aligned}
H_c &= 0: \; u_c(c,1-n) = \lambda \\
H_n &= 0: \; u_l(c,1-n) = \lambda F_n(k,n) \\
\dot{\lambda} &= \rho\lambda - \lambda\left[F_k(k,n) - \delta\right]
\end{aligned}$$

3) [40 points]

3.i) Use the foc w.r.t. c and the one w.r.t. n to arrive to one equation linking c, n and k (so c is an implicit function of k, n) Your answer should involve derivatives of u and F only.

Answer:

$$\frac{u_l(c,1-n)}{u_c(c,1-n)} = F_n(k,n)$$

From now on, assume that u is additively separable in ℓ and c, in particular assume that

$$u(c,1-n) = \frac{c^{1-\sigma}-1}{1-\sigma} - B\, n$$

where B is a positive constant and $\sigma > 0$.

3.ii) Use the equation in 3.i to define a function $n(c,k)$ that solves it. From now on denote the elasticities of $n(c,k)$ as follows:

$$\begin{aligned}\gamma_k &= \frac{k}{n(c,k)}\frac{\partial n(c,k)}{\partial k}\\ \gamma_c &= -\frac{c}{n(c,k)}\frac{\partial n(c,k)}{\partial c}\end{aligned}$$

Notice the $-$ in the definition of η_c. Show that

$$\begin{aligned}\gamma_c &> 0\\ \gamma_k &= 1.\end{aligned}$$

Write down two lines explaining the intuition behind the $\eta_k = 1$ result.

Answer: the equation defining $n(c,k)$ becomes

$$Bc^{\sigma} = F_n\left(\frac{k}{n(c,k)}, 1\right).$$

Since the LHS is independent of k, so has to be the RHS. Since F_n is monotone in k this implies that $(k/n)(c,k)$ is independent of k. That is, $n(c,k) = k\hat{n}(c)$. This immediately implies

$$\gamma_k = \frac{k}{n(c,k)}\frac{\partial n(c,k)}{\partial k} = \frac{k}{k\hat{n}(c)}\hat{n}(c) = 1$$

For the other elasticity, differentiate the expression defining n w.r.t c:

$$B\sigma c^{\sigma-1} = F_{nk}\frac{-k}{n^2}n_c$$

since $F_{nk} > 0$ given the assumption of CRS, this implies $n_c < 0$. Hence

$$\gamma_c = \frac{c}{n(c,k)}(-n_c) > 0$$

3.iii) Using the assumption that u is additively separable in ℓ and c, use the $H_c = 0$ and $\dot{\lambda}$ equations derived in 2) to derive an equation for $\dot{c}$ as a function of c and

k/n. This equation should use derivatives of the functions F and the constants ρ, σ and δ.

Answer: Rewrite the $\dot{\lambda}$ equation as

$$-\dot{\lambda}/\lambda = F_k\left(\frac{k}{n}, 1\right) - (\delta + \rho)$$

and from the $H_c = 0$ equation we obtain

$$\sigma\frac{\dot{c}}{c} = -\dot{\lambda}/\lambda$$

thus

$$\dot{c} = \frac{c}{\sigma}\left[F_k\left(\frac{k}{n}, 1\right) - (\delta + \rho)\right]$$

3.iv) What value(s) of σ, if any, allow this model to have a balanced growth path with labor augmenting technology growing at a constant rate, where consumption grows at the same rate than technology and labor supply n_t is constant? (Recall, this was a problem set!)

Answer: $\sigma = 1$ (i.e. $u(c, 1-n) = \log c \; - \; B\, n$).

4) [30 points] Use the law of motion of capital and the answers to 3.i, 3.iii to write a system of two differential equations in c and k. [Hint: replace n by $n(c,k)$]. In particular:

4.i) Write $\dot{c} = f(c,k)$. Give an expression for f using the $F(\cdot,\cdot)$ and $n(\cdot,\cdot)$ and the parameters ρ, δ and σ.

4.ii) Show that $f(c,k)$ does not depend on k. [Hint: Use your answer to 3.ii.]

4.iii) Write $\dot{k} = g(c,k)$. Give an expression for g using the functions $F(\cdot,\cdot)$ and $n(\cdot,\cdot)$ and the parameter δ.

Answer:

$$\begin{aligned} \dot{c} &= f(c,k) \equiv \frac{c}{\sigma}\left[F_k\left(\frac{k}{n(c,k)},1\right)-(\rho+\delta)\right] \\ \dot{k} &= g(c,k) \equiv F(k,n(k,c))-\delta k-c \end{aligned}$$

Further, using the answer to 3.ii we know that $k/n(c,k)$ is independent of k, hence, $f(c,k)$ does not depend on k.

Let k^*, c^*, n^* be the steady state values. Recall that

$$F_k\left(\frac{k^*}{n^*},1\right)=\rho+\delta$$

5) [40 points] Let $c(k)$ be the optimal consumption, or equivalently the saddle path of this system. We want to find an expression for

$$c_k \equiv \frac{dc(k^*)}{dk},$$

the derivative of the optimal consumption function evaluated at the steady state, or equivalently the slope of the saddle path. We will use that

$$c_k=\frac{dc(k)}{dk}=\frac{dc/dt}{dk/dt}=\frac{\dot{c}(k)\,|_{k=k^*}}{\dot{k}(k)\,|_{k=k^*}}$$

where $\dot{c}$ is the RHS of the $\dot{c}$ equation where c has been replaced by $c(k)$, i.e. $f(c(k),k)$ and $\dot{k}(k)$ is the RHS of the $\dot{k}$ equation where c has been replaced by the function $c(k)$, i.e. $g(c(k),k)$. In particular

$$c_k=\frac{\dot{c}(k)\,|_{k=k^*}}{\dot{k}(k)\,|_{k=k^*}}=\frac{f(c(k),k)\,|_{k=k^*}}{g(c(k),k)\,|_{k=k^*}}.$$

Since at steady state $\dot{c}=\dot{k}=0$ we use L'Hopital's rule to compute c_k by totally differentiating f and g w.r.t. k and evaluating the derivatives at steady state, i.e.

$$c_k=\frac{f(c(k),k)\,|_{k=k^*}}{g(c(k),k)\,|_{k=k^*}}=\frac{f_c(c(k^*),k^*)\,c_k+f_k(c(k^*),k^*)}{g_c(c(k^*),k^*)\,c_k+g_k(c(k^*),k^*)}$$

5.i) To do this compute the following derivatives $f_c\left(c\left(k^*\right),k^*\right)$, $f_k\left(c\left(k^*\right),k^*\right)$, $g_c\left(c\left(k^*\right),k^*\right)$, $g_k\left(c\left(k^*\right),k^*\right)$ in terms of the parameters

$$\varepsilon/\sigma,\ \rho,\ s_l,\ \gamma_c,\ \left(y^*/k^*\right),\ \left(y^*/c^*\right)$$

where

$$\begin{aligned}
\varepsilon &\equiv -k^*\frac{F_{kk}\left(k^*,n^*\right)}{F_k\left(k^*,n^*\right)} = -\frac{k^*}{n^*}\frac{F_{kk}\left(\frac{k^*}{n^*},1\right)}{F_k\left(\frac{k^*}{n^*},1\right)} \\
s_l &\equiv \frac{F_n\left(k^*,n^*\right)\ n^*}{F\left(k^*,n^*\right)} \\
\frac{y^*}{k^*} &\equiv \frac{F\left(k^*,n^*\right)}{k^*} \\
\frac{y^*}{c^*} &\equiv \frac{F\left(k^*,n^*\right)}{c^*}
\end{aligned}$$

In particular find the expressions for

$$f_c\left(c\left(k^*\right),k^*\right) =$$

in terms of ε/σ and γ_c

$$f_k\left(c\left(k^*\right),k^*\right) =$$

in terms of a real number (use your answer to 4.ii)

$$g_c\left(c\left(k^*\right),k^*\right) =$$

in terms of (y^*/c^*), s_l and γ_c

$$g_k\left(c\left(k^*\right), k^*\right) \;=\;$$

in terms of ρ, (k^*/y^*) and s_l.

Answers:

$$g\left(c,k\right) \equiv F\left(k, n\left(k,c\right)\right) - \delta k - c$$

$$\begin{aligned} f_c\left(c\left(k^*\right), k^*\right) &\equiv -\frac{c^*}{\sigma} F_{kk}\left(\frac{k^*}{n^*}, 1\right) \frac{k^*}{\left(n^*\right)^2} n_c = \frac{1}{\sigma}\left[\frac{-\left(\frac{k^*}{n^*}\right) F_{kk}\left(\frac{k^*}{n^*}, 1\right)}{F_k\left(\frac{k^*}{n^*}, 1\right)}\right]\left(\rho+\delta\right)\frac{n_c c^*}{n^*} \\ &= -\frac{\varepsilon}{\sigma}\left(\rho+\delta\right)\gamma_c \end{aligned}$$

$$f_k\left(c\left(k^*\right), k^*\right) = 0$$

$$g_c\left(c\left(k^*\right), k^*\right) = F_n n_c - 1 = \frac{F_n^* n^*}{F^*}\frac{n_c^*}{n^*} c^* \frac{F^*}{c^*} - 1 = -\left(s_l \gamma_c^* \frac{y^*}{c^*} + 1\right)$$

$$g_k\left(c\left(k^*\right), k^*\right) = F_k^* - \delta + F_n^* n_k = \rho + \frac{F_n^* n^*}{F^*}\frac{n_k^* k^*}{n^*}\frac{F^*}{k^*} = \rho + \frac{y^*}{k^*} s_l$$

5.ii) Show that c_k solves a quadratic equation:

$$\left(\rho + \left(\frac{y^*}{k^*}\right) s_l - \left(1 + \frac{y^*}{c^*}\; s_l\; \gamma_c\right) c_k\right) c_k = -\; c_k \;\; \frac{\varepsilon}{\sigma}\left(\delta+\rho\right)\; \gamma_c$$

Argue that the solution is given by

$$c_k = \frac{\frac{\varepsilon}{\sigma}\;\left(\rho+\delta\right)\;\gamma_c + \rho + \left(\frac{y^*}{k^*}\right)\; s_l}{1 + \frac{y^*}{c^*}\; s_l\;\gamma_c}$$

Hint: your answer should use 5.i), and and brief -one line- argument on the root you selected)

Answer:

$$\begin{aligned} c_k &= \frac{f_c\left(c\left(k^*\right), k^*\right) c_k}{g_c\left(c\left(k^*\right), k^*\right) c_k + g_k\left(c\left(k^*\right), k^*\right)} \\ &= \frac{-\frac{\varepsilon}{\sigma}\left(\rho+\delta\right)\gamma_c c_k}{\rho + \frac{y^*}{k^*}s_l - \left(1+\frac{y^*}{c^*}s_l\gamma_c^*\right)c_k} \end{aligned}$$

thus

$$\left(\rho + \frac{y^*}{k^*}s_l - \left(1+\frac{y^*}{c^*}s_l\gamma_c\right)c_k\right)c_k = -\frac{\varepsilon}{\sigma}\left(\rho+\delta\right)\gamma_c c_k$$

This equation has two roots, one is zero, which is *not* the slope of the saddle path (we know that is has strictly positive slope) and the other root is the slope of the saddle path, and given by

$$c_k = \frac{\frac{\varepsilon}{\sigma}\ \left(\rho+\delta\right)\ \gamma_c + \rho + \left(\frac{y^*}{k^*}\right)\ s_l}{1+\frac{y^*}{c^*}\ s_l\ \gamma_c}$$

6) [10 points] From now on, let F be Cobb-Douglas with capital share θ. Find an expression for γ_c.[Hint: use the definition of $n\left(\cdot,\cdot\right)$ and the functional forms for u and F, and don't forget the $-$ sign in the definition of γ_c.]

Answer: The function $n\left(c,k\right)$ solves

$$Bc^{\sigma} = \left(1-\theta\right)k^{\theta}n\left(c,k\right)^{-\theta}$$

thus

$$n\left(c,k\right) = k\left[\frac{\left(1-\theta\right)}{B}\right]^{1/\theta}c^{-\sigma/\theta}$$

This implies:

$$\gamma_c = \frac{\sigma}{\theta}$$

7) [30 points] Let ψ the elasticity of the optimal decision rule:

$$\psi \equiv \frac{k^*}{c^*} c_k$$

7.i) Write ψ in terms of

$$\varepsilon/\theta,\ \sigma/\theta,\ \delta,\ \ \rho,\ s_l,\ \ (y^*/k^*)\,,\ \ (y^*/c^*)$$

Remark: the steady state values (y^*/k^*) and (y^*/c^*) only depend on ρ, δ and θ (and not on ε and σ).

7.ii) How does ψ depend on ε? Write two lines with the intuition for this result.

7.iii) How does ψ depend on σ? Write two lines with the intuition for this result.

Answer:

7.i)

$$\psi = \frac{k^*}{c^*} c_k = \frac{(y^*/c^*)}{(y^*/k^*)} \frac{\frac{\varepsilon}{\theta}\ (\rho+\delta)\ +\rho + \left(\frac{y^*}{k^*}\right)\ s_l}{1+\frac{y^*}{c^*}\ s_l\ \frac{\sigma}{\theta}}$$

Given the remark, (y^*/c^*) and (y^*/k^*) do not depend on ε or σ. In fact, they are

$$\frac{y^*}{k^*} = \frac{k^{*\theta} n^{*1-\theta}}{k^*} = \frac{1}{\theta}\theta k^{*\theta-1} n^{*1-\theta} = \frac{\rho+\delta}{\theta}$$

and from the $\dot{c}=0$ equation

$$\begin{aligned} c^* &= k^{*\theta} n^{*1-\theta} - \delta k^* \\ &= k^* \left[\frac{1}{\theta}\theta k^{*\theta-1} n^{*1-\theta} - \delta\right] \\ &= k^* \left[\frac{\rho+\delta}{\theta} - \delta\right] \\ &= k^* \left[\frac{\rho+\delta\,(1-\theta)}{\theta}\right] \end{aligned}$$

thus

$$\begin{aligned} \frac{y^*}{c^*} &= \frac{y^*}{k^*}\frac{k^*}{c^*} = \frac{\rho+\delta}{\theta}\frac{\theta}{\rho+\delta\,(1-\theta)} \\ &= \frac{\rho+\delta}{\rho+\delta\,(1-\theta)} \end{aligned}$$

7.ii) ψ is increasing in ε: for a given k consumption is smaller, this is so because the household finds it optimal to invest a lot given the higher concavity of the production function

7.iii) ψ is decreasing in σ: the saddle path is closer to the $\dot{c} = 0$ equation. Agents dislike big intertemporal changes in consumption, hence they stay close to the $\dot{c} = 0$ (and hence to the steady state level of consumption c^*). (To see this better, draw the phase diagram and realize that the $\dot{c} = 0$ equation is horizontal, if k is in the horizontal axis and c in the vertical axis.)

8) [30 points] Let $\tilde{n}(k) = n(c(k), k)$ be the optimal employment decision rule. Define the elasticity of the optimal decision rule with respect to k as

$$\nu \equiv \frac{k^*}{n^*} \frac{d\tilde{n}(k^*)}{dk}$$

8.i) A robust empirical regularity is that hours worked are highly procyclical in the US for the post-WWII period (i.e. total employment increases in booms and decreases in recessions). What sign should ν have in order for the predictions of the model to be consistent with that observation (hint: how should we think about an expansion, as a period where capital is below or above steady state?)

8.ii) Compute ν as a function of γ_c and ψ.

8.iii) Compute ν as a function of

$$\theta, (y^*/k^*), (y^*/c^*), \rho, \delta, \sigma.$$

[More hints: Recall that for a Cobb-Douglas $s_l = 1 - \theta$ and $\varepsilon = 1 - \theta$. Use the expressions for ψ, c_k and γ_c obtained above.]

Answer:

8.i)

$$
\begin{aligned}
\tilde{n}_k &= n_k + n_c c_k \\
\frac{k}{n}\tilde{n}_k &= \frac{k}{n} n_k + \frac{c}{n} n_c \frac{k}{c} c_k = 1 - \gamma_c\, \psi
\end{aligned}
$$

hence

$$
\nu = 1 - \gamma_c\, \psi
$$

8.ii) Recall

$$
\begin{aligned}
\psi &= \frac{k^*}{c^*} c_k = \frac{(y^*/c^*)}{(y^*/k^*)} \frac{\frac{1-\theta}{\theta}\ (\rho + \delta)\ + \rho + \left(\frac{y^*}{k^*}\right)\ s_l}{1 + \frac{y^*}{c^*}\ s_l\ \frac{\sigma}{\theta}} \\
&= \frac{(y^*/c^*)}{(y^*/k^*)} \frac{\frac{1-\theta}{\theta}\ (\rho + \delta)\ + \rho + \left(\frac{y^*}{k^*}\right)\ (1-\theta)}{1 + \frac{y^*}{c^*}\ (1-\theta)\ \frac{\sigma}{\theta}}
\end{aligned}
$$

Thus

$$
\begin{aligned}
\nu &= 1 - \gamma_c \frac{(y^*/c^*)}{(y^*/k^*)} \frac{\frac{1-\theta}{\theta}\ (\rho + \delta)\ + \rho + \left(\frac{y^*}{k^*}\right)(1-\theta)}{1 + \frac{y^*}{c^*}\ (1-\theta)\ \frac{\sigma}{\theta}} \\
&= 1 - \frac{\sigma}{\theta} \frac{(y^*/c^*)}{(y^*/k^*)} \frac{\frac{1-\theta}{\theta}\ (\rho + \delta)\ + \rho + \left(\frac{y^*}{k^*}\right)\ (1-\theta)}{1 + \frac{y^*}{c^*}\ (1-\theta)\ \frac{\sigma}{\theta}} \\
&= 1 - \frac{(y^*/c^*)}{(y^*/k^*)} \frac{\frac{1-\theta}{\theta}\ (\rho + \delta)\ + \rho + \left(\frac{y^*}{k^*}\right)\ (1-\theta)}{\frac{\theta}{\sigma} + \frac{y^*}{c^*}\ (1-\theta)}
\end{aligned}
$$

9) [40 points] Let $\nu(\sigma)$ be the elasticity of the optimal labor decision rule as a function of σ.

9.i) How does $v(\sigma)$ varies with σ? Write down two lines explaining the intuition behind this result. (Hint: mention income and substitution effects)

9.ii) What is the sign of $\lim_{\sigma\to\infty} v(\sigma)$?

9.iii) What is the sign of $\lim_{\sigma\to 0} v(\sigma)$?

9.iv) Find an expression for σ, as a function of $\theta, \rho,$ and c^*/k^* for which $\nu = 0$.

Answer:

9.i) $\nu(\sigma)$ is strictly decreasing in σ.

9.ii)

$$\begin{aligned}
\lim_{\sigma\to\infty} v(\sigma) &= 1-\frac{(y^*/c^*)}{(y^*/k^*)}\,\frac{\frac{1-\theta}{\theta}\ (\rho+\delta)\ +\rho+\left(\frac{y^*}{k^*}\right)\ (1-\theta)}{\frac{y^*}{c^*}\ (1-\theta)}\\
&= 1-\frac{(y^*/c^*)}{(y^*/k^*)}\,\frac{\frac{1}{\theta}(\rho+\delta)\ +\frac{\rho}{1-\theta}+\left(\frac{y^*}{k^*}\right)}{\frac{y^*}{c^*}}\\
&= 1-\frac{k^*}{y^*}\ \left[\frac{1}{\theta}(\rho+\delta)\ +\frac{\rho}{1-\theta}+\left(\frac{y^*}{k^*}\right)\right]\\
&= 1-\frac{k^*}{y^*}\ \left[\frac{1}{\theta}(\rho+\delta)\ +\frac{\rho}{1-\theta}\right]-1\\
&= -\frac{k^*}{y^*}\ \left[\frac{1}{\theta}(\rho+\delta)\ +\frac{\rho}{1-\theta}\right]<0.
\end{aligned}$$

9.iii)

$$\lim_{\sigma\to 0} v(\sigma)=1>0.$$

9.iv) $\nu=0$ if and only if

$$\nu=0 \text{ iff } \frac{k^*}{c^*}\ \frac{\frac{1-\theta}{\theta}\ (\rho+\delta)\ +\rho+\left(\frac{y^*}{k^*}\right)\ (1-\theta)}{\frac{\theta}{\sigma}+\frac{y^*}{c^*}\ (1-\theta)}=1$$

or

$$\frac{k^*}{c^*}\ \left[\frac{1-\theta}{\theta}\ (\rho+\delta)\ +\rho+\left(\frac{y^*}{k^*}\right)\ (1-\theta)\right]=\frac{\theta}{\sigma}+\frac{y^*}{c^*}\ (1-\theta)$$

or

$$\frac{k^*}{c^*}\ \left[\frac{1-\theta}{\theta}\ (\rho+\delta)\ +\rho\right]+\left(\frac{k^*}{c^*}\right)\left(\frac{y^*}{k^*}\right)\ (1-\theta)=\frac{\theta}{\sigma}+\frac{y^*}{c^*}\ (1-\theta)$$

or

$$\frac{k^*}{c^*}\ \left[\frac{1-\theta}{\theta}\ (\rho+\delta)\ +\rho\right]=\frac{\theta}{\sigma}$$

or

$$\sigma=\theta\frac{c^*}{k^*}\frac{1}{\left[\frac{1-\theta}{\theta}\ (\rho+\delta)\ +\rho\right]}$$