Technical appendix to:

Fixed Term Employment Contracts

in an Equilibrium Search Model

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Appendix A: Analysis of the Island Planning Problem

The next set of results establish that the fixed point $V = H[V]$, the fixed point of the corresponding Bellman equation, is differentiable and that its derivatives are indeed given by V_j^*, define in equation (3). The results in the next three lemmas and two propositions are analogous to standard manipulations of first order conditions, except for the fact that V may not be differentiable.

Consider the problem of the planner of an island that receives U workers per period and that starts with workers $(T_1, T_2, ..., T_{J-1}, T_J)$ where T_i is the number of workers with tenure $i = 1, 2, ..., J$. Define E as the set of possible workers tenure profiles, $E = [0, U]^{J-1} \times R_+$. The planners value function $V : E \times Z$ solves

$$\begin{aligned} & H[V](T_1, T_2, ..., T_{J-1}, T_J, z) \\ = & \max_{\{E_i\}_{i=0}^{J}} \left\{ F\left(\sum_{i=0}^{J} E_i, z \right) + \sum_{i=0}^{J-1} \theta [T_i - E_i] + (\theta - \tau)[T_J - E_J] \right. \\ & \left. + \beta \int V(E_0, E_1, ..., E_{J-2}, E_{J-1} + E_J, z') Q(z, dz') \right\} \end{aligned} \tag{11}$$

subject to

$$\begin{aligned} 0 & \leq E_0 \leq U, \\ 0 & \leq E_i \leq T_i \text{ for } i = 1, 2, ..., J. \end{aligned}$$

The fixed point of H gives the stationary version of island planning problem defined in 37.

Proposition 5 *H maps concave functions into concave ones.*

The proof of this Proposition is standard, so we omit it..

We use the following notation for subgradients. Let $G : X \to R$ a concave function. We use $\partial G(x)$ to denote its subgradient at x (if it is clear the value of x from the context we simply use ∂G). In our case $X \subset R^n$, we use $\partial G_{x_i}(x)$ for $i = 1, 2, ..., n$ (and ∂G_{x_i} when it is clear) to denote the projection of $\partial G(x)$ into the subspace of the $x_i's$. Abusing notation, we use $G_{x_i}(x)$ (and G_i when it is clear) to denote a generic element of $\partial G_{x_i}(x)$, so that $G_{x_i}(x) \in \partial G_{x_i}(x)$.

The next proposition gives a useful result, ordering the subgradients of V

Proposition 6 *Consider a function V satisfying*

$$V_{T_1} \geq V_{T_2} \geq \cdots \geq V_{T_{J-1}} \geq V_{T_J}, \tag{12}$$

$$V_{T_1} \leq V_{T_J} + \tau \tag{13}$$

for all z and $T > 0$, where

$$\left(V_{T_1}, V_{T_2}, \ldots, V_{T_{J-1}}, V_{T_J} \right) \in \partial V(T, z).$$

Then,

$$H[V]_{T_1} \geq H[V]_{T_2} \geq \cdots \geq H[V]_{T_{J-1}} \geq H[V]_{T_J}, \tag{14}$$

$$H[V]_{T_1} \leq H[V]_{T_J} + \tau \tag{15}$$

for all z and $T > 0$, where

$$\left(H[V]_{T_1}, H[V]_{T_2}, \ldots, H[V]_{T_{J-1}}, H[V]_{T_J} \right) \in \partial H[V](T, z).$$

Intuitively it follows from the assumption that workers are perfect substitutes and from the fact that $\tau > 0$.

The following proposition and corollaries are important to characterize the solution of the problem and to reduce its dimensionality.

Proposition 7 *Let V satisfy (12). Then the policies for $H[V]$ satisfy the following. Let $E = (E_0, E_1, ..., E_{J-1}, E_J) \in [0, U]^J \times R_+$ be feasible given T. Consider an alternative $\tilde{E} = \left(\tilde{E}_0, \tilde{E}_1, ..., \tilde{E}_{J-1}, \tilde{E}_J \right)$ such that: i) it is feasible for T, ii)*

$$\sum_{j=0}^{J-1} E_j = \sum_{j=0}^{J-1} \tilde{E}_j \text{ and } E_J = \tilde{E}_J,$$

and iii) there is a j' such that $\tilde{E}_j \geq E_j$ for all $j \leq j' \leq J-1$ and that $\tilde{E}_j = 0$for all j, $j' < j \leq J-1$. Then $\tilde{E}$ is weakly preferred to E.

Proof. Replacing any policy by one with these properties can not decrease output but can decrease the separation cost τ. ■

Corollary 8 *The optimal policy can be chosen with the following property:*
(*) If $E_j < T_j$ for some j, $1 \leq j \leq J-1$, then $E_{j'} = 0$ for all $j' : j < j' \leq J-1$.

Corollary 9 *If $T \in \mathcal{E}$ and T' is given by the optimal policy*

$$T' = (T_1', T_2', ..., T_J') = (E_0, E_1, ..., E_{J-2}, E_{J-1} + E_J)$$

then $T' \in \mathcal{E}$.

The next set of results establish that the fixed point $V = H[V]$ is differentiable and that its derivatives are indeed given by V_j^*. The results in the next three lemmas and two propositions are analogous to standard manipulations of first order conditions, except for the fact that V may not be differentiable.

Let define the function $\hat{R}(E, z)$, as follows: $\hat{R} : R_+^{J+1} \times Z \to R$

$$\begin{aligned} \hat{R}(E, z) &= F\left(\sum_{i=0}^{J} E_i, z\right) - \theta \sum_{i=0}^{J-1} E_i - (\theta - \tau) E_J \\ &\quad + \beta \int V(E_0, E_1, \ldots, E_{J-2}, E_{J-1} + E_J, z') Q(z, dz'). \end{aligned}$$

The first lemma shows a standard saddle-type result for the problem defining $H[V]$.

Lemma 10 *Let V be concave. Fix T, z and let*

$$\begin{aligned} H[V](T, z) &= \max_E \left\{ \hat{R}(E, z) + \hat{\theta} T : 0 \leq E \leq T \right\}, \\ E(T, z) &= \arg\max_E \left\{ \hat{R}(E, z) : 0 \leq E \leq T \right\}. \end{aligned} \tag{16}$$

Then

$$\hat{\theta} + \lambda^* = (H[V]_0, H[V]_1, ..., H[V]_J) \in \partial H[V](T, z)$$

if and only if λ^* is a Lagrange multiplier, i.e.

$$\begin{aligned} \hat{R}(E^*, z) + \lambda (T - E^*) &\geq \hat{R}(E^*, z) + \lambda^* (T - E^*) \\ &\geq \hat{R}(E, z) + \lambda^* (T - E) \end{aligned} \tag{17}$$

for all non-negative E, λ, where $\hat{\theta} = (\theta, ..., \theta, \theta - \tau)$, $E^* = E(T, z)$ and $U = T_0$.

Notice that since $\hat{R}$ is concave and the restrictions are linear, $E(T, z)$ solves problem (16) if and only if there (E^*, λ^*) is a saddle as in equation (17) -see, for example, "Analytical Method in Economics", Takayama, Theorem 2.9-.

The next lemma shows the Kuhn-Tucker conditions for this problem.

Lemma 11 *Let V be concave. A necessary and sufficient condition for $E^* = \{E_i^*\}_{i=0}^J$ solves*

$$E^* \in \arg\max_E \hat{R}(E, z) \ \ s.t. \ 0 \leq E \leq T$$

given T, z is that there exists a $\left\{\hat{R}_i\right\}_{i=0}^{J} \in \partial \hat{R}(E^*, z)$ such that (E^*, λ^*) is a saddle where,

$$\lambda_i^* = \hat{R}_i^*. \tag{18}$$

Given our previous results we can now write the analogous to the Euler equations.

Proposition 12 *Let V be concave. Fix T,z. Then, $0 \leq E^* \leq T$ is an optimal choice given T,z if and only if for all $\{H[V]_i(T,z)\}_{i=0}^{J} \in \partial H[V](T,z)$ there is a $\left\{\hat{R}_i\right\}_{i=0}^{J} \in \partial \hat{R}(E^*,z)$ such that*

$$\begin{aligned}
H[V]_i(T,z) &= \hat{R}_i(E^*,z) + \theta \text{ for } i = 0,\dots,J-1 \\
\hat{R}_i(E^*,z) &\geq f\left(\sum_{i=0}^{J} E_i^*, z\right) - \theta \\
&+\beta \int V_{i+1}\left(E_0^*,\dots,E_{J-2}^*,E_{J-1}^*+E_J^*,z'\right) Q(z,dz') \\
\text{with} &= \text{ if } E_i^* > 0 \\
\hat{R}_i(E^*,z) &\geq 0, \\
0 &= (H[V]_i(T,z) - \theta)(T_i - E_i^*), \text{ and} \\
H[V]_J(T,z) &= \hat{R}_J(E^*,z) + \theta - \tau \ , \\
0 &= (H[V]_J(T,z) - (\theta - \tau))(T_J - E_J^*), \\
\hat{R}_J(E,z) &\geq f\left(\sum_{i=0}^{J} E_i^*, z\right) - (\theta - \tau) \\
&+\beta \int V_J\left(E_0^*,\dots,E_{J-2}^*,E_{J-1}^*+E_J^*,z'\right) Q(z,dz') \\
\text{with} &= \text{ if } E_J^* > 0 \\
\hat{R}_J(E^*,z) &\geq 0
\end{aligned}$$

where we let $U = T_0$.

The next lemma shows that employment is bounded below, and hence marginal productivity is bounded above.

Lemma 13 *There is an $e > 0$ such that for all T,z*

$$\sum_{i=0}^{J} E_i(T,z) \geq e > 0.$$

By this lemma, the solution for V_j^* is well defined because, since $f\left(\sum_{i=0}^{J} E_{i,s}^*, z_s\right)$ are uniformly bounded.

Proposition 14 *Let V be the fixed point of H. Assume that $U > 0$. Then V is differentiable with respect to T_i when $T_i > 0$.*

Appendix B: Analysis of the Simplified Island Planning Problem

The planner's value function $v:[0,J\cdot U]\times R_{+}\times Z$ has to satisfy the functional equation h:

$$\begin{aligned} & h[v](t,p,z) \\ = & \max_{e_t,e_p}\{F(e_t+e_p,z)+\theta[t-e_t]+(\theta-\tau)[p-e_p] \\ & +\beta\int v(t',p',z')Q(z,dz')\Big\} \end{aligned} \tag{19}$$

subject to

$$\begin{aligned} 0 &\leq e_t\leq t, \\ 0 &\leq e_p\leq p, \end{aligned}$$

and where the law of motion is given by

$$\begin{aligned} t' &= \min\{U+e_t,JU\} \\ p' &= e_p+\max\{U+e_t-JU,0\} \end{aligned}$$

Proposition 15 *Consider V and v such that*

$$v(T_1+T_2+...+T_{J-1},T_J,z)=V(T_1,T_2,...,T_{J-1},T_J,z) \tag{20}$$

for all $(T_1,T_2,...,T_{J-1},T_J)\in\mathcal{E}$. Then

$$h[v](T_1+T_2+...+T_{J-1},T_J,z)=H[V](T_1,T_2,...,T_{J-1},T_J,z) \tag{21}$$

for all $(T_1,T_2,...,T_{J-1},T_J)\in\mathcal{E}$.

Proof. By Proposition 7 and its corollaries, $h[v]=H[V]$ in $\mathcal{E}$. ■

Lemma 16 *Assume that V satisfies (12). Consider T and $\hat{T}$ and V such that*

$$T_1+T_2+...+T_{J-1}=\hat{T}_1+\hat{T}_2+...+\hat{T}_{J-1} \text{ and } T_J=\hat{T}_J. \tag{22}$$

for any $\hat{T}\in\mathcal{E}$ and $T\in E$ then

$$H[V](T,z)\leq H[V]\left(\hat{T},z\right).$$

Proof. If follows directly from the definition of $\mathcal{E}$ and the assumed property (12). ■

Proposition 17 *Let v be the function corresponding to V as in (20) defined for $T\in\mathcal{E}$. Assume that $V(\cdot,z)$ is concave, and that V satisfies (12). Then $h[v](\cdot,z)$ is concave in t,p.*

Remark 18 *The previous proposition is not obvious since the feasible set of the problem defined by the right hand side of $h[v]$ is not convex.*

We now introduced the R, which is the objective function being maximized in $h[v]$. The "derivatives' of R are used to define the functions $\hat{t}$ and $\hat{p}$.

Definition 19 *Given v, define $R(e_t,e_p,z)$ as*

$$R(e_t,e_p,z)=F(e_t+e_p,z)-\theta e_t-(\theta-\tau)e_p$$

$$+\beta\int v(U+\min\{e_t,(J-1)U\},e_t+e_p-\min\{e_t,(J-1)U\},z')Q(z,dz')$$

Consider an island planner with no temporary workers ($t = 0$) and a given z. The quantity $\hat{p}(z)$ is the number of permanent workers that leaves the island's planner indifferent between firing "one" permanent worker and keeping all $\hat{p}(z)$ of them.

Definition 20 *Let R be defined as in (19). For each z define $\hat{p}(z)$, such that*

$$0 \in \partial R_{e_p}(0, \hat{p}(z), z) \ .$$

Consider an island planner with $0 < p < \hat{p}(z)$, so it does it not want to fire any permanent worker for that z. The quantity $\hat{t}(p, z)$ is the number of temporary workers that leaves the island's planner indifferent between firing "one" transitory worker and keeping all $\hat{t}(p, z)$ of them. Formally:

Definition 21 *Let R be defined as in (19). For each p, z define $\hat{t}(p, z)$ as follows*:
(i) if $R_{et} > 0$ for all $R_{et} \in \partial R_{et}(U \cdot J, p, z)$, then $\hat{t}(p, z) = J \cdot U$,
(ii) if $R_{et} < 0$ for all $R_{et} \in \partial R_{et}(0, p, z)$, then $\hat{t}(p, z) = 0$,
(iii) otherwise $\hat{t}(p, z)$ solves $0 \in \partial R_{e_t}\left(\hat{t}(p, z), p, z\right)$.

The remaining of this section shows that $\hat{p}, \hat{t}$ exists, that they are unique, and that $\hat{t}$ is decreasing in p. The proofs are complicated by the fact that R is not differentiable.

Proposition 22 *Let v be functions corresponding to V as in (20), assume that V is concave and satisfies (12). The function $R(\cdot, z)$ is strictly concave.*

Define $M : [0, U \cdot J] \to R_+$ as

$$M(e_t) \equiv \min\{e_t, (J-1)U\}$$

notice that

$$\begin{aligned} & e_p + \max\{e_t - (J-1)U, 0\} \\ = \ & e_p + e_t - \min\{e_t, (J-1)U\} \\ = \ & e_p + e_t - M(e_t). \end{aligned}$$

Remark 23 *It is standard to show that $h[v]$ is increasing in t, p and z if v has that properties.*

Remark 24 *Assume that V satisfies (12) and (13). Let v be defined as in (20). Denote by $\partial h[v]$ the subgradient of $h[v](t, p, z)$ when v is considered as a function of t and p. A corollary of Proposition (15) and Proposition (6) is that*

$$h[v]_p \le h[v]_t \le h[v]_p + \tau,$$

for all $\left(h[v]_t, h[v]_p\right) \in \partial h[v](t, p, z)$.

Proposition 25 *Fix t, p, z. Assume that v satisfies (12), (13), and is concave. Define v as in (20). Let $\left(h[v]_t, h[v]_p\right) \in \partial h[v](t, p, z)$. Then $h[v]_p \ge \theta - \tau$. Moreover, there exists a $\bar{p}(z)$ such that for all $p \ge \bar{p}(z)$ and t: $h[v]_p = \theta - \tau$ for any $h[v]_p \in \partial h[v]_p(t, p, z)$.*

Given v define

$$b(e_t, e_p, z) \equiv \int v(U + M(e_t), e_t + e_p - M(e_t), z')\, Q(z, dz')$$

as a function of e_t and e_p and z. Let ∂B be its subgradient with respect to (e_t, e_p).

Lemma 26 *Assume that v is concave and that it satisfies*

$$v_p \leq v_t \leq v_p + \tau,$$

for all t, p, z. Define v as in (20). Fix any z, e_t, e_p. Let $\left(b_{e_t}, b_{e_p}\right) \in \partial b\left(e_t, e_p, z\right)$. Then

$$b_{e_p} \leq b_{e_t} \leq b_{e_p} + \tau.$$

Let $\partial R\left(e_t, e_p, z\right)$ be the subgradient of R when considered as a function of (e_t, e_p).

Lemma 27 *Assume that v is concave and that is satisfies*

$$v_p \leq v_t \leq v_p + \tau,$$

for all t, p, z. Fix any z, e_t, e_t. For all $\left(R_{e_p}, R_{e_t}\right) \in \partial R\left(e_t, e_p, z\right)$

$$R_{e_p} \geq R_{e_t} + \tau\left(1 - \beta\right).$$

Corollary 28 *Let e_p, e_t be the optimal choice of employment for Problem (19). If $e_p < p$ and $t > 0$, then $e_t = 0$. If this were not true, i.e. if $e_p < p$ and $e_t > 0$, then $R_{e_p} = R_{e_t} = 0$, which contradicts Lemma 27.*

Lemma 29 *Let v be functions corresponding to V as in (20), assume that V is concave and satisfies (12). Let R be defined as in (19).*
For each z there is a unique $\hat{p}$ satisfying (20). Moreover, $0 < \hat{p}\left(z\right) < \bar{p}\left(z\right) < +\infty$.

Using the concavity of R and strict concavity of F we define $\hat{t}$ as follows.

Lemma 30 *Let v be functions corresponding to V as in (20), assume that V is concave and satisfies (12). Let R be defined as in (19).*
Then for each (p, z), $0 < p < \hat{p}\left(z\right)$, there exists a unique $\hat{t}$ that satisfies (21).

Proof. The existence and uniqueness of $\hat{t}$ in follows from the strict concavity of R. ∎

Proposition 31 *Assume that V is concave and that satisfies (13) and (12). Let v be given by V as in (20). Assume, without loss of generality that v is concave in (t, p). Then,*
i) The optimal decision rules of $h\left[v\right]$ are described by the set of Inaction for R as

$$\begin{aligned} e_t\left(t, p, z\right) &= \min\left\{t, \hat{t}\left(p, z\right)\right\}, \\ e_p\left(t, p, z\right) &= \min\left\{p, \hat{p}\left(z\right)\right\} \end{aligned}$$

for all t, p, z.
ii) $H\left[V\right]$ is concave, satisfies (13) and (12).
iii) $h\left[v\right]$ and $H\left[V\right]$ satisfy (20) and $h\left[v\right]$ is concave.

Proof. It follows from the definition of $\hat{t}$, $\hat{p}$ and various of the previous results. ∎

Lemma 32 *Let V be concave, and satisfy (12) and (13). Let v be defined as in (20). Let $\hat{t}$, $\hat{p}$ and I be defined as in (29), (**??**), (5). Then, the subgradients of $h\left[v\right]$ are as follows:*
If $t \neq iU$ for $i = 1, 2, .., J-1$, then $h\left[v\right]\left(t, p, z\right)$ is differentiable with respect to t.
If $(t, p) \in Int\left(I\left(z\right)\right)$:

$$h\left[v\right]_t\left(t, p, z\right) = f\left(t + p, z\right) + \beta \int b_{e_t}\left(t, p, z'\right) Q\left(z, dz'\right) > \theta,$$

If $(t, p) \in Int\left(I\left(z\right)^C\right)$:

$$h\left[v\right]_t\left(t, p, z\right) = \theta > f\left(t + p, z\right) + \beta \int b_{e_t}\left(t, p, z'\right) Q\left(z, dz'\right),$$

If (t, p): $t = \hat{t}\left(p, z\right) < JU$:

$$\left[\underline{h}\left[v\right]_t\left(t, p, z\right), \bar{h}\left[v\right]_t\left(t, p, z\right)\right] = \left[\theta,\ f\left(t + p, z\right) + \beta \bar{b}_{et}\left(t, p, z\right)\right]$$

Definition 33 *We say that $\partial v_t\,(t,p,z)$ is decreasing in p if it satisfies the following property. If $p < p'$, then define $\underline{v_t}', \bar{v}_t', \underline{v_t}$ and $\bar{v}_t$ satisfying*

$$\left[\underline{v_t}', \bar{v}_t'\right] = \partial v_t\,(t, p', z)\,,$$

and

$$\left[\underline{v_t}, \bar{v}_t\right] = \partial v_t\,(t, p, z)\,.$$

Then

$$\underline{v}_t' \leq \underline{v}_t \text{ and } \bar{v}_t' \leq \bar{v}_t$$

Notice that if v is differentiable at (t,p,z) this property simply says that $\partial v\,(t,p,z)\,/\partial t$ is decreasing in p.

Lemma 34 *. Let V be concave, and satisfy (12), and (13). Let v be defined as in (20). Assume that the subgradient of v_t is decreasing in p, i.e. it satisfies the condition 33. Let $\hat{t}\,(p,z)$ be defined as in (**??**) for the optimal rule that attains the right hand side of $h\,[v]$. Then, the subgradient of $h\,[v]_t$ is decreasing in p too, i.e. it satisfies the condition 33 and $\hat{t}\,(p,z)$ is weakly decreasing in p.*

Finally

Proposition 35 *Let v be the fixed point of h. Let $\hat{t}$ be defined as in definition* **??**. *Then $\hat{t}\,(p,z)$ is decreasing in p. Moreover, if $\hat{t}$ is not a multiple of U, then $\hat{t}$ is strictly decreasing in t.*

Appendix C: Proofs

Proof of Theorem 1. To show this proposition we characterize the competitive equilibrium of a particular decentralization of the economy. Since the 1st welfare theorem holds, characterizing this equilibrium gives us a characterization of the efficient allocations. We call this equilibrium "auxiliary competitive equilibrium" or "ACE" for short. See Appendix D below for a definition of the ACE. The characterization of a stationary ACE coincides with conditions i) to vi) of Theorem 1.

We start by providing some of the necessary conditions that an ACE must satisfy.

Lemma 36 *Let* $\{\theta_t,\ \lambda_t(z^t,X),\ E_{j,t}(z^t,X),\ T_{j,t}(z^t,X),\ S_{j,t}(z^t,X),\ U_t,\ L_t;\ all\ t, z^t, j, X\}$ *be an AC equilibrium. Then, there is sequence* $\{\sigma_t\}$ *where* σ_t *is the value of search at t, for which:*
i) without loss of generality, $\theta_t(z^t,X)=\theta_t$*,*
ii)

$$\begin{aligned}\sigma_t &= \beta\sum_X\sum_{z^{t+1}}\lambda_{t+1}\left(z^{t+1},X\right)\eta\left(X|z_0\right)q_t\left(z^t\right)\\ \theta_t &= \max\left\{\omega+\beta\theta_{t+1},\sigma_t\right\}\\ 0 &= L_t\left[\theta_t-\omega-\beta\theta_{t+1}\right]\end{aligned}$$

and
iii) for all z^t, X*,*

$$\begin{aligned}\theta_t &\le \lambda_t\left(z^t,X\right),\\ 0 &= \left[\lambda_t\left(z^t,X\right)-\theta_t\right]\left[T_{0,t}\left(z^t,X\right)-E_{0,t}\left(z^t,X\right)\right].\end{aligned}$$

The proof of this Lemma follows directly from the linearity in the problem of firms of type II.

This Lemma shows, among other things, that the value to a firm of type I of reallocation (firing) a worker does not depend on the characteristic of the island, so that θ_t does not depend on (z^t,X) and that the value of search σ_t is related to the value of "selling" (assigning) a worker to the different islands randomly, i.e. in proportion to the number of island of each type.

We will show that the ACE allocation can be obtained by solving a particular dynamic programing problem given two numbers (θ,U) and by checking two appropriate equilibrium conditions. We develop this characterization in a sequence of results.

The solution of the dynamic programing problem will give the equilibrium quantities chosen by firms of type I and the equilibrium prices $\lambda_t(z^t,X)$. This problem has the interpretation of the maximization problem solved for a coalition of firms of type I that are endowed with a flow $\mathbf{U}=\{U_t\}_{t=0}^{\infty}$ of newly arrived workers. We refer to this problem as the "island planner problem", i.e. the problem of a planner in charge of the island employment decision by tenure. The planner chooses how many workers of each tenure to employ and how many to send back, obtaining θ_t for each of them, net of the cost τ.

Definition 37 *Let* $V_t : R_+^J\times Z\times R_+^\infty\to R$

$$\begin{aligned}&V_t\left(T_1,...T_J;z_t,\mathbf{U}\right)\\ =\ &\max_{E_j,j=0,...J}\{\ F\left(\sum_{j=0}^{J}E_j,z_t\right)\\ &+\sum_{j=0}^{J}\left[T_j-E_j\right]\theta_t-\tau\left[T_J-E_J\right]\\ &\beta\sum_{z_{t+1}\in Z}V_{t+1}\left(E_0,...,E_{J-1}+E_J;z_{t+1},\mathbf{U}\right)\ \}Q\left(z_{t+1}|z_t\right)\ \}\end{aligned}$$

subject to

$$\begin{aligned}T_0 &= U_t\\ E_j &\le T_j\ j=0,1,...,J.\end{aligned}$$

where $\mathbf{U}=\{U_t;\ all\ t\ge 0\}\in R_+^\infty$*.*

The next Lemma links the island planning problem with the equilibrium quantities chosen by type I firms and the prices $\{\lambda_t\}$.

Lemma 38 *Let* $\left\{\theta_t^*,\, \lambda_t^*\left(z^t, X\right),\, E_{j,t}^*\left(z^t, X\right),\, T_{j,t}^*\left(z^t, X\right),\, S_{j,t}^*\left(z^t, X\right),\, U_t^*,\, L_t^*;\ all\ t, z^t, j, X\right\}$ *be an auxiliary competitive equilibrium given initial conditions* $U_{-1}^*, \eta^*(X|z_0)$. *Define* $\hat{V}_t$ *for* $\{U_t^*, \theta_t^*\}$ *and let* $\hat{E}_{j,t}(T, z)$ *be its optimal policy. Then,* $\left\{E_{j,t}^*\left(z^t, X\right)\right\}$ *solves* V_t *for all the initial conditions* X, *i.e.*

$$\hat{E}_{j,t}\left(T_t^*\left(z^t, X\right), z_t\right) = E_{j,t}^*\left(z^t, X\right) \text{ for all } t, z^t, X$$

and

$$\lambda_t^*\left(z^t, X\right) = \partial V_t\left(T_t^*\left(z^t, X\right), z_t; \mathbf{U}^*\right) \text{ for all } t, z^t, X$$

where $\partial V_t(T, z_t; \mathbf{U}^*)$ *is and element of the subgradient of* $V_t\left(T, z_t; \left\{U_0^*, ..., U_{t-1}^*, \cdot, U_{t+1}^*, ...\right\}\right)$ *with respect to* U_t^*.

The proof of this Lemma follows from comparing the island planning problem with the problem of firms of type I in a competitive equilibrium, and from the definition of a subgradient.

The next Lemma gives the characterization of ACE.

Lemma 39 . *Let some arbitrary initial distribution* $\eta^*(X|z_0)$ *be given. Let also some arbitrary sequence* $\{U_t^*, \theta_t^* : all\ t\}$ *be given. Let* $\hat{E}_{j,t}(T, z)$ *be the optimal policy of island planning problem (37) defined for* $\{U_t^*, \theta_t^*\}$. *Define* $\left\{E_{j,t}^*\right\}$ *as*

$$E_{j,t}^*\left(z^t, X\right) = \hat{E}_j\left(T_t^*\left(z^t, X\right), z_t\right)$$

where $T_{t,j}^*$ *has been generated by* $\left\{\hat{E}\right\}$ *and the initial condition* X, *i.e.*

$$\begin{aligned} T_{0,j}^* &= X_j \\ T_{t,j}^*\left(z^t, X\right) &= \hat{E}_{j-1}\left(T_{t-1}^*\left(z^{t-1}, X\right), z_{t-1}\right) \text{ for } j = 1, ..., J-1 \\ T_{t,J}^*\left(z^t, X\right) &= \hat{E}_{J-1}\left(T_{t-1}^*\left(z^{t-1}, X\right), z_{t-1}\right) + \hat{E}_J\left(T_{t-1}^*\left(z^{t-1}, X\right), z_{t-1}\right) \end{aligned}$$

and let $\partial V_t(T, z_t; U^*)$ *be an element of the subgradient of* $V_t\left(T, z_t; \left\{U_0^*, ..., U_{t-1}^*, \cdot, U_{t+1}^*, ...\right\}\right)$ *with respect to* U_t^*.

i) Define $\left\{\lambda_{t+1}^*\right\}$ *as*

$$\lambda_t^*\left(z^t, X\right) = \partial V_t\left(T_t^*\left(z^t, X\right), z_t; U^*\right).$$

ii)Define, the value of search $\{\sigma_t\}$ *as*

$$\sigma_t = \beta \sum_{z^{t+1}} \sum_X \lambda_{t+1}^*\left(z^{t+1}, X\right) q_{t+1}\left(z^{t+1}\right) \eta^*(X|z_0) \text{ for all } t.$$

iii)Define $\{L_t^*\}$ *as*

$$L_t^* = N - U_t^* - \sum_{z^t} \sum_X E_{j,t}^*\left(z^t, X\right) q_t\left(z^t\right) \eta^*(X|z_0) \geq 0 \text{ for all } t.$$

iv) Suppose that the following optimal labor force participation conditions are satisfied

$$\theta_t^* = \max\left\{\sigma_t, \omega + \beta\theta_{t+1}^*\right\},$$

$$L_t^*\ \left[\theta_t - \omega + \beta\theta_{t+1}\right] = 0$$

for all t. *Then* $\left\{\theta_t^*,\, \lambda_t^*\left(z^t, X\right),\, E_{j,t}^*\left(z^t, X\right),\, T_{j,t}^*\left(z^t, X\right),\, S_{j,t}^*\left(z^t, X\right),\, U_t^*,\, L_t^*;\ all\ t, z^t, j, X\right\}$ *is a auxiliary competitive equilibrium given the initial conditions* U_{-1}^* *and* η^*.

The proof of this Lemma follows by construction and by the definition of competitive equilibrium and the properties of Problem (37).

Since the first welfare theorem hold for this economy, the characterization of the allocation for an ACE in the previous Lemma applies to the efficient allocations.

Now we define stationary ACE in terms of the objects used our previous characterization of the ACE.

Definition 40 *We say that the auxiliary competitive equilibrium* $\{\theta_t, \lambda_t, L_t, U_t, E_{jt}, S_{jt}\}$ *for initial measure* η *is a stationary equilibrium if there are constants,* θ, U, L, *and functions,* $E_j^* : R_+^{J+1} \times Z \to R$, $j = 0, ..., J$, $\lambda^* : R_+^{J+1} \times Z \to R$, *for which*

$$\begin{aligned}
\theta_t &= \theta, \text{ all } t \\
U_t &= U, \text{ all } t \\
L_t &= N, \text{ all } t \\
E_{i,t}\left(z^t, X\right) &= E_j^*\left(T_t\left(z^t, X\right), z_t\right), \text{ all } t, z^t \\
\lambda_t\left(z^t, X\right) &= \lambda^*\left(T_t\left(z^t, X\right), z_t\right), \text{ all } t, z^t
\end{aligned}$$

and where defining $T_j' : R_+^{J+1} \times Z \to R$ *as*

$$\begin{aligned}
T_0'(T, z) &= U, \\
T_j'(T, z) &= E_{j-1}^*(T, z) \text{ for } j = 1, ..., J-1, \\
T_J'(T, z) &= E_J^*(T, z) + E_{J-1}^*(T, z)
\end{aligned}$$

and letting μ *be an invariant distribution of the joint process* (T, z), *with transition given by* (T', Q), *we have*

$$\eta(T|z)\,\zeta(z) = \mu(T, z)$$

where $\zeta(z)$ *is the invariant distribution of* z.

Finally, since a stationary ACE is a particular type of ACE, then by the previous application of the 1st welfare theorem, the stationary version of conditions i) to iv) in Lemma 39 characterizes a stationary efficient allocation. Since the stationary version of conditions i) to iv) in Lemma 39 coincide with conditions i) to iv) of this Theorem, we have finished its the proof.

■

Proof. of Proposition 6 We first show that (15). Consider two states $T > 0$ and $T' > 0$, where T' is obtained from T by increasing the number of workers with tenure J by δ and by decreasing decreasing the number of workers with tenure 1 by δ:

$$\begin{aligned}
T_j' &= T_j \text{ for } j = 2, ..., J-1 \\
T_1' &= T_1 - \delta \text{ and } T_J' = T_J + \delta
\end{aligned}$$

It suffices to show that there is a feasible policy for T' that produces a reduction in total payoff at most by τ and thus

$$H[V(T', z)] - H[V(T, z)] \geq -\tau.$$

To establish this consider two cases, depending on whether in the original \plan more than δ workers with tenure 1 were fired or not. Let δ be a positive number smaller than $T_1/2$. In the case where more than δ workers with tenure 1 were fired, then reduce the firing of workers with tenure 1 by δ and increase the firing of workers with tenure J by δ. Then there is a reduction in current payoff of τ, and no change in the future state. In the second case, let

$$\begin{aligned}
& \frac{1}{\delta}\left(H[V(T', z)] - H[V(T, z)]\right) \\
\geq\ & \frac{1}{\delta}\beta E\left[V\left(\tilde{E}_0, ..., \tilde{E}_{J-2}, \tilde{E}_{J-1} + \tilde{E}_J, z'\right)|z\right] \\
& -\frac{1}{\delta}\beta E\left[V\left(E_0, ..., E_{J-2}, E_{J-1} + E_J, z'\right)|z\right]
\end{aligned}$$

where

$$\begin{aligned}
\tilde{E}_j &= E_j \text{ for } j = 2, ..., J-1 \\
\tilde{E}_1 &= E_1 - \delta \\
\tilde{E}_J &= E_J + \delta
\end{aligned}$$

which is feasible given the stated assumptions. Thus using the properties of directional derivatives and subgradients of concave functions

$$\begin{aligned}
& \lim_{\delta\to 0}\frac{1}{\delta}V\left(\tilde{E}_0,...,\tilde{E}_{J-2},\tilde{E}_{J-1}+\tilde{E}_J,z'\right)-V\left(E_0,...,E_{J-2},E_{J-1}+E_J,z'\right) \\
= & \min_{(V_1,...,V_J)\in\partial V}\left\{\left(V_J-V_1\right)\left(E_0,...,E_{J-2},E_{J-1}+E_J,z'\right)\right\} \\
= & -\tau+\min_{(V_1,...,V_J)\in\partial V}\left\{\left(V_J-V_1\right)\left(E_0,...,E_{J-2},E_{J-1}+E_J,z'\right)+\tau\right\} \\
= & -\tau+\min_{(V_1,...,V_J)\in\partial V}\left\{\lim_{\varepsilon\downarrow 0}\left(V_J-V_1\right)\left(E_0+\varepsilon,...,E_{J-2}+\varepsilon,E_{J-1}+E_J+\varepsilon,z'\right)+\tau\right\} \\
= & -\tau+\lim_{\varepsilon\downarrow 0}\min_{(V_1,...,V_J)\in\partial V}\left\{\left(V_J-V_1\right)\left(E_0+\varepsilon,...,E_{J-2}+\varepsilon,E_{J-1}+E_J+\varepsilon,z'\right)+\tau\right\} \\
\geq & -\tau
\end{aligned}$$

where we use theorem 24.4, page 233, of Rockafellar (1997) which shows that the graph of ∂f is closed for a concave function on $R^{n\cdot}$, the hypothesis that (13) holds for all subgradients with $T>0$, and where we denote

$$\begin{aligned}
& \left(V_J-V_1\right)\left(E_0,...,E_{J-2},E_{J-1}+E_J,z'\right) \\
\equiv & V_J\left(E_0,...,E_{J-2},E_{J-1}+E_J,z'\right)+\tau-V_1\left(E_0,...,E_{J-2},E_{J-1}+E_J,z'\right).
\end{aligned}$$

Finally since for all subgradients:

$$H\left[V\right]_J\left(T,z\right)-H\left[V\right]_1\left(T,z\right)\geq\lim_{\delta\to 0}\frac{1}{\delta}\left(H\left[V\left(T',z\right)\right]-H\left[V\left(T,z\right)\right]\right)$$

then

$$H\left[V\right]_J\left(T,z\right)-H\left[V\right]_1\left(T,z\right)\geq -\tau.$$

The argument to show that (14) follows from a similar argument, where we let

$$T'_j=T_j+\delta \text{ and } T'_{j+1}=T_{j+1}-\delta$$

for $j=1,...,J-1$. ■

Proof of Lemma 10. Let λ^* be a Lagrange multiplier, then $\lambda^*\left(T-E^*\right)=0$. Consider T', and $E'=E\left(T',z\right)$, then

$$\begin{aligned}
& H\left[V\right]\left(T,z\right)-\hat{\theta}T \\
= & \hat{R}\left(E\left(T\right),z\right) \\
\geq & \hat{R}\left(E\left(T'\right),z\right)+\lambda^*\left(E\left(T\right)-E\left(T'\right)\right) \\
\geq & \hat{R}\left(E\left(T'\right),z\right)+\lambda^*\left(T-T'\right) \\
= & H\left[V\right]\left(T',z\right)-\hat{\theta}T'+\lambda^*\left(T-T'\right)
\end{aligned}$$

thus $\hat{\theta}+\lambda^*$ is a subgradient of $H\left[V\right]$. Let $\hat{\theta}+\lambda^*$ be a subgradient of $H\left[V\right]\left(T,z\right)$. Since workers can always be sent back and get $\hat{\theta}$, then $\lambda^*\geq 0$. Also,

$$H\left[V\right]\left(T,z\right)=H\left[V\right]\left(E^*,z\right)+\hat{\theta}\left[T-E^*\right]$$

for $E^*=E\left(T,z\right)$. Then, by definition of subgradient

$$\hat{\theta}\left[T-E^*\right]=H\left[V\right]\left(T,z\right)-H\left[V\right]\left(E^*,z\right)\geq\left(\hat{\theta}+\lambda^*\right)\left(T-E^*\right)$$

or

$$0=\hat{R}\left(T,z\right)-\hat{R}\left(E^*,z\right)\geq\lambda^*\left(T-E^*\right)$$

but $E^*\leq T$ so $\lambda^*\left(T-E^*\right)=0$. This equality, together with the definition of a subgradient imply that λ^* is a Lagrange multiplier. ■

Proof of 11. Let (E^*, λ^*) be a saddle satisfying (18). Then, by theorem 2.9 in Takayama E^* is optimal. Let E^* be optimal. Then, by theorem 2.9 in Takayama there are $\lambda^* \geq 0$ such that (E^*, λ^*) is a saddle. It rests to show that $\lambda_i^* = \hat{R}_i^*$ for some subgradient. From the definition of a saddle,

$$\hat{R}(E^*, z) + \lambda^*(T - E^*) \geq \hat{R}(E, z) + \lambda^*(T - E)$$

or

$$\hat{R}(E^*, z) \geq \hat{R}(E, z) + \lambda^*(E^* - E).$$

which is the definition of a subgradient. ■

Proof of Proposition 12. Let E^* be optimal. Take any $\{H[V]_i(T, z)\}_{i=0}^J \in \partial H[V](T, z)$. By lemma 10 λ^* is a Lagrange multiplier, where

$$\{H[V]_i(T, z)\}_{i=0}^J = \lambda^* + \hat{\theta}.$$

By lemma 18, $\lambda_i^* = \hat{R}_i^*$ for some subgradient. Then $\hat{R}_i \geq 0$, and

$$0 = \hat{R}_i^*(T_i - E_i^*) = (H[V]_i(T, z) - \theta)(T_i - E_i^*).$$

Let $\{H[V]_i(T, z)\}_{i=0}^J \in \partial H[V](T, z)$, and let R_i^* be a subgradient of $\hat{R}^*$ evaluated at some $0 \leq E^* \leq T$ such that the above conditions are satisfied. Define λ^* as

$$\lambda^* = \{H[V]_i(T, z)\}_{i=0}^J - \hat{\theta} = \{R_i^*\}_{i=0}^J$$

where the last equality follow by the assumed properties. We will show that (E^*, λ^*) is a saddle. From the above conditions,

$$0 = \lambda_i^*(T_i - E_i^*)$$

Hence,

$$\hat{R}(E^*, z) + \lambda(T - E^*) \geq \hat{R}(E^*, z) + \lambda^*(T - E^*), \text{ for every } \lambda \geq 0$$

Since, by the above conditions, λ^* is a subgradient of $\hat{R}^*$ evaluated at $0 \leq E^* \leq T$, it follows that

$$\hat{R}(E, z) \leq \hat{R}(E^*, z) + \lambda^*(E - E^*), \text{ for every } E$$

Hence,

$$\hat{R}(E^*, z) + \lambda^*(T - E^*) \geq \hat{R}(E, z) + \lambda^*(T - E), \text{ for every } E$$

so that E^* is optimal.

If $E_i^* > 0$,

$$\begin{aligned}\hat{R}_i(E^*, z) &= f\left(\sum_{i=0}^J E_i^*, z\right) - \theta \\ &+ \beta \int V_{i+1}\left(E_0^*, \ldots, E_{J-2}^*, E_{J-1}^* + E_J^*, z'\right) Q(z, dz')\end{aligned}$$

follows since $\partial(g + h)(x) = \partial g(x) + \partial h(x)$, see Rockafeller, Thm 23.8 and since F is differentiable with derivative f. When $E_i^* = 0$, the subgradient of F are any numbers greater than f, and hence the previous expression hold with inequality. ■

Proof of Lemma 13. By contradiction , for all $e > 0$, there is a T, z such that

$$\sum_{i=0}^J E_i(T, z) \leq e,$$

Take $e < U$ and such that

$$f(e, \underline{z}) > \theta.$$

where $\underline{z} = min\{z : z \in Z\}$. Since $T_0 = U > 0$, $E_0(T, z) < T_0$.From 12

$$0 = [H[V]_0(T, z) - \theta][T_0 - E_0]$$

thus

$$H[V]_0(T,z) = \theta$$

but

$$H[V]_0(T,z) = \hat{R}_0(E^*,z) + \theta$$

so $\hat{R}_0(E^*,z) = 0$. Since

$$\begin{aligned} 0 &= \hat{R}_0(E^*,z) \geq f\left(\sum_{i=0}^{J} E_i^*, z\right) - \theta \\ &\quad + \beta \int V_1\left(E_0^*, \ldots, E_{J-2}^*, E_{J-1}^* + E_J^*, z'\right) Q(z, dz') \\ &> \beta \int V_1\left(E_0^*, \ldots, E_{J-2}^*, E_{J-1}^* + E_J^*, z'\right) Q(z, dz') \\ &\geq 0. \end{aligned}$$

■

Proof of Proposition 14. Let T, z be such that $T_i > 0$. Assume that $\left\{E_{j,s}^*\right\}$ is optimal. Take a subgradient $V_i(T,z) = H[V]_i(T,z)$.

First consider the case where $E_{i,s}^* = 0$, then

$$\left[H[V]_i(T,z) - \theta_i(s)\right]\left[T_{i,s} - E_{i,s}^*\right] = 0$$

thus, its unique solution is $H[V]_i(T,z) = \theta_i(s)$, provided that $T_{i,s} > 0$. Thus, as an special case, if $E_{i,0}^* = 0$, then $\theta_i(0)$ is the derivative of V.

Now consider the case where $E_{i,s}^* > 0$. We use the formulae in Proposition 12 and replace its value repeatedly, solving it forward until $\hat{\tau}_j = s$, the first time that for this cohort employment is smaller than the number of workers present at the location. Since $E_{i,s}^* > 0$ at each iteration

$$\begin{aligned} H[V]_i(T,z) &= f\left(\sum_{i=0}^{J} E_i^*, z\right) \\ &\quad + \beta \int V_{i+1}\left(E_0^*, \ldots, E_{J-2}^*, E_{J-1}^* + E_J^*, z'\right) Q(z, dz'). \end{aligned}$$

Notice that in this case, we argue above that $V_i = \theta_i(s)$. Thus, we find that unique solution of $H[V]_i(T,z)$ is $V_i^*(T,z)$. Hence the subgradient is unique, and thus $V(T,z)$ is differentiable. ■

Proof. of Proposition 17. Take (t_1, p_1) and (t_2, p_2) and consider $(t_\lambda, p_\lambda) = \left(\lambda t_1 + (1-\lambda) t_2,\ \lambda p_1 + (1-\lambda) p_2\right)$. Let the unique corresponding elements in $\mathcal{E}$ for (t_1, p_1) and (t_2, p_2) be T_1 and T_2. Consider $T_\lambda = \lambda T_1 + (1-\lambda) T_2$, which is not necessarily on $\mathcal{E}$. Let $\hat{T}_\lambda$ be the unique element in $\mathcal{E}$ that corresponds to T_λ. Note that (t_λ, p_λ) satisfies

$$t_\lambda = \sum_{j=1}^{J-1} \hat{T}_\lambda \text{ and } p_\lambda = \hat{T}_J.$$

Then,

$$\begin{aligned} & \lambda h[v](t_1, p_1, z) + (1-\lambda) h[v](t_2, p_2, z) \\ = & \lambda H[V](T_1, z) + (1-\lambda) H[V](T_2, z) \\ \leq & H[V](T_\lambda, z) \\ \leq & H[V]\left(\hat{T}_\lambda, z\right) \\ = & h[v](t_\lambda, p_\lambda, z), \end{aligned}$$

where the first equality follows from Proposition 15, the first inequality follows from concavity of V and Proposition 5 , the second inequality follows from Lemma 16, and the last equality follows from Proposition 15. ■

Proof. of Proposition 22. First define

$$\begin{aligned}\hat{R}(E,z) &= F\left(\sum_{j=0}^{J}E_j,z\right)-\theta\sum_{j=0}^{J-1}E_j-(\theta-\tau)E_J\\ &+\beta\int V(U,E_0,\ldots,E_{J-2},E_{J-1}+E_J,z')\,Q(z,dz').\end{aligned}$$

Since V and F are concave, then $\hat{R}$ is concave. Now take $\left(e_t^i,e_p^i\right)$ for $i=1,2$ and consider $\left(e_t^\lambda,e_p^\lambda\right)=\left(\lambda e_t^1+(1-\lambda)e_t^2,\lambda e_p^1+(1-\lambda\right.$ Let the unique corresponding elements to $\left(e_t^i,e_p^i\right)$in $[0,U]^J\times R_+$ that satisfies property (*) be denoted by $\tilde{E}^i$ for $i=1,2$. Define $E^\lambda=\lambda\tilde{E}^1+(1-\lambda)\tilde{E}^2$. Note that

$$\sum_{j=0}^{J-1}E_j^\lambda=e_t^\lambda \text{ and } E_J=e_p^\lambda.$$

Define $\tilde{E}^\lambda$ as the unique element in $[0,U]^J\times R_+$ that satisfies property (*) and such that

$$\sum_{j=0}^{J-1}E_j^\lambda=\sum_{j=0}^{J-1}\tilde{E}_j^\lambda \text{ and } E_J=\tilde{E}_J.$$

Then

$$\begin{aligned}&\lambda R\left(e_p^1,e_t^1,z\right)+(1-\lambda)R\left(e_p^2,e_t^2,z\right)\\ =\ &\lambda\hat{R}\left(\tilde{E}^1,z\right)+(1-\lambda)\hat{R}\left(\tilde{E}^2,z\right)\\ \le\ &\hat{R}\left(E^\lambda,z\right)\\ \le\ &\hat{R}\left(\tilde{E}^\lambda,z\right)\\ =\ &R\left(e_p^\lambda,e_t^\lambda,z\right),\end{aligned}$$

where the first equality follows construction of $\tilde{E}^i$ and since by assumption v and V satisfies (20), the first inequality follows from the concavity of $\hat{R}$, the second inequality follows by assumption (12) and Proposition 7 and its corollaries, and the last equality follows from the same argument than in Proposition 15. ■

Proof. of Proposition 25 Define the operator $\bar{h}$ as

$$\begin{aligned}\bar{h}[v](t,p,z) &= \max_{0\le e_t,0\le e_p}\{F(e_t+e_p,z)+\theta[t-e_t]+(\theta-\tau)[p-e_p]+\\ &+\beta\int v(U+M(e_t),e_p+e_t-M(e_t),z')\,Q(z,dz')\Big\}\end{aligned}$$

Comparing this problem with (**??**) the constraints $e_t\le t$ and $e_p\le p$ were removed, hence

$$h[v](t,p,z)\le\bar{h}[v](t,p,z).$$

The optimal policies e_t,e_p do not depend on t and p, thus the function $\bar{h}[v]$ is linear with derivatives

$$\begin{aligned}\bar{h}[v]_p(t,p,z) &= \theta-\tau\\ \bar{h}[v]_t(t,p,z) &= \theta\end{aligned}$$

for all t,p,z. By concavity of $h[v]$,

$$\begin{aligned}h[v](t,0,z) &\le h[v](t,p,z)+h[v]_p(0-p) \text{ or}\\ h[v](t,p,z) &\ge h[v]_p p+h[v](t,0,z)\end{aligned}$$

where $\left(h[v]_t,h[v]_p\right)\in\partial h[v](t,p,z)$. Rearranging and using the linearity of $\bar{h}[v]$:

$$h[v]_p p+h[v](t,0,z)\le h[v](t,p,z)\le\bar{h}[v](t,p,z)=\bar{h}[v](t,0,z)+[\theta-\tau]p$$

for all p. Thus by monotonicity of $h[v]$ and $\bar{h}[v]$ on t :

$$h[v]_p(t,p,z)\,p + h[v](0,0,z) \le \bar{h}[v]((J-1)U,0,z) + [\theta-\tau]\,p$$

$$\sup_{t\in[0,U(J-1)]} h[v]_p(t,p,z)\,p + h[v](0,0,z) \le \bar{h}[v]((J-1)U,0,z) + [\theta-\tau]\,p$$

Hence

$$\begin{aligned} &\lim_{p\to\infty}\inf \frac{h[v](0,0,z)-\bar{h}[v](U(J-1),0,z)}{p} \\ = \quad & 0 \le \lim\inf_{p\to\infty}\left([\theta-\tau] - \sup_{t\in[0,U(J-1)]} h[v]_p(p,t,z)\right) \end{aligned}$$

or

$$\lim_{p\to\infty}\sup\left[\sup_{t\in[0,U(J-1)]} h[v]_p(p,t,z)\right] \le \theta-\tau.$$

On the other hand, for the original problem (**??**) for (p_0,t,z). A feasible policy for $p \ge p_0$ is to set $e_p^0 = e_p(p_0,t,z)$, in which case each additional unit of p yields $\theta-\tau$. Hence the right derivative of $h[v](t,p_0,z)$ is greater or equal than $\theta-\tau$. Since $h[v]$ is concave, then $h[v]_p(t,p_0,z) \ge \theta-\tau$ for all (t,p_0,z).

Combining the two inequalities, for large enough p, $h[v]_p(p,t,z) = \theta-\tau$ for all t. ■

Proof. of Lemma 26. Consider two cases. First $e_t < (J-1)U$. In this case $M(e_t) = e_t$, which implies that

$$b(e_t,e_p,z) = \int v(U+e_t,e_p,z')\,Q(z,dz'),$$

thus

$$\begin{aligned} b_{e_t} &= \int v_t dQ, \\ b_{e_p} &= \int v_p dQ \end{aligned}$$

where

$$(v_t,v_p) \in \partial v(U+e_t,e_p,z')$$

for the corresponding elements. Second, if $e_t > (J-1)U$,

$$b(e_t,e_p,z) = \int v(JU, e_p+e_t-(J-1)U, z')\,Q(z,dz'),$$

thus

$$\begin{aligned} b_{e_t} &= \int v_p dQ, \\ b_{e_p} &= \int v_p dQ. \end{aligned}$$

Since, by assumption,

$$v_p \le v_t \le v_p + \tau,$$

we have shown the required result, except for the case where $e_t = (J-1)U$. This case follows by continuity, since the graph of the subgradient of a concave function is closed (Rockafellar, 1997, Theorem 24.4, page 233). ■

Proof. of Lemma 27. By the definition of R :

$$\begin{aligned} R_{e_p} &= f(e_t,e_p,z) - (\theta-\tau) + \beta b_{e_p}, \\ R_{e_t} &= f(e_t,e_p,z) - \theta + \beta b_{e_t} \end{aligned}$$

where

$$\left(b_{e_t}, b_{e_p}\right) \in \partial b(e_t,e_p,z).$$

Then

$$R_{e_p} - R_{e_t} = \tau + \beta \left[b_{e_p} - b_{e_t} \right] \geq \tau \left(1 - \beta \right)$$

where the inequality follows from the previous lemma. ■

Proof. of Lemma 29. The existence of $\hat{p}$ follows by the concavity of R with respect to p, the Inada conditions on F and from Proposition 25, which shows that $v_p = \theta - \tau$ for large p. The uniqueness of p follows by the strict concavity of F. That $\hat{p} < \bar{p}$ follows from concavity of R with respect of e_p and Lemma 27. ■

Proof. of Lemma 32. The first statement follows by considering the case where $T \in \mathcal{E}$ so that there is a $i \in \{1, 2, ..., J-1\}$ and T_i such that

$$T_1, ..., T_J = (U, .., U, T_i, 0, ..., 0, T_J)$$

for $T_i \in (0, U)$

$$V\left(U, ..., U, T_i, ..., 0, T_J\right) = v\left(\left(i-1\right)U + T_i, T_J\right) \text{ for all } T_i \in \left(0, U\right)$$

Thus

$$V_i\left(U, ..., U, T_i, ..., 0, T_J\right) = v_t\left(\left(i-1\right)U + T_i, T_J\right) \text{ for } i = 1, 2, ..., J-1.$$

The second and third claims follows from the form of the optimal decision rules, i.e. the definition of the range of inaction and the strict concavity of R. The third follows since, for $t \geq \hat{t}\left(p, z\right)$ it is feasible to fire any extra temporary workers, so that we know the right derivative of $h\left[v\right]$ with respect to t. ■

Proof. of Lemma 34. First we establish that $\hat{t}\left(p, z\right)$ is decreasing in p. Then we use this result, to show that $h\left[v\right]_t$ is decreasing in p.

By definition of $\hat{t}$,

$$0 \in R_{et}\left(\hat{t}\left(p, z\right), p, z\right)$$

for the case when $0 < \hat{t} < JU$. The main idea is to show that $R_{et}\left(t, p, z\right)$ is decreasing in p, and then use that, by concavity, $R_{et}\left(t, p, z\right)$ is decreasing in t.

The subgradient R_{et} is given by

$$R_{e_t}\left(t, p, z\right) = f\left(t + p, z\right) - \theta + \beta b_{et}\left(t, p, z\right)$$

where $b\left(t, p, z\right)$ is given by

$$b\left(t, p, z\right) = \int v\left(U + \min\left\{t, U\left(J-1\right)\right\}, t + p - \min\left\{t, U\left(J-1\right)\right\}, z'\right) Q\left(z, dz'\right)$$

We can then write b by cases as

$$\begin{aligned} b\left(t, p, z\right) &= \int v\left(U + t, p, z'\right) Q\left(z, dz'\right) \text{ if } t \leq U\left(J-1\right) \\ b\left(t, p, z\right) &= \int v\left(UJ, t + p - U\left(J-1\right), z'\right) Q\left(z, dz'\right) \text{ for } t \geq U\left(J-1\right) \end{aligned}$$

and hence its subgradients are

$$\begin{aligned} b_{et}\left(t, p, z\right) &= \int v_t\left(U + t, p, z'\right) Q\left(z, dz'\right) \text{ if } t < U\left(J-1\right) \\ b_{et}\left(t, p, z\right) &= \left[\int \underline{v}_p\left(UJ, p, z'\right) Q\left(z, dz'\right), \int \bar{v}_t\left(UJ, p, z'\right) Q\left(z, dz'\right)\right] \text{ if } t = U(J-1) \\ b_{et}\left(t, p, z\right) &= \int v_p\left(UJ, t + p - U\left(J-1\right), z'\right) Q\left(z, dz'\right) \text{ for } t > U\left(J-1\right) \end{aligned}$$

Now we are ready to show that $R_{et}\left(t, p, z\right)$ is strictly decreasing in p. Consider first the case where $t < U\left(J-1\right)$. In this case it follows from the hypothesis that v_t is decreasing in p and the strict concavity of f. Consider the case where $t > \left(J-1\right)U$. In this case it follows from the concavity of v, so that v_p is decreasing, and the strict concavity of f. Finally, for the case where $t = \left(J-1\right)U$, we combine the previous two arguments for the right and left derivatives.

Having established that $R_{et}\left(t, p, z\right)$ is strictly decreasing in p, then it follows that $\hat{t}$ is decreasing in p since R_{et} is decreasing in t by concavity of R.

The cases where $\hat{t} = UJ$ or $\hat{t} = 0$ are similar.

Now we turn to show that $h[v]_t$ is decreasing in p. We consider three cases. First, let $(t, p) \in Int(I(z))$. In this case,

$$\begin{aligned} h[v]_t(t,p,z) &= f(t+p,z) + \beta b_{et}(t,p,z) \\ &= R_{et}(t,p,z) + \theta \end{aligned}$$

and thus $h[v]_t$ is decreasing in p since, as shown above, $R_{et}(t,p,z)$ is decreasing in p. In the case where $(t,p) \in Int\left(I(z)^C\right)$, then $h[v]_t = \theta$, and hence $h[v]$ is differentiable, and its derivative constant, so that it is weakly decreasing in p. Finally, consider the case where (t,p,z) is such that $t = \hat{t}(p,z)$. As shown above $\hat{t}$ is weakly decreasing in p, thus for $p' > p,\ t \geq \hat{t}(p',z)$. Also, the derivative subgradient of $h[v]_t$ are

$$\left[\underline{h}[v]_t(t,p,z), \bar{h}[v]_t(t,p,z)\right] = \left[\theta,\ f(t+p,z) + \beta\bar{b}_{et}(t,p,z)\right]$$

If $\hat{t}(p',z) = \hat{t}(p,z)$, then, using the expression for the left derivative of $h[v]_t$, it follows since f is concave and since, as shown above, b_{et} is decreasing in p. If $\hat{t}(p',z) < \hat{t}(p,z)$, then, it must be that the point $\left(\hat{t}(p,z), p', z\right)$ is in the interior of the complement of the range of inaction, and thus $h[v]_t\left(\hat{t}(p,z), p', z\right) = \theta$. Thus $h[v]_t$ has decreased in this case too, since the subgradient has collapsed to its right derivative. ■

Proof. of Proposition 35. That $\hat{t}$ is decreasing in t follows using Lemma 34. Notice that starting with $V^0 = 0$ and $v^0 = 0$ satisfies all the hypothesis of this lemma. Since all these properties are preserved in the limit, they hold for the fixed point. That see that $\hat{t}$ is strictly consider first the case where $t < U(J-1)$. In this case if follows by using that in a neighborhood of those points $v(t+U,p,z')$ is differentiable with respect to t -see Proposition **??**-, that it satisfies

$$\theta = f\left(\hat{t}(p,z) + p, z\right) + \beta \int v_t\left(\hat{t}(p,z) + U, p, z'\right) Q(z, dz'),$$

that $v_t(t+U,p,z')$ is decreasing in p, and that f is strictly decreasing. A similar argument holds when $t > U(J-1)$, where

$$\theta = f\left(\hat{t}(p,z) + p, z\right) + \beta \int v_p\left(JU, p + \hat{t}(p,z) - JU, z'\right) Q(z, dz').$$

■

Proof of Theorem 3 and Proposition 4.

To prove the theorem it is convenient to define a sequential economy that corresponds to the island planning problem taking as given U, θ.. This economy has a firm whose problem corresponds to that one of the firm with value function B in the RCE and a family whose problem has solution that gives the workers value function W in the RCE.

I) We define this economy in a standard Arrow-Debreu sequential way. This definition allows to use the 1st and 2nd welfare theorem to link the allocation that solves the island sequence planning problem with an allocation that solves the firms and workers problem in the island economy as well as to link it with the equilibrium wages w.

The commodities for the sequential island economy with initial state X, z_0 is given by processes for employment by tenure E and consumption C

$$(E, C) = \left\{C_t\left(z^t\right), E_{jt}\left(z^t\right) : j = 0, ..., J,\ z^t \in Z\right\}.$$

We use g_{jt} to denote the labor choice of the firms in a sequential island problem. We use the h_t and s_t for hiring and firing of permanent workers. The net output of firms is to produce the following date t history z^t amount of consumption good

$$F\left(\sum_{j=0}^{J} g_{jt}\left(z^t\right), z_t\right) - \tau\ s_t\left(z^t\right) \tag{23}$$

The choices of g for the firms are subject to the restrictions that $g_{j,-1}(z_{-1}) = X_j$ for $j = 0, ..., J$, the law of motion of the permanent workers

$$g_{Jt}\left(z^t\right) = g_{Jt-1}\left(z^{t-1}\right) + g_{J-1t-1}\left(z^{t-1}\right) - s_t\left(z^t\right) + h_t\left(z^t\right) \tag{24}$$

and the non-negativity of hiring and firing

$$s_t\left(z^t\right) \geq 0\ ,\ h_t\left(z^t\right) \geq 0, g_{jt}\left(z^t\right) \geq 0$$

for all $j = 0, ..., J,\ z^t, t \geq 0.$

We use e to denote the labor choice and c for the consumption choice of the household in the sequential island problem. This household "owns" as an endowment a stream of U unemployed workers per period, that arrive to the island every period. The household is risk neutral in terms of consumption $c_t\left(z^t\right)$. Its decision is to assign a worker to work on the island or to permanently work outside the island, which gives value θ per worker, in units of the final good. The utility function of the household is

$$\sum_{t=0}\sum_{z^t}\beta^t Q\left(z^t|z_0\right)\times$$

$$\times\left[c_t\left(z^t\right)+\theta\left(\sum_{j=0}^{J-1}\left[e_{j-1t-1}\left(z^{t-1}\right)-e_{jt}\left(z^t\right)\right]+\left[e_{Jt-1}\left(z^{t-1}\right)+e_{J-1t-1}\left(z^{t-1}\right)-e_{Jt}\left(z^t\right)\right]\right)\right]$$

The household is subject to the following restrictions:

$$e_{j,-1}\left(z_{-1}\right) = X_j \text{ for } j = 0, ..., J$$

and for all t, z^t non-negative e_{jt} subject to:

$$\begin{aligned} e_{0,t}\left(z^t\right) &\leq U, \\ e_{jt}\left(z^t\right) &\leq e_{j-1t-1}\left(z^{t-1}\right) \text{ for } j = 1, 2, ..., J-1 \\ e_{Jt}\left(z^t\right) &\leq e_{Jt-1}\left(z^{t-1}\right)+e_{J-1t-1}\left(z^{t-1}\right) \end{aligned} \tag{25}$$

Market clearing for the sequential economy is given by

$$\begin{aligned} e_{jt}\left(z^t\right) &= g_{jt}\left(z^t\right) \\ c_t\left(z^t\right) &= F\left(\sum_{i=0}^{J} g_{it}\left(z^t\right),\ z_t\right)-\tau s_t\left(z^t\right) \end{aligned}$$

for all $j = 0, ..., J,$ and all $t,\ z^t$. Prices in this sequence island economy are given by intertemporal consumption prices, $P_t\left(X, z^t\right)$ and wages by tenure $w_{jt}\left(X, z^t\right)$ in terms of date t history z^t consumption goods. Given the household preferences for consumption we impose

$$P_t\left(X, z^t\right) = \beta^t Q\left(z^t|z_0\right)$$

With these prices the problem for the firm is to maximize profits, i.e.

$$\begin{aligned} & B_0\left(x_J, X, z_0\right) \\ = \ & \max_{\{g\}}\sum_{t=0}\beta^t\sum_{z^t}\left[F\left(\sum g_{jt}\left(z^t\right), z_t\right)-\sum_{j=0}^{J} g_{jt}\left(z^t\right)w_{jt}\left(X, z^t\right)-\tau s_t\left(z^t\right)\right]Q\left(z^t|z_0\right) \end{aligned}$$

subject to

$$g_{J-1} = x_J$$

and the law of motion for s, h and g. Let $\beta^t\xi_t\left(z^t\right)Q\left(z^t|z_0\right)$ be the multiplier of the restriction (24). The first order conditions for the firm's problem are:

$$f\left(\sum g_{jt}\left(z^t\right), z_t\right)-w_{jt}\left(X, z^t\right) \leq 0$$

for $j = 0, ..., J-2$ with equality if $g_{jt}\left(z^t\right) > 0$. For $j = J-1$

$$f\left(\sum g_{jt}\left(z^t\right), z_t\right)-w_{J-1t}\left(X, z^t\right)+\beta\sum_{z_{t+1}}\hat{\xi}_{t+1}\left(z^t, z_{t+1}\right)Q\left(z_{t+1}|z_t\right) \leq 0$$

with equality if $g_{J-1t}(z^t) > 0$. For $j = J$

$$\hat{\xi}_t(z^t) = f\left(\sum_{j=0}^{J} g_{jt}(z^t), z_t\right) - w_{Jt}(X, z^t) + \beta \sum_{z_{t+1}} \hat{\xi}_{t+1}(z^t, z_{t+1}) Q(z_{t+1}|z_t) \tag{26}$$

if $g_{Jt}(z^t) > 0$. The first order conditions for $h_t(z^t)$ is

$$\hat{\xi}_t(z^t) \leq 0$$

with equality if $h_t(z^t) > 0$. The first order conditions for $s_t(z^t)$ is

$$-\tau - \hat{\xi}_t(z^t) \leq 0$$

with equality if $s_t(z^t) > 0$.The last three inequalities imply (26). The slackness condition for (24) gives:

$$g_{J,t-1}(z^{t-1}) + g_{J-1,t-t}(z^{t-1}) > g_{J,t}(z^t) \text{ then } \hat{\xi}_t(z^t) = -\tau$$

$$g_{J,t-1}(z^{t-1}) + g_{J-1,t-t}(z^{t-1}) < g_{J,t}(z^t) \text{ then } \hat{\xi}_t(z^t) = 0.$$

Now we turn to the household problem in a sequential island economy. Letting $\beta^t Q(z^t|z_0)\hat{\nu}_{j,t}(z^t)$ be the Lagrange multiplier for (25) the first order conditions of the household problem are equivalent to

$$W_{jt}(X, z^t) = \max\left\{\theta,\ w_{jt}(X, z^t) + \beta \sum_{z_{t+1}} W_{j+1t+1}(X, z^t, z_{t+1}) Q(z_{t+1}|z_t)\right\},$$

for $j = 0, ..., J-1$ and

$$W_{Jt}(X, z^t) = \max\left\{\theta,\ w_{Jt}(X, z^t) + \beta \sum_{z_{t+1}} W_{Jt+1}(X, z^t, z_{t+1}) Q(z_{t+1}|z_t)\right\}$$

where

$$W_{jt}(X, z^t) = \hat{v}_{jt}(X, z^t) + \theta$$

with slackness

$$e_{jt}(z^t) < e_{j-1t-1}(z^{t-1}) \text{ then } W_{jt}(X, z^t) = \theta,$$

and $e_{jt}(z^t) > 0$,

$$W_{jt}(X, z^t) = w_{jt}(X, z^t) + \beta \sum_{z_{t+1}} W_{j+1t+1}(X, z^t, z_{t+1}) Q(z_{t+1}|z_t)$$

for $j = 0, ..., J-1$ and analogously for $j = J$.

To see why this is the case, write the Lagrangian of the household problem as

$$\sum_{t=0} \beta^t \sum_{z^t} Q(z^t|z_0) \times$$

$$\begin{aligned}
&\Bigg\{\sum_{j=0}^{J} e_{jt}(z^t) w_{jt}(X, z^t) + \\
&\theta\left(\sum_{j=0}^{J-1}\left[e_{j-1t-1}(z^{t-1}) - e_{jt}(z^t)\right] + \left[e_{J\ t-1}(z^{t-1}) + e_{J-1t-1}(z^{t-1}) - e_{Jt}(z^t)\right]\right) \\
&+\hat{\nu}_{0t}(z^t)\left[U - e_{0t}(z^t)\right] + \\
&+\sum_{j=1}^{J-1} \hat{\nu}_{jt}(z^t)\left[e_{j-1t-1}(z^{t-1}) - e_{jt}(z^t)\right] + \hat{\nu}_{jt}(z^t)\left[e_{J-1t-1}(z^{t-1}) + e_{Jt-1}(z^{t-1}) - e_{Jt}(z^t)\right]\Bigg\}
\end{aligned}$$

The first order conditions of this problem are as follows. For $e_{jt}(z^t)$:

$$w_{jt}(X,z^t) - (\theta + \hat{\nu}_{jt}(z^t)) + \beta \sum_{z_{t+1}} (\theta + \hat{v}_{j+1\ t+1}(z^t, z_{t+1})) Q(z_{t+1}|z_t) \leq 0$$

with = if $e_{jt}(z^t) > 0$ for $j = 0, 1, ..., J-1$ and ,

$$w_{Jt}(X,z^t) - (\theta + \hat{\nu}_{Jt}(z^t)) + \beta \sum_{z_{t+1}} (\theta + \hat{v}_{J\ t+1}(z^t, z_{t+1})) Q(z_{t+1}|z_t) \leq 0$$

with = if $e_{Jt}(z^t) > 0$.

The slackness conditions are: if $e_{jt}(z^t) < e_{j-1t-1}(z^{t-1})$ then $\hat{\nu}_{jt}(z^t) = 0$ for $j = 0, ..., J-1$ and if $e_{Jt}(z^t) < e_{J-1t-1}(z^{t-1}) + e_{J-1t-1}(z^{t-1})$ then $\hat{\nu}_{Jt}(z^t) = 0$.

To compare a competitive equilibrium with the planning problems it helps define a sequential island planning problem. In this problem the planner maximizes the expected discounted value of net output (23) subject to the feasibility constraints (24) and (25). This is the sequential version of the recursive island planning problem. Let $V_0(X, z_0)$ be the value attained by this planning problem.

Let $\beta^t \xi_t(z^t) Q(z^t|z_0)$ be the multiplier of the constraint (24) and $\beta^t \nu_{jt}(z^t) Q(z^t|z_0)$ the multiplier of the constraints (25). The first order conditions of the island sequential planning problem are equivalent to:

$$\theta + \nu_{jt}(z^t) = \max \left\{ \theta,\ f\left(\sum E_{jt}(z^t), z_t\right) + \beta \sum_{z_{t+1}} (\theta + v_{j+1\ t+1}(z^t, z_{t+1})) Q(z_{t+1}|z_t) \right\}$$

with $\nu_{jt}(z^t) = 0$ if $E_{jt}(z^t) < E_{j-1t-1}(z^{t-1})$ and

$$\theta + \nu_{jt}(z^t) = f\left(\sum E_{jt}(z^t), z_t\right) + \beta \sum_{z_{t+1}} (\theta + v_{j+1\ t+1}(z^t, z_{t+1})) Q(z_{t+1}|z_t)$$

if $E_{jt}(z^t) > 0$ for $j = 0, .., J-2$. For $j = J-1$ we have

$$\begin{aligned} & \theta + \nu_{J-1t}(z^t) \\ = \ & \max \left\{ \theta,\ f\left(\sum E_{jt}(z^t), z_t\right) + \beta \sum_{z_{t+1}} (\theta + v_{J\ t+1}(z^t, z_{t+1}) + \xi_{t+1}(z^t, z_{t+1})) Q(z_{t+1}|z_t) \right\} \end{aligned}$$

with $v_{J-1t}(z^t) = 0$ if $E_{J-1t}(z^t) < E_{J-1t-1}(z^{t-1})$ and

$$\begin{aligned} & \theta + \nu_{J-1t}(z^t) \\ = \ & f\left(\sum E_{jt}(z^t), z_t\right) + \beta \sum_{z_{t+1}} (\theta + v_{J\ t+1}(z^t, z_{t+1}) + \xi_{t+1}(z^t, z_{t+1})) Q(z_{t+1}|z_t) \end{aligned}$$

if $E_{J-1t}(z^t) > 0$. For $j = J$ we have

$$\begin{aligned} & \theta + \nu_{Jt}(z^t) + \xi_t(z^t) \\ = \ & \max \left\{ \theta,\ f\left(\sum E_{jt}(z^t), z_t\right) + \beta \sum_{z_{t+1}} (\theta + v_{J\ t+1}(z^t, z_{t+1}) + \xi_{t+1}(z^t, z_{t+1})) Q(z_{t+1}|z_t) \right\} \end{aligned}$$

with $v_{Jt}(z^t) = 0$ if $E_{Jt}(z^t) < E_{Jt-1}(z^{t-1}) + E_{J-1t-1}(z^{t-1})$ and

$$\begin{aligned} & \theta + \nu_{Jt}(z^t) + \xi_t(z^t) \\ = \ & f\left(\sum E_{jt}(z^t), z_t\right) + \beta \sum_{z_{t+1}} (\theta + v_{J\ t+1}(z^t, z_{t+1}) + \xi_{t+1}(z^t, z_{t+1})) Q(z_{t+1}|z_t) \end{aligned}$$

if $E_{Jt}(z^t) > 0$.

To see why this is the case, write the Lagrangian for the planning problem is:

$$V_0(X, z_0) = \max_{\{E\}} \sum_{t=0} \beta^t \sum_{z^t} Q(z^t|z_0) \times$$

$$
\begin{aligned}
&\Big\{F\left(\sum E_{jt}\left(z^t\right), z_t\right) - \tau S_t\left(z^t\right) \\
&+\theta\left(\sum_{j=0}^{J-1}\left[E_{j-1t-1}\left(z^{t-1}\right) - E_{jt}\left(z^t\right)\right] + \left[E_{J\ t-1}\left(z^{t-1}\right) + E_{J-1t-1}\left(z^{t-1}\right) - E_{Jt}\left(z^t\right)\right]\right) \\
&+\nu_{0t}\left(z^t\right)\left[U - E_{0t}\left(z^t\right)\right] + \sum_{j=1}^{J-1}\nu_{jt}\left(z^t\right)\left[E_{j-1}\left(z^{t-1}\right) - E_{jt}\left(z^t\right)\right] \\
&+\nu_{Jt}\left(z^t\right)\left[E_{J-1t-1}\left(z^{t-1}\right) + E_{Jt-1}\left(z^{t-1}\right) - E_{Jt}\left(z^t\right)\right] \\
&\xi_t\left(z^t\right)\left[-E_{Jt}\left(z^t\right) + E_{Jt-1}\left(z^{t-1}\right) + E_{J-1t-1}\left(z^{t-1}\right) - S_t\left(z^t\right) + H_t\left(z^t\right)\right]\Big\}
\end{aligned}
$$

The f.o.c. are:

$$
f\left(\sum E_{jt}\left(z^t\right), z_t\right) - \theta - \nu_{jt}\left(z^t\right) + \beta \sum_{z_{t+1}}\left(\theta + v_{j+1\ t+1}\left(z^t, z_{t+1}\right)\right) Q\left(z_{t+1}|z_t\right) \leq 0
$$

with equality if $E_{jt}\left(z^t\right) > 0$.

$$
\begin{gathered}
f\left(\sum E_{jt}\left(z^t\right), z_t\right) - \theta - \nu_{J-1t}\left(z^t\right) + \\
\beta \sum_{z_{t+1}}\left(\theta + v_{J\ t+1}\left(z^t, z_{t+1}\right)\right) Q\left(z_{t+1}|z_t\right) + \beta \sum_{z_{t+1}} \xi_{t+1}\left(z^t, z_{t+1}\right) Q\left(z_{t+1}|z_t\right) \leq 0
\end{gathered}
$$

with equality if $E_{J-1t}\left(z^t\right) > 0$

$$
\begin{gathered}
f\left(\sum E_{jt}\left(z^t\right), z_t\right) - \theta - \xi_t\left(z^t\right) - \nu_{Jt}\left(z^t\right) + \\
\beta \sum_{z_{t+1}}\left(\theta + v_{J\ t+1}\left(z^t, z_{t+1}\right)\right) Q\left(z_{t+1}|z_t\right) + \beta \sum_{z_{t+1}} \xi_{t+1}\left(z^t, z_{t+1}\right) Q\left(z_{t+1}|z_t\right) \leq 0
\end{gathered}
$$

with equality if $E_{Jt}\left(z^t\right) > 0$.

The first order condition for $H_t\left(z^t\right)$ is

$$
\xi_t\left(z^t\right) \leq 0
$$

with equality if $H_t\left(z^t\right) = 0$. The first order condition for $S_t\left(z^t\right)$ is

$$
-\tau - \xi_t\left(z^t\right) \leq 0
$$

with equality if $S_t\left(z^t\right) > 0$.

(II) We now show i), the 1st welfare theorem, and iii). We start with an island RCE $\{w, W, B, G\}$. Pick an arbitrary state $(T, z) = (X, z_0)$ in the support of μ. We construct the sequential CE with (X, z_0) as initial condition as follows. Let wages be:

$$
w_{jt}\left(X, z^t\right) = w_j\left(D^{t-1}\left(X, z^{t-1}\right), z_t\right).
$$

and let multipliers and employment be

$$
\begin{aligned}
\theta + \hat{v}_{jt}\left(X, z^t\right) &= W_j\left(D^{t-1}\left(X, z^{t-1}\right), z_t\right) \\
e_{jt}\left(X, z^t\right) &= G_j\left(D^{t-1}\left(X, z^{t-1}\right), z_t\right)
\end{aligned}
$$

where

$$
\begin{aligned}
D^t\left(X, z^t\right) &= G\left(T, z_t\right) \quad \text{for} \\
T &= \left(U, D_0^{t-1}\left(X, z^{t-1}\right), ..., D_{J-2}^{t-1}\left(X, z^{t-1}\right), D_{J-1}^{t-1}\left(X, z^{t-1}\right) + D_J^{t-1}\left(X, z^{t-1}\right)\right) \text{ and} \\
D^{-1}\left(X, z_0\right) &= X
\end{aligned} \tag{27}
$$

It is immediate to verify that $\{e, \hat{\nu}\}$ solves the f.o.c. of the household problem in a sequential island equilibrium, and hence it solves the household problem. For future reference we define

$$
W_{j0}\left(X, z_0\right) = \hat{\nu}_{j0}\left(z_0, X\right) + \theta.
$$

Define the Lagrange multiplier and employment for the firms problem as:

$$\begin{aligned}\hat{\xi}_t\left(X, z^t\right) &= B_p\left(D_J^{t-1}\left(X, z^{t-1}\right), D^{t-1}\left(X, z^{t-1}\right), z_t\right) \\ g_{jt}\left(X, z^t\right) &= G_j\left(D^{t-1}\left(X, z^{t-1}\right), z_t\right)\end{aligned}$$

It is immediate to verify that $\left\{g, \hat{\xi}\right\}$ solves the firms order conditions of the firm's problem in an island sequential CE, and hence it solves the firm's problem. Let B_0 be the value of the firm in the sequential island CE. For future reference, from the envelope theorem, we have

$$\partial B_0\left(x_J, X, z_0\right)/\partial x_J = \hat{\xi}_0\left(X, z_0\right)$$

evaluated at $x_J = X_J$.

By the 1st welfare thm. applied to the sequential island economy, $\{e\} = \{g\}$ is a P.O. allocation, and hence solves the sequential island planning problem. By inspection, the Lagrange multipliers $\{\xi_t, \nu_{jt}\}$ that satisfy the first order conditions of the sequential planning problem are identical to the Lagrange multipliers for the firm's problem $\left\{\hat{\xi}_t\right\}$ and to the Lagrange multipliers $\{\hat{\nu}_{jt}\}$ of the households problem in the sequential CE. From these first order conditions:

$$W_{j0}^{*}\left(X, z_0\right) = \hat{\nu}_{j0}\left(z_0, X\right) + \theta = \nu_{j0}\left(z_0, X\right) + \theta = \partial V_0\left(X, z_0\right)/\partial X_j$$

for $j = 0, ..., J-1$ and

$$\begin{aligned}W_{J0}\left(X, z_0\right) + \partial B_0\left(x_J, X, z_0\right)/\partial x_J &= \hat{\nu}_{J0}\left(X, z_0\right) + \theta + \hat{\xi}_0\left(X, z_0\right) \\ &= \nu_{J0}\left(X, z_0\right) + \theta + \xi_0\left(X, z_0\right) = \partial V_0\left(X, z_0\right)/\partial X_J\end{aligned}$$

evaluated at $x_J = X_J$. The allocation described by G is, by hypothesis, recursive, so it solves the recursive island planning problem with initial condition X, z_0. Repeating this argument for each initial condition (X, z_0) we show that

$$\begin{aligned}V_0\left(T, z\right) &= V\left(T, z\right), \\ B_0\left(T_J, T, z\right) &= B\left(T_J, T, z\right), \\ W_{j0}\left(T, z\right) &= W_j\left(T, z\right)\end{aligned}$$

for all $(T, z.)$ Hence we have shown the first welfare theorem for the recursive representation of the island problem, and that (6), condition iii) of the theorem, holds.

(III). We now show ii), the 2nd welfare theorem, condition iii) of Theorem 3 and condition (b) of Proposition 4. We start with a solution of the recursive planning problem, and with $\nu\left(T, z\right)$ and $\xi\left(T, z\right)$ which, by the envelope theorem satisfy

$$\frac{\partial V\left(T, z\right)}{\partial T_j} = \theta + \nu_j\left(T, z\right).$$

for $j = 0, ..., J-1$ and

$$\frac{\partial V\left(T, z\right)}{\partial T_J} = \theta + \nu_J\left(T, z\right) + \xi\left(T, z\right)$$

If if were the case that there are more than one pair ν_J, ξ for a given T, z, utilize a selection that only depends on (T, z). >From the principle of optimality, the solution of the recursive island problem V is the same as the value function for the sequential island problem V_0, so that $V\left(T, z\right) = V_0\left(T, z\right)$.

Choose any initial state X, z_0 to be used as initial condition to the sequential island problem. Define

$$\begin{aligned}\nu_{jt}\left(X, z^t\right) &= \nu_j\left(D^{t-1}\left(X, z^{t-1}\right), z_t\right) \\ \xi_t\left(X, z^t\right) &= \xi\left(D^{t-1}\left(X, z^{t-1}\right), z_t\right) \\ E_{jt}\left(X, z^t\right) &= G\left(D^{t-1}\left(X, z^{t-1}\right), z_t\right)\end{aligned}$$

where D^{t-1} is defined as in (27) using the optimal decision rule from the recursive planning problem. By comparing the first order conditions of the recursive island planing problem with the first order conditions of the sequential island planning problem, it can be seen that $\{E_t, \nu_{jt}, \xi_t\}$ so defined solve the f.o.c. of the sequence island's planning problem. Next define wages as follows:

$$w_{jt}\left(X, z^t\right) = f\left(\sum_{j=0}^{J} E_{jt}\left(z^t, X\right), z_t\right) \tag{28}$$

for $j = 0, 1, 2, ..., J-2$, for $j = J-1$ let

$$w_{J-1t}\left(X, z^t\right) = f\left(\sum_{j=0}^{J} E_{jt}\left(X, z^t\right), z_t\right) + \beta \sum_{z^{t+1}} \xi_{t+1}\left(X, z^t, z_{t+1}\right) Q\left(z_{t+1}|z_t\right) \tag{29}$$

Finally, for $j = J$

$$w_{Jt}\left(X, z^t\right) = f\left(\sum_{j=0}^{J} E_{jt}\left(X, z^t\right), z_t\right) - \xi_t\left(X, z^t\right) + \beta \sum_{z^{t+1}} \xi_{t+1}\left(X, z^t, z_{t+1}\right) Q\left(z_{t+1}|z_t\right) \tag{30}$$

The function $B_0\left(x_J, X, z_0\right)$ is defined as the solution of the firm problem for wages w_{jt} in the sequential island equilibrium.Given wages w_{jt}, the functions W_{jt} are defined as:

$$W_{jt}\left(X, z^t\right) = \nu_{jt}\left(X, z^t\right) + \theta \tag{31}$$

for $j = 0, ..., J$.

Define the candidate multipliers for the sequential firm problem as $\hat{\xi}_t = \xi_t$. Given wages w_{jt}, and multipliers $\hat{\xi}_t$, it is easy to verify that the allocation $g_{jt} = E_{jt}$, and its implied $\{s_t, h_t\}$ solve the first order conditions of the firms in the island sequential economy. To verify this, one uses the first order conditions for the island planner problem in the island sequential economy. >From the envelope condition it is immediate that

$$\partial B_0\left(x_J, X, z_0\right)/\partial x_J = \hat{\xi}_0\left(X, z_0\right)$$

where $x_J = X_J$.

Define the candidate multipliers for the sequential household problem $\hat{v}_{jt} = v_{jt}$.Given wages w_{jt} and multipliers $\hat{v}_{jt}$ it is easy to verify that the allocation $e_{jt} = E_{jt}$ solve the first order conditions of the household problem in the island sequential economy. To verify this, one uses the first order conditions for the island planner problem in the island sequential economy.

Thus we have established that the sequential allocation constructed out of the solution of the recursive island planning problem from an initial state X, z_0 can be decentralized as a sequential island competitive equilibrium. Finally, we define the elements of the recursive competitive island equilibrium as follows:

$$\begin{aligned} w_j\left(X, z_0\right) &= w_{j0}\left(X, z_0\right), \\ W_j\left(X, z_0\right) &= W_{j0}\left(X, z_0\right), \\ B\left(X_J, X, z_0\right) &= B_0\left(X_J, X, z_0\right) \end{aligned}$$

By repeating this construction for all (X, z_0) in the support of μ, we construct the functions w, W and B. These functions constitute a RCE since they are constructed from the sequential island competitive equilibrium.

>From the previous arguments we have:

$$\begin{aligned} \frac{\partial V\left(T, z\right)}{\partial T_j} &= \theta + \nu_j\left(T, z\right) = \theta + v_{j0}\left(z, T\right) \\ &= \theta + \hat{\nu}_{j0}\left(z, T\right) = W_{j0}\left(T, z\right). \end{aligned} \tag{32}$$

for $j = 0, ..., J-1$ and

$$\begin{aligned} \frac{\partial V\left(T, z\right)}{\partial T_J} &= \theta + \nu_J\left(T, z\right) + \xi\left(T, z\right) = \theta + \nu_{J0}\left(, z\right) + \xi_0\left(z, T\right) \\ &= \theta + \hat{\nu}_{J0}\left(z, T\right) + \hat{\xi}_0\left(z, T\right) = W_{J0}\left(T, z\right) + \frac{\partial}{\partial x_J} B_0\left(T_J, T, z\right), \end{aligned} \tag{33}$$

and thus condition iii) is satisfied.

(IV) We establish conditions (b) of Proposition 4. Since in (II) and (III) we have shown the 1st and 2nd welfare theorems, we can, without loss of generality, start with an efficient allocation and examine the equilibrium wages w that we constructed in (III) in equations (28), (29) and (30). The multiplier $\xi \in [-\tau, 0]$ and $\xi_t(X, z^t) = -\tau$ if $S_t(z^t) > 0$, i.e. if permanent workers are being fired. Thus, the inequalities in (b) follows from these definitions and the properties of ξ.

(V). We establish condition c) of Proposition 4. Since in (II) and (III) we have shown the 1st and 2nd welfare theorems, we can, without loss of generality, start with an efficient allocation and examine the equilibrium value function for workers W that we constructed in (III) in equation (31). Using equations (32), (33) we have that

$$\begin{aligned}\frac{\partial V(T,z)}{\partial T_j} &= W_j(T,z), \text{ for } j = 0, ..., J-1 \\ \frac{\partial V(T,z)}{\partial T_J} &= W_J(T,z) + \xi(T,z) \ .\end{aligned}$$

In Proposition (6) we have shown that

$$V_{T_1} \geq V_{T_2} \geq \cdots \geq V_{T_{J-1}},$$

Thus

$$W_1 \geq W_2 \geq \cdots \geq W_{J-1}.$$

Finally since W are part of an equilibrium, they satisfy

$$\begin{aligned}W_{J-1}(T,z) &= w_{J-1}(T,z) + \beta E[W_J(A(T,z), z') | z] \\ W_J(T,z) &= w_J(T,z) + \beta E[W_J(A(T,z), z') | z]\end{aligned}$$

and since we have already established (c), $w_J \geq w_{J-1}$, and thus we have $W_{J-1} \leq W_J$. This finishes the proof of IV). ■

Appendix D: Definition of Auxiliary Competitive Equilibrium ("ACE").

This appendix defines the competitive equilibrium (ACE) used in the proof of Theorem 1. There are two types of firms, type I and II, and families. The are as many markets to "buy" and "sell" workers as islands of type z^t, X.

Preferences of the family.

The families own all firms of both type and consume final consumption goods. They are risk neutral, and discount at rate β.

$$\sum_t \beta^t \sum_{z^t} C_t\left(z^t\right) q_t\left(z^t\right)$$

Notice that firms do not "own" they all labor. The "labor" is allocated initially to the two types of firms.

To simplify the notation we anticipate that, given the risk neutrality of households, the price for final goods sold at date t, state z^t is $\beta^t q_t\left(z^t\right)$.

Problem of Firms type I.

There is a continuum of firms of type I in each island of type X by "buying" workers from a central location at price $\lambda_t\left(z^t, Z\right)$. They start at period $t = 0$ with a profile of workers given by their type X. Workers that are "bought" in this period are given tenure $j = 0$. The operate the technology F. They can sale workers to the central location they obtained a price $\theta_t\left(z^t, X\right)$. If they "sale" workers with tenure J or higher in the island, they lose τ per worker.

The sequence problem for the firms in the islands who “buys” workers at price λ_t and sell them at price θ_t. He also pays the separation cost τ.

For each $(X|z_0)$ they maximize:

$$\sum_{t=0} \beta^t \sum_{z^t} \left\{ F\left(\sum_{j=1}^{J} E_{j,t}\left(z^t, X\right), z_t\right) - T_{0,t}\left(z^t, X\right)\lambda_t\left(z^t, X\right) \right\} q_t\left(z^t\right) \tag{34}$$

$$+\sum_{t=0} \beta^t \sum_{z^t} \left\{ \sum_{j=0}^{J} \left[T_{j,t}\left(z^t, X\right) - E_{j,t}\left(z^t, X\right)\right]\theta_t\left(z^t, X\right) - \left[T_{J,t}\left(z^t, X\right) - E_{J,t}\left(z^t, X\right)\right]\tau \right\} q_t\left(z^t\right)$$

by choice of $\{E_{jt}, T_{jt}\}_{t\geq 0}$ subject to to the technological constraints in hiring and firing:

$$E_{j,t}\left(z^t, X\right) \leq T_{j,t}\left(z^t, X\right) \text{ for } j = 0, 1, .., J$$

$$T_{j,t}\left(z^t, X\right) = E_{j-1,t-1}\left(z^{t-1}, X\right) \text{ for } j = 1, 2, ..., J-1$$

$$T_{J,t}\left(z^t, X\right) = E_{J-1,t-1}\left(z^{t-1}, X\right) + E_{J,t-1}\left(z^{t-1}, X\right)$$

given initial conditions

$$T_{j,0}\left(z^0, X\right) = X_j \text{ for } j = 1, 2, ..., J.$$

Problem for firms type II.

The sequence problem for the firms that produce home goods and reallocation of workers. They sell workers to each islands, subject to the undirected search technology, and buy them back workers from islands. The firms also operate the home production technology. "Purchases" are denoted by $S_{jt}\left(z^t, X\right)$ with price $\theta_t\left(z^t, X\right)$ and "sales" are denoted by $Y_t\left(z^t, X\right)$ at the price $\lambda_t\left(z^t, X\right)$.

Firms type II maximize:

$$\sum_t \beta^t \omega L_t + \sum_t \beta^t \sum_{z^t} \sum_X Y_t\left(z^t, X\right)\lambda_t\left(z^t, X\right)\eta\left(X|z_0\right) q_t\left(z^t\right) \tag{35}$$

$$-\sum_t \beta^t \sum_{z^t} \sum_X \left[\sum_{j=0}^{J} S_{j,t}\left(z^t, X\right)\right]\theta_t\left(z^t, X\right)\eta\left(X|z_0\right) q_t\left(z^t\right)$$

by choice of $\{Y_t, S_{jt}, U_t\}_{t\geq 0}$ subject to the undirected search technology, so that they cannot sell different quantities to different islands, which is written as

$$U_{t-1} = Y_t\left(z^t, X\right) \text{ for all } t, z^t, X.$$

and the flow constraint stating that workers "bought" can be allocated to either increase the stock producing at home or to search:

$$U_t + L_t - L_{t-1} \leq \sum_x \sum_{z^t} \sum_{j=0}^{J} S_{jt}\left(z^t, X\right)\eta\left(X|z_0\right) q_t\left(z^t\right) \text{ for all } t$$

and where U_{-1} and L_{-1} are given.

Market clearing:

For final goods:

$$\begin{aligned} & \tau \sum_{x} \sum_{z^t} \left[T_{J,t}\left(z^t, X\right) - E_{J,t}\left(z^t, X\right) \right] q_t\left(z^t\right) \eta\left(X|z_0\right) + C_t \\ = \quad & L_t \omega + \sum_{x} \sum_{z^t} F\left(\sum_{j=1}^{J} E_{j,t}\left(z^t, X\right), z_t \right) q_t\left(z^t, X\right) \eta\left(X|z_0\right) \end{aligned}$$

for the market of new (tenure $j = 0$) workers:

$$U_{t-1} = T_{0,t}\left(z^t, X\right) \text{ for all } t, z^t, X$$

for the market of incumbent (tenure $j > 0$) workers:

$$S_{j,t}\left(z^t, X\right) = E_{j,t}\left(z^t, X\right) - T_{j,t}\left(z^t, X\right) \text{ for all } j, t, z^t, X.$$

Appendix E: Lagrangian for the Recursive Island Planning Problem.
It is helpful to rewrite the Recursive Island Planning Problem using the Lagrange ν and ξ for the constraints:

$$V(T,z) = \max_{g\geq 0, s\geq 0, h\geq 0} \min_{\nu\geq 0,\xi\geq 0} \{ F\left(\sum g_j, z\right) - \tau s + \beta \sum_{z'} V(U, g_0, g_1, ..., g_{J-2}, g_{J-1} + g_J) Q(z'|z)$$

$$+\theta\left(\sum_{j=0}^{J-1}[T_j - g_j] + [T_J - g_J]\right)$$

$$+\nu_0 [U - g_0] + \sum_{j=1}^{J-1} \nu_j [T_j - g_j] + \nu_J [T_J - g_J]$$

$$\xi [-g_J + T_J - s + h] \}$$

It is immediate to obtain the following envelope conditions:

$$\frac{\partial V(T,z)}{\partial T_J} = \theta + \nu_J + \xi$$

for $j = 0, ..., J-1$

$$\frac{\partial V(T,z)}{\partial T_j} = \theta + \nu_j.$$

Appendix F: Binding contracts and tenure at the firm level (a formal description)

There are competitive markets in the island. At each date t and event z^t the set of commodities traded is $S(z^t)$. A commodity $s \in S(z^t)$ is a stopping time indicating the time at which a worker will be dismissed under each possible continuation sequence $z_{t+1}^{\infty} = \{z_{t+1}, z_{t+2}, ...\}$ following the history z^t. Formally, $S(z^t)$ is the set of all functions

$$s\left(z^t; z_{t+1}^{\infty}\right) : Z^{\infty} \to \{t+1, t+2, ..., \infty\}$$

satisfying that

$$\begin{aligned} s\left(z^t; z_{t+1}^{\infty}\right) &= T \Rightarrow s\left(z^t; \hat{z}_{t+1}^{\infty}\right) = T, \\ \text{for all } \hat{z}_{t+1}^{\infty} \text{ such that } &: \{z_{t+1}, z_{t+2}, ..., z_T\} = \{\hat{z}_{t+1}, \hat{z}_{t+2}, ..., \hat{z}_T\}. \end{aligned}$$

When a worker arrives for the first time to the island at date and event z^t, he is a "newly arrived worker" and can supply only one stopping time in the set $S(z^t)$. The worker cannot supply a new stopping time before the previous stopping time is actually executed, i.e. before the worker is separated from his previous employer. The first time that the worker separates he becomes an "incumbent worker" for the rest of his stay in the island. An incumbent worker at date and event z^t can also supply any one stopping time in the set $S(z^t)$ as long as he has no outstanding stopping time from a previous sale. "Newly arrived workers" and "incumbent workers" sell different commodities, though. The stopping time sold by an "incumbent worker" at date and event z^t entails a cost τ at date $s\left(z^t; z_{t+1}^{\infty}\right)$, for every possible realization z_{t+1}^{∞}. On the contrary, the stopping time sold by a "newly arrived worker" at date and event z^t entails a cost τ at date $s\left(z^t; z_{t+1}^{\infty}\right)$, only if the realization z_{t+1}^{∞} is such that $s\left(z^t; z_{t+1}^{\infty}\right) \geq t + J$.

Each stopping time, being a different commodity, has a price associated with it. We express the price of the stopping times traded at time and event z^t in terms of the final consumption good at that time and event, and denote them for each $s \in S(z^t)$ by $P^A(z^t, s)$ and $P^I(z^t, s)$ for the "newly arrived" and "incumbent" stopping times, respectively. Workers and firms take the prices $P^A(z^t, s)$ and $P^I(z^t, s)$ for all $t \geq 0$, $z^t \in Z^t$, and $s \in S(z^t)$ as given.

The problem of an "incumbent" worker at time and event z^t, if she has no outstanding stopping times at the time, is the following:

$$I\left(z^t\right) = \max\left\{\theta, \max_{s \in S(z^t)} \left\{P^I\left(z^t, s\right) + E\left[\beta^{s-t} I\left(z^s\right)\right]\right\}\right\} \tag{36}$$

where the expectation is taken with respect to all possible realizations $z_{t+1}^{\infty} = \{z_{t+1}, z_{t+2}, ...\}$, conditional on z^t. This equation states that an incumbent worker can choose to leave the island, obtaining θ, or sell the stopping time $s \in S(z^t)$ that provides the highest value. A stopping time $s \in S(z^t)$ provides $P^I(z^t, s)$ units of the consumption good during the current period and the value $I(z^s)$ of being an incumbent worker at the (random) stopping time s. Observe that, since the worker maximizes the present expected value of his earnings, equation (36) implicitly assumes linear preferences.[11]

The problem of "a newly arrived worker" at time t state z^t is given by

$$A\left(z^t\right) = \max\left\{\theta, \max_{s \in S(z^t)} \left\{P^A\left(z^t, s\right) + E\left[\beta^{s-t} I\left(z^s\right)\right]\right\}\right\}.$$

This problem is analogous to the "incumbent" worker problem, except that the "newly arrived worker faces a different price for the stopping time that she sells and becomes an "incumbent" worker at the end of the stopping time (i.e. she changes its type).

We let $N^A(z^t, s)$ be the quantity of newly arrived workers hired with contract $s \in S(z^t)$ at time and event z^t. Likewise, we let $N^I(z^t, s)$ be the quantities of incumbent workers hired with contract $s \in S(z^t)$ at time and event z^t. The firm chooses $N^A(z^t, s)$ and $N^I(z^t, s)$ for every z^t and $s \in S(z^t)$ to maximize expected discounted profits, taking as given the prices $P^A(z^t, s)$ and $P^I(z^t, s)$ and the fact that the stopping times of the different types of workers entail potentially different separation costs at termination. Without loss of generality, we assume that the firm never employed any workers previous to $t = 0$. This will has no consequence in the analysis given our focus on steady state equilibria.

The problem of the representative firm is the following:

$$\max_{N^A, N^I} \sum_{t=0} \sum_{z^t \in Z^t} \beta^t \left[F(n_t\left(z^t\right), z_t) - \sum_{s \in S(z^t)} \left(P^A\left(z^t, s\right) N^A\left(z^t, s\right) + P^I\left(z^t, s\right) N^I\left(z^t, s\right)\right) - T_t\left(z^t\right)\right] \mu_t\left(z^t\right)$$

[11] The linear preferences assumption in this "island-economy" is justified by the existence of perfect insurance markets in the original economy.

subject to:

$$n_t\left(z^t\right)=\sum_{i=0}^{t}\left\{\sum_{s\in S\left(z_0^i\right):\ s\left[z_0^i;\left(z_{i+1}^t,z_{t+1}^\infty\right)\right]\ >\ t,\text{ for every } z_{t+1}^\infty}\left[N^A\left(z^i,s\right)+N^I\left(z^i,s\right)\right]\right\} \tag{37}$$

$$T_t\left(z^t\right)=\tau\sum_{i=0}^{t-1}\left\{\sum_{s\in S\left(z_0^i\right):\ s\left[z_0^i;\left(z_{i+1}^t,z_{t+1}^\infty\right)\right]\ =\ t,\text{ for every } z_{t+1}^\infty}N^I\left(z^i,s\right)\right\}+\tau\sum_{i=0}^{t-J}\left\{\sum_{s\in S\left(z_0^i\right):\ s\left[z_0^i;\left(z_{i+1}^t,z_{t+1}^\infty\right)\right]\ =\ t,\text{ for every } z_{t+1}^\infty}N^A\left(z^i,s\right)\right\} \tag{38}$$

where z_j^i in equations (37) and (38) denotes the partial history $\{z_j, z_{j+1}, ..., z_{i-1}, z_i\}$ embodied in z^t. The firm maximizes the expected discounted value of profits, which are given by output minus the purchase of the stopping times supplied both by "new arrival" and "incumbent" workers, minus separation costs. The employment of the firm at time and event z^t, is given by equation (37). This equation says that total employment is the sum of all the workers, both "new arrivals" and "incumbents", that were hired between periods zero and t and that have been never fired along the history z^t. Equation (38) describes the separation costs at time and event z^t as the sum of two terms. The first term is the sum of all "incumbent" workers that have been hired between periods 0 and $t-1$, which have been contracted to separate at date t if event z^t took place. The second term is the sum of all "newly arrived" workers that have been hired between periods 0 and $t-J$, which have been contracted to separate at date t if event z^t took place. Observe that those "newly arrived" workers that have been hired between periods $t-J+1$ and $t-1$ and separate at date t and event z^t are not included in equation (38) because they separate during the trial period stipulated by the fixed term contracts and, thus, are not subject to separation costs.

The market clearing conditions are as follows. If $N^A\left(z^t,s\right)>0$ at some time and event z^t and some $s\in S(z^t)$, then

$$A\left(z^t\right)=P^A\left(z^t,s\right)+E\left[\beta^{s-t}\ I\left(z^s\right)\right]$$

Also,

$$\sum_{s\in S(z^t)}N^A\left(z^t,s\right)<U\ \Rightarrow A\left(z^t\right)=\theta.$$

The conditions for "incumbent" workers are similar. If $N^I\left(z^t,s\right)>0$ at some time and event z^t and some $s\in S(z^t)$, then

$$I\left(z^t\right)=P^I\left(z^t,s\right)+E\left[\beta^{s-t}\ I\left(z^s\right)\right]$$

Also,

$$\sum_{s\in S(z^t)}N^I\left(z^t,s\right)<X^I\left(z^t\right)\Rightarrow I\left(z^t\right)=\theta.$$

where $X^I\left(z^t\right)$ is the number of "incumbent" workers available for hire at the beginning of time and event z^t, which is given as follows:

$$X^I\left(z^t\right)=\sum_{i=0}^{t-1}\left\{\sum_{s\in S\left(z_0^i\right):\ s\left[z_0^i;\left(z_{i+1}^t,z_{t+1}^\infty\right)\right]\ =\ t,\text{ for every } z_{t+1}^\infty}\left[N^I\left(z^i,s\right)+N^A\left(z^i,s\right)\right]\right\} \tag{39}$$

Finally, the hiring of each type of workers cannot exceed the amount initially available:

$$\sum_{s\in S(z^t)}N^A\left(z^t,s\right)\le U \tag{40}$$

$$\sum_{s\in S(z^t)}N^I\left(z^t,s\right)\le X^I\left(z^t\right) \tag{41}$$

Observe that the supply of stopping time is indivisible: Workers can supply only one stopping time $s\in S(z^t)$, and only in the case that the worker has no previous stopping time outstanding. However, the linear preferences assumed, together with the convex production possibility set of the firm, guarantee that the welfare theorems hold. The competitive allocation is then obtained as the solution to the social planner's problem, which is to maximize

$$\sum_{t=0}\sum_{z^t\in Z^t}\beta^t\left[F(n_t\left(z^t\right),z_t)+\theta\left(U-\sum_{s\in S(z^t)}N^A\left(z^t,s\right)\right)+\theta\left(X^I\left(z^t\right)-\sum_{s\in S(z^t)}N^I\left(z^t,s\right)\right)-T_t\left(z^t\right)\right]\mu_t\left(z^t\right)$$

subject to equations (37), (38), (39), (40) and (41).

A few remarks are in order. Clearly, the social planner will never want to separate a "newly arrived" worker and rehire him as a an "incumbent" before the trial period for the fixed term contracts is over. The reason is that being rehired as "incumbent" makes the worker liable to separation costs, while maintaining his "newly arrived" status saves on separation costs during the trial period. Also, the social planner will never want to separate a "newly arrived" worker after the trial period is over and rehire him under an "incumbent" contract because this entails incurring the separation cost τ without any benefit. As a consequence, the planner will choose the stopping times for "newly arrived" workers in such a way that they separate only when they are to leave the island (and receive the value θ). This means that $N^I(z^t, s) = 0$ for every z^t and every $s \in S(z^t)$.

Being left with only "newly arrived" workers, the planner's problem is formally identical to the Island's Planner problem described in Section ??.[12] This has an important implication: The competitive equilibrium with long term contracts and tenure at the level of the firm described in this Appendix is equivalent to the competitive equilibrium with spot labor contracts and tenure at the level of the island that was described in the main text of the paper. Moreover, for every z^t and $s \in S(z^t)$ such that $N^A(z^t, s) > 0$ the price $P^A(z^t, s)$ must be equal to the expected discounted value of the spot wages obtained (in the equilibrium with spot labor contracts and tenure at the island level) by a worker that arrives to the island at time and event z^t, and follows an employment plan described by the stopping time s.

[12] In particular, it is identical to the problem of an Island's Planner endowed with no worker of positive tenure at $t = 0$.

Appendix G: Calibration of τ.

Heckman and Pages-Serra (2000) propose to summarize employment protection policies into a single statistic. The measure they use is the expected discounted cost at the time that a worker is hired of dismissing that worker in the future as a summary. Their index I is given by

$$I = \sum_{t=1}^{T} \beta^t \delta^{t-1} (1-\delta) \left\{ b_t + a S_t^j + (1-a) S_t^u \right\}$$

where T is the maximum tenure consider in the index, β a time discount factor, δ is the survival rate (prob. of remaining employed next period if employed during the current period), b_t is wage earning during the advance notice period for a worker of tenure t, S_t^j is the severance payment to a worker of tenure t if the dismissal is classified as "justified" (i.e. "fair" or "objective") and S_t^u is the severance payment to a worker of tenure t if the dismissal is "not justified".

Heckman and Pages-Serra use a year as a time period, and the following values: $\beta = 0.92$ (an 8 percent interest rate), $\delta = 0.88$ (a turnover rate of 12 percent, based on data for the US), a value of T of 20 years, and for Spain they advocate to use $a = 0.2$ for the period before 1997, based instead on the information on Bertola, Boeri and Cazes (2000), "Employment protection, the case of Industrialized countries: the case for new indicators", International Labor Review, 139(1):2000. Heckman and Pages-Serra compute their Job security index for Spain for the late 90s. Since we calibrate our model to the period before the broadening applicability of temporary contracts, we recompute their index for the policies in place before the 1984 reform. We use the following values:

- b_t : one month of wages for tenure 1 and 2 and 3 months for higher tenure (from Chapter 2 of OECD Labor Outlook, 1999, Table 2.2)

- a : 0.2 (since their argument applies prior to 1984)

- S_t^j : 2/3 months per year up to a maximum of 12 months (from Chapter 2 of OECD Labor Outlook, 1999, page 96)

- S_t^u : 1 1/2 months per year up to a maximum of 42 months (from Chapter 2 of OECD Labor Outlook, 1999, page 101).

We consider two cases. Case a: with these choices for $b_t, a,$ S_t^u and S_t^j, and using the values for β and δ used by Heckman and Pages-Serra, we obtain that I prior to 1984 equals to 0.42 as a fraction of annual average wages. Case 2, if instead we use $\beta = 0.96$, which is the value we use in our paper, and $\delta = 0.93$, which is closer to the one for Spain prior to 1984 according to Hopenhayn and Cabrales, we obtain a value of I prior to 1984 of 0.56 as a fraction of annual wages.

Finally, since in our benchmark case the firing taxes do not depend on the tenure of the workers, we select the value of τ that so that the value of the index above will give the value we calibrate for Spain prior to the reform. These value solves the equation:

$$I = \sum_{t=1}^{T} \beta^t \delta^{t-1} (1-\delta) \ \tau = \tau \ (1-\delta) \beta \frac{1-(\beta\delta)^T}{1-\beta\delta}$$

or

$$\tau = I \frac{1-\beta\delta}{(1-\delta) \ \beta \ \left(1-(\beta\delta)^T \right)}$$

The value of τ that corresponds to the first case is 0.74, and to the second case is 0.98 of annul wages. We think that for our purposes the choices of the second case better reflect the situation prior to 1984 and hence calibrate the model to τ equivalent to one year of average wages.