# Fixed-Term Employment Contracts in an Equilibrium Search Model

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**Abstract**

We develop a theoretical model of firm dynamics and unemployment and characterize equilibria with tenure dependent separation taxes. The model is a version of the Lucas and Prescott island model with *undirected* search. Two equivalent decentralizations are considered: one with spot labor markets and one with long-term employment relations. We model "temporary contracts" as the special case of a separation tax that only applies to workers with tenure higher than $J$. While in principle these contracts require a $J$ dimensional state space, equilibrium allocations solve a simple dynamic programming problem characterized by two dimensional inaction set(s).

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# 1 Introduction and summary

Temporary contracts have been introduced in several countries that regularly apply large penalties to employers for firing workers.[1] Temporary contracts stipulate a period of time during which workers can be dismissed at zero (or very low) firing costs but if workers are retained beyond this period, regular firing costs apply. An often stated view in favor of temporary contracts is that they provide firms considerable flexibility in the hiring and firing process, despite the presence of large firing costs. The potential effects of these policies on unemployment and worker turnover are also commonly at the center of the economic debate.[2] While the effects on unemployment are unclear, the empirical evidence indicates that temporary contracts have significant effects on worker turnover.[3] For instance, Cabrales and Hopenhayn (1997) report that after the introduction of temporary contracts in Spain during 1984 there was a sharp increase in the firing rate of workers with employment tenure equal to the length of the temporary contracts introduced.[4] This indicates that the reform induced firms to fire workers right before they became subject to regular firing costs. Given the widespread use of temporary contracts and their significant effects on worker turnover it is important to develop models in which this type of policies can be analyzed. This paper provides a basic framework for performing such analysis.

The model considered is an undirected search version of the Lucas-Prescott (1974) model of equilibrium unemployment. This is an important extension because it combines features of the Mc Call (1970) search model (which focuses on the decision problem of an unemployed worker) with features of the Hopenhayn (1992) industry equilibrium model (which focuses on the employment dynamics of firms). The model turns out to be very tractable: In spite of the search frictions and the large amount of heterogeneity across islands we show that the optimal allocations are characterized by a simple dynamic programming problem, which we refer to as the "island planning problem". The equivalence between these optimal allocations and two alternative equilibrium decentralizations is also demonstrated. The

[1]Examples of countries that have introduced or extended the use of temporary contracts since the mid 80's include Belgium, Denmark, Germany, Greece, Italy, Netherlands, Portugal, Spain and Sweden (European Commision, 2010). As a consequence of these reforms, the share of temporary employment in total employment has been trending up over time in all of these countries. Spain is the county with the largest increase: its share of temporary employment in total employment went from 11% in 1983 to about 35% in 1995.

[2]See, for instance, the surveys by Dolado, García-Serrano and Jimeno (2002) and by Bentolilla, Dolado and Jimeno (2008) on the Spanish experience and the voluminous literature cited therein.

[3]See, for example, Bentolilla, Dolado and Jimeno (2008), page 54.

[4]See Güell and Petrongolo (2007) for further analysis and similar results.

framework allows us to analyze an important class of policies that includes firing taxes (as in Hopenhayn and Rogerson, 1993) and temporary employment contracts (as in Cabrales and Hopenhayn, 1997) as special cases. In principle, this class of policies requires keeping track of the distribution of workers across tenure levels in each island. However, we show that under firing taxes and temporary contracts, the undirected search assumption leads to a state space representation that is simple enough to provide a sharp characterization of equilibrium allocations and wages. Interestingly, we find that workers that are about to complete their temporary contracts face the highest firing rates, a feature consistent with the empirical finding reported by Cabrales and Hopenhayn (1997).

The structure of the economy is as follows. Production takes place in a large number of locations (or islands). All islands operate the same decreasing returns to scale technology but are subject to island-specific productivity shocks. Changes in the island-specific productivity shock give raise to changes in labor demand across locations. Moving a worker across locations is costly: It requires one period during which the agent neither works nor enjoys leisure. In addition, agents that search arrive randomly to one of the islands in the economy (i.e., search is undirected). Workers that separate from their islands have to choose between two alternative activities: to work at home (i.e., to leave the labor force) or to search (i.e., to become unemployed).

The employment protection system that we consider is characterized by separation taxes $\tau_j$ that depend on the tenure level $j$ of the worker. However, we assume that there is a tenure level $J$ such that all workers with tenure equal to or greater than $J$ are subject to the same separation taxes. When $J = 1$ this employment protection system reduces to the separation tax regime analyzed by Hopenhayn and Rogerson (1993).

We consider two alternative (yet equivalent) concepts of competitive equilibrium: one with spot labor markets and one with multiperiod employment relations. In the spot labor markets equilibrium, workers and firms solve problems that are natural extensions of the McCall (1970) search model and the Hopenhayn and Rogerson (1993) firing costs model, respectively. This equilibrium concept has two advantages: It can be easily related to the previous literature and it is simple to analyze. However, it has a considerable drawback: It assumes that the separation cost $\tau_j$ is determined by the tenure level $j$ of a worker in an island, not in an individual firm. The equilibrium with multiperiod employment relations is more complex but captures the nature of fixed-term employment contracts in a much more realistic way. In particular, it assumes that the separation cost $\tau_j$ is determined by the tenure level $j$ of a worker in an individual firm.

In the spot labor market equilibrium, workers are differentiated by their tenure levels, they participate in different labor markets and receive different wages. Given the presence of tenure-dependent firing costs, firms solve a modified (S,s) optimization problem. In turn, workers at each tenure level face a standard search problem: They decide whether to stay on the island where they are currently located or become non-employed. Both firms and workers take as given the island-level law of motion for wages across tenure levels. At equilibrium, this island-level process must be such that every tenure-specific labor market clears at each island-wide state. An economy-wide equilibrium is completed by an invariant distribution across island states. This economy-wide distribution is needed to describe the benefits of search and the aggregate demand for labor.

To describe the structure of efficient allocations and their relationship with equilibrium allocations it is useful to consider two cases: 1) when the separation costs are a technological feature of the environment, and 2) when the separation costs are taxes rebated lump-sum to households (the interesting case to consider for policy analysis). In the first case, we show that the first and second welfare theorems hold. In the second case, the welfare theorems do not apply but we can still use a slightly modified version of the planning problem to characterize a competitive equilibrium. In particular, we can break the economy-wide planning problem into a series of island-wide planning problems, one for each island. Each of these island-wide social planners solves a similar problem: to maximize the expected discounted value of output by deciding how many workers to keep and how many workers to take out from the island. In this problem, the island-wide planner takes the constant flow $U$ of new arrivals to the island as given (this flow is independent of the characteristics of the island because of the assumption of undirected search). The island's planner also takes as given the shadow value of returning a worker to non-employment and the tenure-dependent separation costs $\tau_j$. Given the solution to the island-wide planning problem, the economy-wide equilibrium is obtained by finding two unknowns: the equilibrium shadow value of non-employment and the equilibrium number of agents that search $U$.

We use the tenure-dependent separation costs to represent a stylized temporary contracts regime as follows: We assume that temporary workers (those with tenure less than $J$) are subject to no separation costs but that all permanent workers (those with tenure equal or greater than $J$) are subject to a separation cost $\tau > 0$. Under this stylized temporary contracts regime we are able to fully characterize the equilibrium adjustments at the island level. In particular, we show that the first workers to be fired are those about to become permanent workers. After all workers of tenure $J - 1$ have been fired, the next workers to be fired are those with successively lower tenure. Permanents workers are fired only when there are no temporary workers left. Given this ordering of separation decisions, we show that the state of the island planner's problem (which is generally given by a distribution of workers across the $J$ tenure levels) can be reduced to a two-dimensional object: the number of temporary workers and the number of permanent workers. Moreover, we show that the solution to this simplified control problem is characterized by two-dimensional sets of inaction, one set for each value of the idiosyncratic productivity shock, and demonstrate their properties.

The paper is organized as follows. Section 2 relates the paper to the previous literature and discusses some limitations to the analysis. Section 3 describes the economy. Section 4 defines efficient allocations. Section 5 characterizes efficient stationary allocations. Section 6 describes a spot labor markets equilibrium. Section 7 describes a competitive equilibrium with multiperiod employment relations. Finally, Section 8 considers the stylized temporary contracts and characterizes the associated equilibrium adjustments. Two appendices contain some of the more technical analysis.[5]

## 2 Related literature, discussion, and limitations of the analysis

This is not the first paper to introduce undirected search in the Lucas-Prescott (1974) islands model. The paper by Jovanovic (1987) is an early precursor. More recently, Alvarez and Veracierto (1999) used a similar type of structure to evaluate a number of simple policies, Veracierto (2008) considered a version with aggregate productivity shocks to analyze business cycle fluctuations, and Kambourov and Manovskii (2007) used a version with island-specific human capital to study occupational mobility. This paper differs from the previous literature in that it analyzes a different class of policies and in that it provides a systematic characterization of equilibrium and efficient allocations.

Our paper is also closely related to the literature analyzing separation taxes and temporary contracts. Within the class of papers studying separation taxes (e.g. Bentolila and Bertola, 1990, Millard and Mortensen, 1997, etc.) the general equilibrium model of Hopenhayn and Rogerson (1993) is probably the most closely related. However, our analysis extends Hopenhayn and Rogerson's by incorporating search frictions and by evaluating the effects of separation taxes on unemployment. Within the extensive literature studying the effects of temporary contracts (e.g. Blanchard and Landier, 2002, Nagypal, 2002, etc.) the papers that are more similar in spirit to ours are Bentolila and Saint Paul (1992), Cabrales and Hopenhayn (1997), Aguiregabiria and Alonso-Borrego (2004), Veracierto (2007) and Alonso-Borrego et al. (2005), since they all study labor demand models with dynamic adjustment costs. An important difference with Bentolila and Saint Paul (1992), Cabrales and Hopenhayn (1997), and Aguiregabiria and Alonso-Borrego (2004) is that these papers consider partial equilibrium models and do not introduce unemployment. Veracierto (2007) introduces unemployment but in a partial equilibrium model with exogenous wages. In principle, Alonso-Borrego et al. (2005) is the most closely related paper since it

[5] Detailed proofs to all the claims made in the appendices are provided in Alvarez and Veracierto (2010).

performs a general equilibrium analysis with search frictions.[6] However, its model is very different. In particular, it assumes that: 1) agents face exogenous borrowing constraints, 2) employment contracts specify a constant wage rate as long as the employment relation lasts, 3) fixed-term contracts last only one model period, and 4) the matching process is subject to congestion externalities. While some of these assumptions are meant to provide realism, they complicate the interpretation of the results quite significantly. For instance, it is unclear to what extent the main result in the paper (which is that higher firing costs reduce unemployment and improve welfare) depends on the ad-hoc wage contracts assumed.[7]

In principle, the model in this paper could be used to evaluate the quantitative effects of firing taxes and temporary contracts. The fact that for $J = 1$ the fixed-term contracts are equivalent to the case of firing taxes and that for large $J$ they are equivalent to the laissez-faire case suggests a very natural experiment: To evaluate how quickly the effects of firing taxes are undone by increasing the length of the temporary contracts $J$.[8] However, we would like to mention a series of caveats that such type of analysis would be subject to.

To start with, it is important to bear in mind that our model has complete markets. As a consequence, government interventions cannot improve the set of mutually beneficial trades between private parties, as may happen in models in which the bilateral contractual arrangements are restricted in arbitrary ways (e.g. Alvarez and Veracierto, 2001, and Alonso-Borrego, 2005). Moreover, due to convexity and lack of externalities, the laissez-faire equilibrium in our model is Pareto optimum and hence any form of government intervention can only reduce welfare. Thus, performing policy analysis in this type of setting would amount to measuring Harberger triangles, i.e. to measuring the efficiency costs of the policy considered, abstracting from any potential benefits.[9] A more complete assessment of firing taxes and temporary contracts would have to consider potential benefits as well.

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[6] Blanchard and Landier (200s2) also perform a general equilibrium analysis with search frictions but they do so in a framework closely related to Mortensen and Pissarides (1994), in which wages are determined through Nash bargaining and the matching process is subject to externalities. In such a context, Blanchard and Landier find that introducing labor market flexibility through fixed-term contracts may lead to lower welfare levels. As we explain in the next paragraph, this can never take place in the framework that we consider.

[7] In Alvarez and Veracierto (2001), on which Alonso-Borrego (2005) is based, higher firing taxes also reduce unemployment and improve welfare. In that setting, we show that the rigid wage contracts assumed play a crucial role in generating the results while the borrowing constraints play no important role.

[8] See Alvarez and Veracierto (2008) for a numerical example of this type of analysis.

[9] In fact, this is precisely the approach that Hopenhayn and Rogerson (1993) and others have previously followed in evaluating the effects of firing taxes.

With regard to the potential benefits of firing taxes, introducing ad-hoc borrowing constraints and ad-hoc wage contracts have already been shown to change the results quite substantially (see Alvarez and Veracierto, 2001, and Alonso-Borrego et al., 2005). Potential benefits would also be obtained by incorporating a hold-up problem that precludes workers from optimally investing in firm-specific human capital. In this scenario, firing penalties could improve welfare by reducing the renegotiation ability of firms after workers have made their firm-specific investments.

A commonly stated argument for introducing temporary contracts is that they provide an employment buffer that firms can use to adjust to their idiosyncratic shocks without having to incur firing costs. This is exactly the type of effect that our paper focuses on. However, another common argument is that temporary contracts provide firms an initial period of time during which they can learn the idiosyncratic productivity of workers without having to pay firing costs.[10] Unfortunately, while potentially important, we are unable to analyze this type of screening role: Introducing the additional heterogeneity would make our model intractable. We refer the reader to Nagypal (2002) for a partial equilibrium analysis of this mechanism.

It is also important to point out that the literature has identified circumstances under which temporary contracts could have negative effects, even when firing costs are already in place. For instance, workers on temporary contracts may be motivated to exert low effort levels because of the high probability of being fired at the end of their contracts (see Dolado and Stucchi, 2008). Another possibility is that the increased worker reallocation generated by the temporary contracts could diminish on-the-job learning (Nagypal 2002). While learning is surely a feature in real labor markets, introducing it into our model would break the simple characterization of our model and, again, make it intractable.[11]

# 3 Description of the Economy

Production takes place in a continuum (measure one) of different locations, or "islands." On each island consumption goods are produced according to $F(E, z)$, a neoclassical production function, where $E$ is employment and $z$ is a productivity shock that takes values in the set $Z$. The process for $z$ is Markov with transition function $Q(z_{t+1}|z_t)$, and realizations are i.i.d. across islands. We let $f(E, z) \equiv$

[10] For instance, see European Commission (2010), page 140.

[11] If productivity levels were assumed to increase with tenure, it would not longer be clear that the first workers to be fired are the ones about to become permanent workers. As a result, the effects of temporary contracts on worker reallocation could be quite different from those obtained in this paper (and may be inconsistent with the large firing spikes found in the data at the end of the temporary contracts).

$\partial F(E,z)/\partial E$ and assume that $f$ is continuous and strictly decreasing in $E$, strictly increasing in $z$, and that $\lim_{E\to 0} f(E,\underline{z}) = \infty$, where $\underline{z} \equiv \min\{z : z \in Z\}$.

The economy is populated by a representative household. The household is constituted by a continuum of members with names in the interval $[0, N]$, which we call "agents". Agents participate in one of the following three activities: work on an island, perform home production (or, equivalently, enjoy leisure), or search. Non-employed agents, whom we sometimes refer to as "agents being at a central location," either work at home (enjoy leisure) or search for a job. If they work at home during the current period, they start the following period as non-employed. If a non-employed agent searches in the current period, she does not produce during the current period but arrives randomly to one of the islands at the beginning of the next period. We assume that search is undirected, so the probability of arriving to an island of any given type is given by the fraction of islands of that type in the economy. An agent located on an island at the beginning of the period can decide whether to stay on the island or become non-employed. If she stays, she works and starts the following period in the same location.

We let $L_t$ be the number of agents engaged in home production at time $t$, and $U_t$ the fraction engaged in search at time $t$. The period utility function of the representative household consuming $C_t$ units of consumption goods and $L_t$ units of leisure is given by

$$v(C_t, L_t) = u(C_t) + \omega L_t,$$

where $\omega > 0$. Observe that this specification implicitly assumes that agents obtain full consumption insurance within the household. Also, the linearity of leisure in household preferences can represent an economy with indivisible labor and employment lotteries, as in Rogerson (1988).

Up to this point the environment is a modification of the equilibrium search model of Lucas and Prescott (1974) that introduces household production and undirected search, as in Alvarez and Veracierto (1999). We now introduce a tenure-dependent separation cost. In this section we introduce this separation cost as being a technological feature of the environment. Later on we show how to use the efficient allocation of this economy to construct an equilibrium where the separation costs are taxes levied on firms and rebated to households in a lump-sum way.

The tenure-dependent separation cost works as follows: If an agent has worked for $J$ or more periods in a location, $\tau_J$ units of the consumption good are lost at the time that that worker returns to the central location. If the worker returns to the central location after $j < J$ periods, the separation cost is equal to $\tau_j < \tau_J$. In Section 7 we present an equilibrium concept that shows that (a version of) this tenure-dependent separation cost at the island level captures salient features of the temporary employment contracts used in the real world.

# 4 Efficient Allocations: A Formal Definition

Since the separation cost depends on tenure levels, a description of an allocation must include the distribution of workers by tenure on each island. We refer to workers with tenure $j = 1, ..., J-1$ in a location as *temporary workers* and to those with tenure $j \geq J$ as *permanent workers*. Thus the state of a location is given by its productivity shock $z$ and a $J$ dimensional vector $T$ indicating the number of workers with different tenure levels. In the sequential notation, locations are indexed by their state at time $t = 0$, denoted by $X = T_0$. We use $z^t = (z_0, ..., z_{t-1}, z_t)$ for the history of shocks of length $t$ and index each location at time $t$ by $\left(z^t, X\right)$, its history of shocks and its initial state. The initial state of the economy is described by a distribution of locations across pairs $(z_0, X)$ and by $U_{-1}$, the number of agents that searched during $t = -1$. We let $\eta\left(X|z_0\right)$ be the fraction of locations with state $X$ conditional on $z_0$, and $q_0\left(z\right)$ the initial distribution of $z_0$. We assume that $q_0$ equals the unique invariant distribution associated with the transition function $Q$. We denote by $q_t\left(z^t\right)$ the fraction of islands with history $z^t$, which by the Law of Large Numbers satisfies

$$q_{t+1}\left(z^t, z_{t+1}\right) = Q\left(z_{t+1}|z_t\right) q_t\left(z^t\right).$$

We indicate the employment of agents with tenure $j$ at a location $\left(z^t,\ X\right)$ by $E_{jt}\left(z^t, X\right)$, for $j = 0, ..., J,\ z^t \in Z^t$ and $t \geq 0$. Likewise, we denote by $S_{jt}\left(z^t, X\right)$ the separations, i.e., the number of agents with tenure $j$ that return to the central location.

Formally, we say that $\{E_{jt}, S_{jt}, C_t, U_t, H_t\}$, given $\eta$ and $U_{-1}$, is a feasible allocation if the following conditions hold:

i) the island's law of motion

$$\begin{aligned}
E_{j,t}\left(z^t, X\right) &= E_{j-1,t-1}\left(z^{t-1}, X\right) - S_{j,t}\left(z^t, X\right),\ j = 1, 2, ..., J-1, \\
E_{J,t}\left(z^t, X\right) &= E_{J-1,t-1}\left(z^{t-1}, X\right) + E_{J,t-1}\left(z^{t-1}, X\right) - S_{J,t}\left(z^t, X\right), \\
E_{0,t}\left(z^t\right) &= U_{t-1} - S_{0,t}\left(z^t, X\right),
\end{aligned}$$

$S_{j,t}\left(z^t, X\right) \geq 0$ for $t \geq 0,\ z^t \in Z^t, X \in \mathrm{supp}(\eta)$;

ii) the initial conditions

$$\begin{aligned}
E_{j-1,-1} &= X_j \text{ for } j = 1, 2, ..., J-1, \\
E_{J-1,-1} + E_{J,-1} &= X_J,
\end{aligned}$$

iii) the feasibility constraint for the labor market

$$U_t + \sum_{z^t} \sum_{X} \sum_{j=0}^{J} E_{j,t}\left(z^t, X\right) q_t\left(z^t\right) \eta\left(X|z_0\right) + L_t = N$$

$U_t, L_t \geq 0$ for all $t \geq 0$;

and iv) the feasibility constraint for the consumption good

$$C_t = \sum_{z^t} \sum_{X} \left[ F\left( \sum_{j=0}^{J} E_{j,t}\left(z^t, X\right), z_t \right) - \sum_{j=1}^{J} \tau_j \; S_{j,t}\left(z^t, X\right) \right] q_t\left(z^t\right) \eta\left(X|z_0\right)$$

for all $t \geq 0$.

The first constraint states that the number of employed workers of tenure $j \leq J-1$ is given by the number of workers of tenure $j-1$ that were employed on the island during the previous period minus the number of these workers taken out of the island during the current period. The second constraint is analogous to the first constraint for workers of tenure $J$ or higher. It differs from the first one because we don't keep track of workers of tenure $j \geq J$ separately (they are all lumped together into tenure $J$). The third constraint says that the employment of tenure zero workers is given by those just arrived to the island, minus the number of them taken out of the island. The fourth and fifth equations define $E_{j,-1}$ in terms of the initial conditions $X_j$, for $j = 1, ..., J$. The sixth constraint states that the sum of total unemployment, total employment, and agents out of the labor force equals the population $N$. The seventh constraint states that consumption is equal to aggregate output net of separation costs.

Henceforth we define $T_{j,t}\left(z^t, X\right)$ as the number of workers of tenure $j$ available at the beginning of the period $t$ on an island of type $(z^t, X)$, so that

$$T_{j,t}\left(z^t, X\right) = E_{j-1,t-1}\left(z^{t-1}, X\right), \; j = 1, 2, ..., J-1, \tag{1}$$

$$T_{J,t}\left(z^t, X\right) = E_{J-1,t-1}\left(z^{t-1}, X\right) + E_{J,t-1}\left(z^{t-1}, X\right), \tag{2}$$

$$T_{0,t}\left(z^t\right) = U_{t-1} \tag{3}$$

Hence condition i) in the definition of feasibility is equivalent to

$$E_{j,t}\left(z^t, X\right) \leq T_{j,t}\left(z^t, X\right) \text{ for all } j.$$

With these objects at hand we can define a planning problem whose solutions characterize the set of efficient allocations. We say that $\{E_{jt},\ S_{jt},\ T_{j,t},\ C_t,\ U_t,\ L_t\}$ is an efficient allocation if it maximizes

$$\sum_t \beta^t \left[u\left(C_t\right) + \omega L_t\right]$$

over all feasible allocations, given the initial conditions $\eta$ and $U_{-1}$.

Given initial conditions $\eta$, $U_{-1}$, a feasible allocation $\{E_{jt},\ S_{jt},\ T_{j,t},\ C_t,\ U_t,\ L_t\}$ is stationary if $C_t$, $U_t$, $L_t$ and the cross sectional distribution $\eta_t$ are constant over time, where $\eta_t$ is given by

$$\eta_{t+1}\left(K|z'\right) = \sum_{z^t \in Z^t} \sum_{X} I_K\left(z^t, X\right) \eta_0\left(X|z_0\right) q_t\left(z^t\right) Q\left(z'|z_t\right),$$

and $I_K$ is an indicator defined as

$$I_K\left(z^t, X\right) = \left\{ \begin{array}{c} 1, \text{ if } \left[T_{1,t}\left(z^t, X\right), ..., T_{J,t}\left(z^t, X\right)\right] \in K \\ 0, \text{ otherwise} \end{array} \right\}$$

for all $z^t \in Z^t$, $X \in \text{supp}(\eta)$, and Borel measurable $K \subset R_+^J$. Finally, we say that $\{L, U, C, \eta\}$ is a stationary efficient allocation if there is some efficient allocation $\left\{\hat{E}_{jt}, \hat{S}_{jt}, \hat{T}_{j,t}, \hat{C}_t, \hat{U}_t, \hat{L}_t\right\}$ with initial condition $\hat{U}_{-1}$, $\hat{\eta}$ which is stationary and for which

$$\hat{U}_{-1} = \hat{U}_t = U,\ \hat{L}_t = L,\ \hat{C}_t = C \text{ and } \hat{\eta}_t = \eta, \text{ for all } t \geq 0.$$

# 5 Characterization of Efficient Stationary Allocations

An efficient allocation is interior if agents are engaged in all three activities: search, home production, and work. Our characterization of interior efficient stationary allocations consists on the solution to two equations in two unknowns: $(U, \theta)$, where $U$ is unemployment and $\theta$ is the shadow value of being non-employed. One equation states that the shadow value of search equals the expected value of randomly arriving to an island next period, according to the invariant distribution. The second equation ensures that agents are indifferent between doing search and home production. The first equation is quite complex: It involves solving a dynamic programing problem and using the invariant distribution generated by its optimal policies. We refer to this dynamic programing problem as the island planning problem.

## 5.1 The Island Planning Problem

The state of this problem is given by $(T, z)$, where $T$ is a vector describing the number of workers across tenure levels $j = 1, 2, ..., J$ at the beginning of the period and $z$ is the current productivity shock. The island planner receives $U$ workers with tenure $j = 0$ every period. The planner decides how many workers to employ at each tenure level and returns workers to the central location at a shadow value given by $\theta$. The planner incurs a cost $\tau_j$ per worker with tenure $j$ that is returned to the central location. Formally,

$$\begin{aligned} V(T, z; U, \theta) &= \max_{\{E_j\}} \left\{ F\left(\sum_{j=0}^{J} E_j, z\right) + \theta\left([U - E_0] + \sum_{j=1}^{J} [T_j - E_j]\right) - \sum_{j=1}^{J} \tau_j [T_j - E_j] \right. \\ &\quad \left. + \beta \sum_{z'} V\left(E_0, E_1, ..., E_{J-2}, E_{J-1} + E_J, z'; U, \theta\right) Q\left(z'|z\right) \right\} \end{aligned}$$

subject to $0 \leq E_j \leq T_j$ for $j = 1, .., J$ and $0 \leq E_0 \leq U$. We let $G(T, z; U, \theta)$ be the optimal employment decision and $T' = A(T, z)$ the implied transition function with $T'_{j+1} = G_j(T, z)$ for $j = 0, ..., J-2$ and $T'_J = G_J(T, z) + G_{J-1}(T, z)$.

The value function $V$ turns out to be differentiable. The standard proof by Benveniste and Scheikman does not apply because the optimal choice of $E$ is not interior, but in Appendix A we construct an alternative proof and find expressions for the derivatives of $V$. Intuitively, the marginal value of an additional worker of tenure $j$ is given by the sum of two terms. The first term is the expected discounted sum of the marginal product of labor over periods in which no worker of the same cohort has ever been sent back to the central location. The second term is the expected discounted shadow value (net of separation costs) the first time that a worker of the same cohort is sent back to the central location. Formally, for $T_j > 0$, $\partial V(T,z)/\partial T_j = V_j^*(T,z)$, where $V_j^*$ is defined as follows. Denote the current date by 0 and define the stopping time $n_j$ as the first date $s$ at which the number of workers with current tenure $j$ is reduced. We let $E_{i,s}^*$ be the optimal employment level $s$ periods from now of workers with tenure level $i$, and $T_{i,s}$ be the beginning-of-period number of workers $s$ periods from now with tenure level $i$, so that

$$n_j = \text{ first date } s \text{ at which } E^*_{\min\{J,\ j+s\},s} < T_{\min\{J,\ j+s\},s}.$$

Now we are ready to define $V_j^*(T,z)$ as:

$$V_j^*(T,z) = \sum_{s=0}^{\infty} \beta^s E_0 \left[ f\left( \sum_{i=0}^{J} E_{i,s}^*, z_s \right) | n_j > s \right] + \ E_0\left[\beta^{n_j}\ \theta\right] - E_0\left[\beta^{n_j}\ \tau_{\min\{J,j+n_j\}}\right] \tag{4}$$

This implies that if some workers of tenure $j$ are sent back, i.e., if $E_j = G_j(T,z) < T_j$, then the marginal value of all workers of this tenure level is $V_j^*(T,z) = \theta$ for $j = 0$ and equal to $\theta - \tau_j$ for $j > 0$.

### 5.2 Efficient Allocations as Solutions to the Island Planning Problem

It is intuitive to see that if $U$ is the economy-wide efficient unemployment level and $\theta$ is the economy-wide shadow value of non-employment, the employment decisions of the island planners' problem recover the economy-wide efficient employment decisions. To see why, notice that each island faces the same value for $U$, since search is undirected, and the same value of $\theta$, since workers are identical once they leave the island and arrive to the central location.

As stated above, the shadow value of non-employment is equal to the discounted expected value of randomly arriving (with zero tenure) to an island under the invariant distribution. To find the shadow value of workers with tenure zero at each island, we define the problem of an island's planner that faces a flow of unemployed workers equal to $\hat{U}$ for one period but that reverts to the constant flow $U$ thereafter:

$$\hat{V}\left(T, z; \hat{U}, \theta\right) = \max_{E_j} \left\{ F\left(\sum_{j=0}^{J} E_j, z\right) + \theta\left(\left[\hat{U} - E_0\right] + \sum_{j=1}^{J} [T_j - E_j]\right) - \sum_{j=1}^{J} \tau_j [T_j - E_j] \right. \quad (5)$$
$$\left. + \beta \sum_{z'} V\left(E_0, E_1, ..., E_{J-2}, E_{J-1} + E_J, z'; U, \theta\right) Q\left(z'|z\right) \right\}$$

subject to $0 \leq E_j \leq T_j$ for $j = 1, .., J$ and $0 \leq E_0 \leq \hat{U}$. Using this problem we define the value of an extra zero-tenure worker in a location with state $(T, z)$ as:

$$\lambda(T, z; U, \theta) = \frac{\partial \hat{V}\left(T, z; \hat{U}, \theta\right)}{\partial \hat{U}} |_{\hat{U}=U} \ . \quad (6)$$

The next theorem gives a characterization of the stationary efficient allocations.

**Theorem 1** . *Let $(U, \theta)$ be an arbitrary pair. Let $V(\cdot; U, \theta)$ be the solution of the island planning problem, and let $G(\cdot; U, \theta)$, $\lambda(\cdot; U, \theta)$ be the the associated optimal policies and shadow value for zero-tenure workers, respectively. Suppose that:*
*i) $\mu(\cdot; U, \theta)$ is a stationary distribution for the process $(T, z)$ with transition functions given by $Q(z'|z)$ for $z'$ and $A(T, z)$ for $T'$;*
*ii) the value of search $\sigma$ is given by*

$$\sigma = \beta \int \lambda(T, z\ ; U, \theta)\, \mu(dT \times dz\ ; U, \theta);$$

*iii) the number of agents engaged in home production $L$ satisfies*

$$L = N - U - \int \left[\sum_{j=0}^{J} G_j(T, z\ ; U, \theta)\right] \mu(dT \times dz\ ; U, \theta) \geq 0;$$

*iv) the labor force participation decisions are optimal, in the sense that*

$$\theta = \max\left\{\sigma,\ \frac{\omega}{u'(C)} + \beta\theta\right\}, \quad \text{and} \ \ 0 = L\ \left[\theta - \frac{\omega}{u'(C)} - \beta\theta\right].$$

*v) aggregate consumption satisfies that*

$$C = \int \left\{ F\left(\sum_{j=0}^{J} G_j(T, z\ ; U, \theta)\right) - \sum_{j=1}^{J} \tau_j\ \ [T_j - G_j(T, z\ ; U, \theta)] \right\} \mu(dT \times dz\ ; U, \theta)$$

*Finally, define $\eta(T, z) = \mu(T|z)$ as the distribution of $T$ conditional on $z$. Then $\{L, U, C, \eta\}$ is an efficient stationary allocation.*

Conditions (i) and (ii) have been explained above. Condition (iii) defines the number of agents doing home production as total population minus the sum of unemployment and employment, and states that home production must be nonnegative. The first equation in condition (iv) states that the value of non-employment must be the best of two alternatives: the value of search, which is $\sigma$, and the value of doing home production during the current period and being non-employed the following period, which is $\omega/u'(C) + \beta\theta$. The second equation in condition (iv) is a complementary slackness condition for home production. Condition (v) is that aggregate consumption must be feasible. The proof of Theorem 1 is lengthy but standard (see Alvarez and Veracierto, 2010, for the details).

Theorem 1 implies that characterizing efficient stationary allocations is reduced to solving two equations in two unknowns and checking that an inequality is satisfied. Given an arbitrary pair $(U, \theta)$, the functions $V(\cdot, U, \theta)$, $G(\cdot, U, \theta)$, $\lambda(\cdot, U, \theta)$, and the distribution $\mu(\cdot, U, \theta)$ can be found using standard recursive techniques. Defining $\sigma(U, \theta)$, $L(U, \theta)$ and $C(U, \theta)$ as the left-hand sides of conditions (ii), (iii) and (v), respectively, the two equations that $U$ and $\theta$ must satisfy are:

$$\theta = \max\left\{\sigma(U,\theta), \frac{\omega}{u'[C(U,\theta)]} + \beta\theta\right\} \quad \text{and} \quad 0 = L(U,\theta)\left[\theta - \frac{\omega}{u'[C(U,\theta)]} - \beta\theta\right],$$

and the inequality that must be satisfied is that $L(U, \theta) \geq 0$. A consequence of this simple characterization is that Theorem 1 can be used for constructing a computational algorithm and establishing the existence and uniqueness of a stationary efficient allocation.

# 6 Spot Labor Markets Equilibrium

The representative household has a continuum of members that share their employment risks. Given the perfect consumption pooling at the household level, each household member seeks to maximize her own expected discounted earnings regardless of risk.[12]

In what follows it will be useful to think of each island as a separate economy and define an "island-level equilibrium," taking as given the flow of new workers to the island, $U$, and the value to a worker of leaving the island, $\theta$. The resulting equilibrium decision rules at the island level define an invariant distribution across island states, which together with $U$ and $\theta$ must satisfy certain conditions to constitute an "economy-wide equilibrium." These conditions are that the consumption market clears, the labor

[12] An equivalent specification would be to identify each household with one individual and introduce complete markets and employment/search lotteries, as in Prescott and Rios-Rull (1992). However, this alternative would require introducing additional notation.

market clears, and the marginal rate of substitution between consumption and leisure equals the flow value of search.

Firms and workers participate in competitive spot labor markets on each island. Wages are indexed by $j$, the workers' tenure on the island and by $(T, z)$, the island-wide state, and are denoted by

$$w(T,z) = (w_0(T,z), w_1(T,z), ..., w_{J-1}(T,z), w_J(T,z)).$$

The law of motion for wages can then be obtained from the island-wide equilibrium employment rule and the associated law of motion for the island-wide state. The equilibrium employment rule is denoted by

$$G(T,z) \equiv (G_0(T,z), G_1(T,z), ..., G_{J-1}(T,z), G_J(T,z)).$$

The law of motion for the endogenous state $T' = A(T,z)$ is then given by

$$A(T,z) = (G_0(T,z), G_1(T,z), ..., G_{J-2}(T,z), G_{J-1}(T,z) + G_J(T,z)).$$

The problem for a worker with tenure $j$ on an island of state $(T, z)$ is to decide whether to become non-employed or stay and work. Becoming non-employed entails a value given by $\theta$. By staying, the worker receives the wage rate $w_j$ during the current period and gains tenure $\min\{j+1, J\}$ for the following period. We denote the value function for a $j$-tenure worker in a $(T, z)$-island as $W_j(T, z)$. This value function must solve

$$W_j(T,z) = \max\left\{\theta,\ w_j(T,z) + \beta \int W_{\min\{j+1,J\}}\left(A(T,z), z'\right) Q\left(z, dz'\right)\right\},$$

for all $(T, z)$ and $j = 0, ..., J$.

Whenever a firm decreases its employment of workers of tenure $j$, it must pay a separation cost $\tau_j$ per unit reduction. The individual state of a firm is then is then given by a vector

$$x = (x_1, x_2, ..., x_{J-1}, x_J),$$

describing the distribution of workers that were employed during the previous period across current tenure levels. The value function $B(x; T, z)$ of a firm on an island with state $(T, z)$ solves:

$$\begin{aligned} B(x;T,z) \quad = \quad & \max_{\{g_j \geq 0\}_{j=0}^{J}} \left\{ F\left(\sum_{j=0}^{J} g_j,\ z\right) - \sum_{i=0}^{J} w_j(T,z) g_j - \sum_{j=1}^{J} \tau_j \max\{x_j - g_J, 0\} \right. \\ & \left. + \beta \sum_{z'} B\left(g_0, g_1, ..., g_{J-2}, g_J + g_{J-1};\ A(T,z), z'\right) Q\left(z|z'\right) \right\} \end{aligned}$$

The optimal decision rule is denoted by $g_j = m_j(x; T, z)$, for $0 \leq j \leq J$, describing the optimal employment level at each tenure $j$. For future reference, notice that $B(x; T, z)$ is decreasing in $x$, since having employed more workers in the previous period makes the firm subject to higher potential separation costs. Thus, provided that $B$ is differentiable, $-\tau_j \leq \partial B/\partial x_j \leq 0$, and $\partial B/\partial x_j = -\tau_j$ if some workers of tenure $j$ are fired, i.e. if $g_j = m_j(x; T, z) < x_j$.

A spot labor market equilibrium (SLM) is given by numbers $\{\theta, U, \sigma, C\}$ and functions $\{w, G, B, m, W\}$ that satisfy the following conditions:

i) Given wages $w(\cdot)$, employment $G(\cdot)$, and the law of motion $A(\cdot)$, the representative firm is representative

$$m_j(T; T, z) = G_j(T, z),$$

for all $(T, z)$ and all $0 \leq j \leq J$;

ii) Given wages $w(\cdot)$, employment $G(\cdot)$ and law of motion $A(\cdot)$, the decision of the representative worker is representative

$$\begin{aligned} W_j(T, z) &> \theta \Rightarrow G_j(T, z) = T_j, \quad \text{for } j > 0 \text{ and} \\ W_0(T, z) &> \theta \Rightarrow G_0(T, z) = U, \end{aligned}$$

and if $G_j(T, z) > 0$, then

$$W_j(T, z) = w_j(T, z) + \beta \int W_{\min\{J, j+1\}}(A(T, z), z') \, Q(z, dz');$$

iii) The law of motion $A$ defines an invariant distribution $\mu$ across states $(T, z)$ as follows

$$\mu(D, z') = \sum_{z \in Z} \left[ \int_{\{T, z:\ A(T, z) \in D\}} \mu(dT \times z) \right] Q(z'|z);$$

iv) Feasibility in the labor market is satisfied

$$N - U - \int \left[ \sum_{j=0}^{J} G_j(T, z) \right] \mu(dT \times dz) \geq 0, \quad U \geq 0,$$

v) Feasibility in the consumption good is satisfied

$$C = \int \left\{ F\left( \sum_{j=0}^{J} G_j(T, z) \right) - \sum_{j=1}^{J} \tau_j \ [T_j - G_j(T, z)] \right\} \mu(dT \times dz)$$

vi) The value of search $\sigma$ and the value of becoming non-employed $\theta$ satisfy

$$\sigma = \beta \int W_0(T, z) \, \mu(dT \times dz), \quad \text{and} \quad \theta = \max\left\{ \frac{\omega}{u'(C)} + \beta\theta, \ \sigma \right\};$$

vii) The labor force participation decision is optimal

$$\begin{aligned} 0 &= \left[ N - U - \int \left[ \sum G_j(T, z) \right] \mu(dT \times dz) \right] \left[ \theta - \frac{\omega}{u'(C)} - \beta\theta \right] \\ 0 &= U \ [\theta - \sigma]. \end{aligned}$$

The next theorem establishes the first and second welfare theorems for this economy and provides a partial characterization of the SLM equilibrium.

**Theorem 2** *Welfare Theorems and equilibrium characterization:*
*i) Let $\{C, U, \theta, w, G, B, m, W, \mu\}$ be a spot labor market equilibrium (SLM). Then, there is an island planner value function $V$, for which $\{V, G, C, U, \theta, \mu\}$ is an stationary efficient allocation.*
*ii) Conversely, let $\{V, G, C, U, \theta, \mu\}$ be a stationary efficient allocation. Then, there are wages and value functions $\{w,\ B,\ m,\ W\}$ for which $\{C, U, \theta, w, E, B, m, W, \mu\}$ is a spot labor market equilibrium (SLM).*
*iii) The functions $B, W$ and $V$ related as in i) and ii) satisfy*

$$\begin{aligned} W_0(T,z) &= \partial V(T,z)/\partial T_0 \quad (7)\\ \partial B(T,T,z)/\partial x_j + W_j(T,z) &= \partial V(T,z)/\partial T_j, \text{ for } j = 1, ..., J \end{aligned}$$

The reasons for the equivalence in i) and ii) are similar to those in Prescott and Mehra (1980). However, in Alvarez and Veracierto (2010) we offer a constructive proof of i) and ii) because our set-up does not directly map into theirs. Condition iii) is obtained by comparing the first order conditions for the planning problem with the optimality conditions for the workers and firms in the spot labor market equilibrium (SLM). Recall that $\partial V/\partial T_j$ is the shadow value of a tenure $j$ worker in the island planning problem. Condition iii) says that the shadow value for the planner of an extra worker of tenure zero is the same as the equilibrium value function $W_0$. Instead the shadow value for the planer of a worker of any other tenure $j > 0$, $\partial V/\partial T_j$, is lower than the equilibrium value function for a worker $W_j$. This difference is exactly the shadow value for the firm of an extra worker of tenure $j > 0$, $\partial B/\partial x_j$, which, due to the separation cost, is a number between $-\tau_j$ and 0.

Observe that when the separation costs $\tau_j$ are taxes rebated to households in a lump-sum way, the equilibrium allocation satisfies the same conditions described in Theorem 1, except that aggregate consumption $C$ is given by the integral of output under the invariant distribution $\mu$ (without subtracting the separation taxes $\tau_j$).

# 7 Multiperiod Employment Relations

The competitive equilibrium of Section 6 had a simple structure with spot labor markets because the separation costs $\tau_j$ applied to workers with tenure at the *island level.* In this section we introduce an alternative and more realistic definition of a competitive equilibrium, where separation costs are determined by the tenure of workers at the *firm level.* This specification ties workers with firms and, hence, requires long-term contracts to achieve efficiency. In fact, we will argue that the competitive equilibrium with long-term contracts and tenure at the firm level supports the same equilibrium allocation as the spot labor markets concept of Section 6). This is an important result: There is no loss of realism in specifying that the separation costs are determined by tenure at the island level rather than tenure at the firm level.

To obtain this equivalence result certain restrictions on the separation costs system are needed. Far from being a weakness of the model, these restrictions arise quite naturally when the tenure-dependent separation costs are interpreted as temporary contracts, since they resemble restrictions observed in actual countries. Indeed, since temporary contracts are often introduced with the purpose of increasing flows out of unemployment, their implementation typically includes eligibility clauses. An example is the Spanish reform of 1984, which significantly broadened the scope of fixed-term contracts but specified that workers had to be registered as unemployed to be eligible for temporary employment contracts (see the Appendix in Cabrales and Hopenhayn, 1997). In Portugal temporary contracts could only be used by new firms, or by firms hiring long-term unemployed or first-time job seekers (see Table 1 in Dolado et al., 2001). Another example is the first approved but later withdrawn 2006 CPE ("first employment contract") legislation in France. This type of contract would have allowed an employer to dismiss a worker younger than 25 during the first two years of the contract, provided that he had never been employed at the time of his hiring

To incorporate this type of eligibility restriction we assume that only workers that searched during the previous period (i.e., that were unemployed) can be hired as temporary workers. If a firm hires a worker that was employed somewhere else on the island during the previous period, the worker immediately becomes subject to the regular separation tax $\tau_J$. In this scenario, the market structure would have to be changed to accommodate the fact that workers would try to exploit the bargaining power that they would gain by staying in the same firm. To avoid this, we assume that firms and workers participate in island-wide competitive markets for binding, long-term, state-contingent wage contracts at the time of the hiring. In what follows we provide a formal description of a competitive equilibrium with multiperiod employment relations. For simplicity, and similarly to Section 6), we will treat each island as a separate economy and define an "island-level equilibrium" taking as given the flow of new workers to the island, $U$, and the value to a worker of leaving the island, $\theta$.

There are competitive markets on the island. At each date $t$, history $z^t$, the set of commodities traded is $S\left(z^t\right)$. A commodity $s \in S(z^t)$ is a stopping time indicating the time at which a worker will be dismissed under each possible continuation sequence $z_{t+1}^{\infty} = \{z_{t+1}, z_{t+2}, ...\}$ following the history $z^t$.

Formally, $S(z^t)$ is the set of all functions

$$s\left(z^t; z_{t+1}^{\infty}\right) : Z^{\infty} \to \{t+1, t+2, ..., \infty\}$$

satisfying

$$\begin{aligned} s\left(z^t; z_{t+1}^{\infty}\right) &= k \Rightarrow s\left(z^t; \hat{z}_{t+1}^{\infty}\right) = k, \\ \text{for all } \hat{z}_{t+1}^{\infty} \text{ such that:} \qquad & \{z_{t+1}, z_{t+2}, ..., z_k\} = \{\hat{z}_{t+1}, \hat{z}_{t+2}, ..., \hat{z}_k\}. \end{aligned}$$

When a worker arrives for the first time to the island at date $t$, history $z^t$, he is a "newly arrived" worker and can supply only one stopping time in the set $S\left(z^t\right)$. The worker cannot supply a new stopping time before the previous stopping time is actually executed, i.e. before the worker is separated from his previous employer. The first time that the worker separates he becomes an "incumbent" worker for the rest of his stay on the island. An "incumbent" worker at date $t$, history $z^t$, can also supply one stopping time in the set $S(z^t)$ as long as he has no outstanding stopping time from a previous sale. "Newly arrived" workers and "incumbent" workers sell different commodities, though. The stopping time sold by an "incumbent" worker at date $t$, history $z^t$, entails a cost $\tau_J$ at date $s\left(z^t; z_{t+1}^{\infty}\right)$, for every possible realization $z_{t+1}^{\infty}$. On the contrary, the stopping time sold by a "newly arrived" worker at date $t$, history $z^t$, entails a cost $\tau_{\min\left\{s\left(z^t; z_{t+1}^{\infty}\right)-t, J\right\}}$ at date $s\left(z^t; z_{t+1}^{\infty}\right)$, for every possible realization $z_{t+1}^{\infty}$. Observe that incumbent workers are always subject to the maximum separation cost $\tau_J$ while newly arrived workers are subject to separation costs that depend on their tenure levels with their current employers.

Each stopping time, being a different commodity, has a different price associated with it. We express the price of the stopping times traded at date $t$, history $z^t$, in terms of the final consumption good at that time and event, and denote them for each $s \in S\left(z^t\right)$ by $P^A\left(z^t, s\right)$ and $P^I\left(z^t, s\right)$ for the "newly arrived" and "incumbent" stopping times, respectively. Workers and firms take the prices $P^A\left(z^t, s\right)$ and $P^I\left(z^t, s\right)$ for all $t \geq 0$, $z^t \in Z^t$, and $s \in S\left(z^t\right)$ as given.

The problem of an "incumbent" worker at date $t$, history $z^t$, if she has no outstanding stopping times, is the following:

$$I\left(z^t\right) = \max\left\{\theta, \max_{s \in S(z^t)} \left\{P^I\left(z^t, s\right) + E\left[\beta^{s-t} I\left(z^s\right)\right]\right\}\right\} \tag{8}$$

where the expectation is taken with respect to all possible realizations $z_{t+1}^{\infty} = \{z_{t+1}, z_{t+2}, ...\}$, conditional on $z^t$. This equation states that an incumbent worker can choose to leave the island, obtaining $\theta$, or sell the stopping time $s \in S\left(z^t\right)$ that provides the highest value. A stopping time $s \in S\left(z^t\right)$ provides $P^I\left(z^t, s\right)$ units of the consumption good during the current period and the value $I\left(z^s\right)$ of being an incumbent worker at the (random) stopping time $s$. Observe that, since the worker maximizes the present expected value of his earnings, equation (8) implicitly assumes linear preferences.[13]

The problem of a "newly arrived" worker at time $t$ state $z^t$ is given by

$$A\left(z^t\right) = \max\left\{\theta,\ \max_{s\in S(z^t)}\left\{P^A\left(z^t,s\right) + E\left[\beta^{s-t}\ I\left(z^s\right)\right]\right\}\right\}.$$

This problem is analogous to the "incumbent" worker problem, except that the "newly arrived" worker faces a different price for the stopping time that she sells and becomes an "incumbent" worker at the end of the stopping time (i.e. she changes her type).

We let $N^A\left(z^t,s\right)$ be the quantity of newly arrived workers hired with contract $s\in S\left(z^t\right)$ at date $t$, history $z^t$. Likewise, we let $N^I\left(z^t,s\right)$ be the quantities of incumbent workers hired with contract $s\in S\left(z^t\right)$ at date $t$, history $z^t$. The firm chooses $N^A\left(z^t,s\right)$ and $N^I\left(z^t,s\right)$ for every $z^t$ and $s\in S\left(z^t\right)$ to maximize expected discounted profits, taking as given the prices $P^A\left(z^t,s\right)$ and $P^I\left(z^t,s\right)$, and the fact that the stopping times of the different types of workers entail potentially different separation costs at termination. Without loss of generality, we assume that the firm never employed any worker previous to $t=0$. This will have no consequence in the analysis given our focus on steady state equilibria.

The problem of the representative firm is the following:

$$\max_{N^A,N^I}\sum_{t=0}\sum_{z^t\in Z^t}\beta^t\left[F(n_t\left(z^t\right),z_t) - \sum_{s\in S(z^t)}\left(P^A\left(z^t,s\right)N^A\left(z^t,s\right) + P^I\left(z^t,s\right)N^I\left(z^t,s\right)\right) - T_t\left(z^t\right)\right]\mu_t\left(z^t\right)$$

subject to:

$$n_t\left(z^t\right) = \sum_{i=0}^{t}\left\{\sum_{s\in S\left(z_0^i\right):\ s\left[z_0^i;\left(z_{i+1}^t,z_{t+1}^\infty\right)\right]\ >\ t,\text{ for every } z_{t+1}^\infty}\left[N^A\left(z^i,s\right)+N^I\left(z^i,s\right)\right]\right\} \tag{9}$$

$$\begin{aligned} T_t\left(z^t\right) \quad = \quad & \tau_J\sum_{i=0}^{t-1}\left\{\sum_{s\in S\left(z_0^i\right):\ s\left[z_0^i;\left(z_{i+1}^t,z_{t+1}^\infty\right)\right]\ =\ t,\text{ for every } z_{t+1}^\infty} N^I\left(z^i,s\right)\right\} \\ & +\sum_{i=0}^{t-1}\tau_{\min\{t-i,J\}}\left\{\sum_{s\in S\left(z_0^i\right):\ s\left[z_0^i;\left(z_{i+1}^t,z_{t+1}^\infty\right)\right]\ =\ t,\text{ for every } z_{t+1}^\infty} N^A\left(z^i,s\right)\right\} \end{aligned} \tag{10}$$

where $z_j^i$ in equations (9) and (10) denotes the partial history $\{z_j,z_{j+1},...,z_{i-1},z_i\}$ embodied in $z^t$. The firm maximizes the expected discounted value of profits, which are given by output minus the purchase of the stopping times supplied both by "newly arrived" and "incumbent" workers, minus separation costs.

[13] The linear preferences assumption in this "island-economy" is justified by the existence of perfect insurance within the household in the original economy.

The employment of the firm at date $t$, history $z^t$, is given by equation (9). This equation says that total employment is the sum of all the workers, both "newly arrived" and "incumbents", that were hired between periods zero and $t$ and have been never fired along the history $z^t$. Equation (10) describes the separation costs at time and event $z^t$ as the sum of two terms. The first term is the sum of all "incumbent" workers that have been hired between periods 0 and $t-1$ and have been contracted to separate at date $t$ if event $z^t$ took place, times the maximum separation cost $\tau_J$. The second term is the sum across all dates $i$ between 0 and $t-1$ of the total number of "newly arrived" workers that have been hired at date $i$ and have been contracted to separate at date $t$ if event $z^t$ took place, times the separation cost $\tau_{\min\{t-i,J\}}$ that correspond to their tenure level at date $t$.

The market clearing conditions are as follows. If $N^A\left(z^t,s\right)>0$ at some time $t$ and event $z^t$ and some $s\in S(z^t)$, then

$$A\left(z^t\right)=P^A\left(z^t,s\right)+E\left[\beta^{s-t}\ I\left(z^s\right)\right]$$

Also,

$$\sum_{s\in S(z^t)} N^A\left(z^t,s\right)<U\ \Rightarrow A\left(z^t\right)=\theta.$$

The conditions for "incumbent" workers are similar. If $N^I\left(z^t,s\right)>0$ at some time $t$ and event $z^t$ and some $s\in S(z^t)$, then

$$I\left(z^t\right)=P^I\left(z^t,s\right)+E\left[\beta^{s-t}\ I\left(z^s\right)\right].$$

Also,

$$\sum_{s\in S(z^t)} N^I\left(z^t,s\right)<X^I\left(z^t\right)\Rightarrow I\left(z^t\right)=\theta,$$

where $X^I\left(z^t\right)$ is the number of "incumbent" workers available for hiring at the beginning of time $t$ and event $z^t$, which is given by:

$$X^I\left(z^t\right)=\sum_{i=0}^{t-1}\left\{\sum_{s\in S\left(z_0^i\right):\ s\left[z_0^i;\left(z_{i+1}^t,z_{t+1}^\infty\right)\right]\ =\ t,\text{ for every } z_{t+1}^\infty}\left[N^I\left(z^i,s\right)+N^A\left(z^i,s\right)\right]\right\}\tag{11}$$

Finally, the hiring of each type of workers cannot exceed the amount initially available:

$$\sum_{s\in S(z^t)} N^A\left(z^t,s\right)\le U\tag{12}$$

$$\sum_{s\in S(z^t)} N^I\left(z^t,s\right)\le X^I\left(z^t\right)\tag{13}$$

Observe that the supply of stopping time is indivisible: Workers can supply one stopping time $s\in S(z^t)$, only if worker has no previous outstanding stopping time. However, the linear preferences assumed, together with the convex production possibility set of the firm, guarantee that the welfare theorems hold. The competitive allocation is then obtained as the solution to the social planner's problem, which is to maximize

$$\sum_{t=0}\sum_{z^t\in Z^t}\beta^t\left[F(n_t\left(z^t\right),z_t)+\theta\left(U-\sum_{s\in S(z^t)}N^A\left(z^t,s\right)\right)\right.$$
$$\left.+\,\theta\left(X^I\left(z^t\right)-\sum_{s\in S(z^t)}N^I\left(z^t,s\right)\right)-T_t\left(z^t\right)\right]\mu_t\left(z^t\right)$$

subject to equations (9), (10), (11), (12) and (13).

A few remarks are in order. Clearly, the social planner will never want to separate a "newly arrived" worker and rehire him as a an "incumbent" before the trial period is over (by "trial period" we refer to the period of time during which a worker has tenure less than $J$). The reason is that being rehired as "incumbent" makes the worker liable to the maximum separation cost $\tau_J$, while maintaining his "newly arrived" status saves on separation costs during the trial period. Also, the social planner will never want to separate a "newly arrived" worker after the trial period is over and rehire him under an "incumbent" contract because this entails incurring the separation cost $\tau_J$ without any benefit. As a consequence, the planner will choose the stopping times for "newly arrived" workers in such a way that they separate only when they are to leave the island (and receive the value $\theta$). This means that $N^I\left(z^t,s\right)=0$ for every $z^t$ and every $s\in S\left(z^t\right)$, i.e. the social planner will never use "incumbent" workers.

Being left with only "newly arrived" workers, the planner's problem is formally identical to the island planning problem described in Section 5.1.[14] This has an important implication: The competitive equilibrium with long-term contracts and tenure at the firm level described here is equivalent to the competitive equilibrium with spot labor contracts and tenure at the island level described in Section 6 . Moreover, for every $z^t$ and $s\in S(z^t)$ such that $N^A\left(z^t,s\right)>0$, the price $P^A\left(z^t,s\right)$ must be equal to the expected discounted value of the spot wages obtained (in the equilibrium with spot labor contracts and tenure at the island level) by a worker that arrives to the island at time $t$ and event $z^t$, and follows the employment plan described by the stopping time $s$. Hence, $A\left(z^t\right)=W_0\left(T\left(z^t,X\right),z_t\right)$, where $T$ is defined by equations (1)-(3) and the Pareto optimal employment process $\{E_t\}$. Given this property, an economy-wide equilibrium can be constructed from the island-level competitive equilibrium described thus far in the same way as it is done in the "spot labor market" competitive equilibrium of Section 6.

[14] In particular, it is identical to the problem of an island's planner endowed with no worker of positive tenure at $t=0$.

# 8 Fixed-term employment contracts

In this section we specialize the tenure-dependent separation costs to the following specification:

$$\tau_j = 0, \text{ for } j = 0, 1, ..., J-1, \tag{14}$$

$$\tau_j = \tau, \text{ for } j = J. \tag{15}$$

That is, we allow for a trial period of length $J-1$ during which a worker can be separated at no cost, but once the trial period is over the worker becomes subject to a regular separation cost $\tau$. This stylized version of fixed-term employment contracts will allow us to provide a sharp characterization of equilibrium adjustments.

## 8.1 Simplified Island Planning Problem

When the separation costs satisfy equations (14)-(15), it can be shown that the solution to the island planning problem described in Section 5.1 has the following three properties (see Appendix A for more details).

First, if $T_j > 0$, then $\partial V(T, z) / \partial T_j \geq \theta$ for $j \leq J$ and $\geq \theta - \tau$ for $j = J$. This must be the case, since the planner always has the option of sending workers back to the central location (Propositions 7 and 9).

Second, if some permanent workers are fired, i.e., if $E_J = G_J(T, z) < T_J$, then all the temporary workers must have been fired, i.e., $E_j = G_j(T, z) = 0$ for all $j = 0, ..., J-1$. A policy with this property saves on the separation cost $\tau$, which is incurred only by the permanent workers (Proposition 13).

Third, the first workers to be fired are the temporary workers with the longest tenure (Proposition 10 and its Corollary). The intuition for this property is that, while all workers are perfect substitutes in production, these workers are the closest to becoming subject to the separation cost $\tau$. Thus this policy saves on potential separation costs. In an economy where all island planners have followed this policy in the past and a constant flow $U$ of tenure $j = 0$ workers has arrived to each island every period, the states $T$ in the ergodic set take a particular form. Formally, the ergodic set is a subset of $\mathcal{E}$, which is given by

$$\mathcal{E} = \left\{ T \in [0, U]^{J-1} \times R_+ : T = (U, ..., U, T_j, 0, ..., 0, T_J), \text{ for some } j: \ 1 \leq j \leq J-1 \right\} \tag{16}$$

This last property is extremely important: It allows us to reduce the dimensionality of the endogenous state of the island planning problem from $J$ to 2. In particular, states for the island planning problem $T$ that belong to $\mathcal{E}$ can be described by two numbers: $t$, the total number of temporary workers (workers with tenure less than $J$), and $p$, the number of permanent workers (workers with tenure greater or equal to $J$ ). We use this feature to consider the island planning problem with a simplified state $(t, p, z)$. In this *simplified island planning problem*, the choices are employment of temporary workers $e_t$ and employment of permanent workers $e_p$. The law of motion for the endogenous state is:

$$t' = U + e_t - \max\{e_t - (J-1)U,\ 0\} \text{ and } p' = e_p + \max\{e_t - (J-1)U,\ 0\}. \tag{17}$$

The number of temporary workers next period $t'$ is equal to the number of temporary workers employed during the current period $e_t$ plus the arrival of new workers $U$, minus the temporary workers that will become permanent next period, max $\{e_t - (J-1)U,\ 0\}$. Likewise, the number of permanent workers next period $p'$ is equal to the number of permanent workers employed during the current period $e_p$ plus the temporary workers that will become permanent next period. The island planner's value function $v : [U, J \cdot U] \times R_+ \times Z \to R$ satisfies the following Bellman equation:

$$v(t,p,z) = \max_{e_t,e_p,t',p'} \left\{ F(e_t + e_p, z) + \theta [t - e_t] + (\theta - \tau)[p - e_p] + \beta \int v(t',\ p', z') Q(z, dz') \right\}$$

subject to

$$0 \le e_t \le t, \quad 0 \le e_p \le p,$$

and the law of motion (17).

Formally, $v$ is related to $V$ for states $T \in \mathcal{E}$ as follows:

$$v(T_1 + T_2 + ... + T_{J-1}, T_J, z) = V(T_1, T_2, ..., T_{J-1}, T_J, z).$$

Since $v$ and $V$ are closely related and $V$ is concave, then $v$ is concave in $(t,p)$, even though the graph of the feasible set for this problem is not convex. From the definition of $v$ and the properties of $V$, we have that $v$ is differentiable with respect to $t$ for all $t > 0$ that are not integer multiples of $U$, and differentiable with respect to $p$ for all $p > 0$. Thus, for all $(t,p,z)$ with $p > 0$

$$\frac{\partial v(t,p,z)}{\partial p} = \frac{\partial V(T,z)}{\partial T_J}$$

and for all $t$ that can be written as $t = (j-1)U + T_j$ with $T_j \in (0, U)$,

$$\frac{\partial v(t,p,z)}{\partial t} = \frac{\partial V(T,z)}{\partial T_j}.$$

At the points $t$ given by $t = j \times U$ for some $j = 1, ..., J-2$, the right derivative of $v$ with respect to $t$ is $\partial V / \partial T_j$, and its left derivative is $\partial V / \partial T_{j+1}$.

The main result of this section is a characterization of the optimal policy. The optimal policy is defined by a two-dimensional set of inaction $I(z)$. For each $z$, the optimal policy $(e_t(t,p,z), e_p(t,p,z))$ is to stay in the set of inaction $I(z)$ or otherwise move to its boundary, as explained below. The boundary of the set of inaction is described by two continuous functions, $\hat{p}$ and $\hat{t}$ defined in $\hat{p} : Z \to R_+$ and

$\hat{t} : R_+ \times Z \to [0,\ J \cdot U]$. The function $\hat{t}$ is decreasing in $p$ and hits zero at a value of $p \leq \hat{p}(z)$. The function $\hat{t}$ is the boundary of the set of inaction for the values $t$ that are strictly positive. Formally, these functions define the set of inaction $I(z)$ as follows:

**Definition 3** *For each $z \in Z$,*

$$I(z) = \left\{(t,p) \in [0, J \cdot U] \times R_+ : p \leq \hat{p}(z),\ \text{and } t \leq \hat{t}(p,z)\ \right\} \tag{18}$$

The optimal policy is as follows: if $p \leq \hat{p}(z)$ and the state is outside the set of inaction $I(z)$, temporary workers are fired until the boundary of $I(z)$ is hit, with no change in permanent workers. If $p > \hat{p}(z)$, all temporary workers are fired, and permanent workers are fired to hit $\hat{p}(z)$. Formally,

$$e_t(t,p,z) = \min\left\{t,\ \hat{t}(p,z)\right\}, \quad \text{and} \quad e_p(t,p,z) = \min\{p,\ \hat{p}(z)\}\ . \tag{19}$$

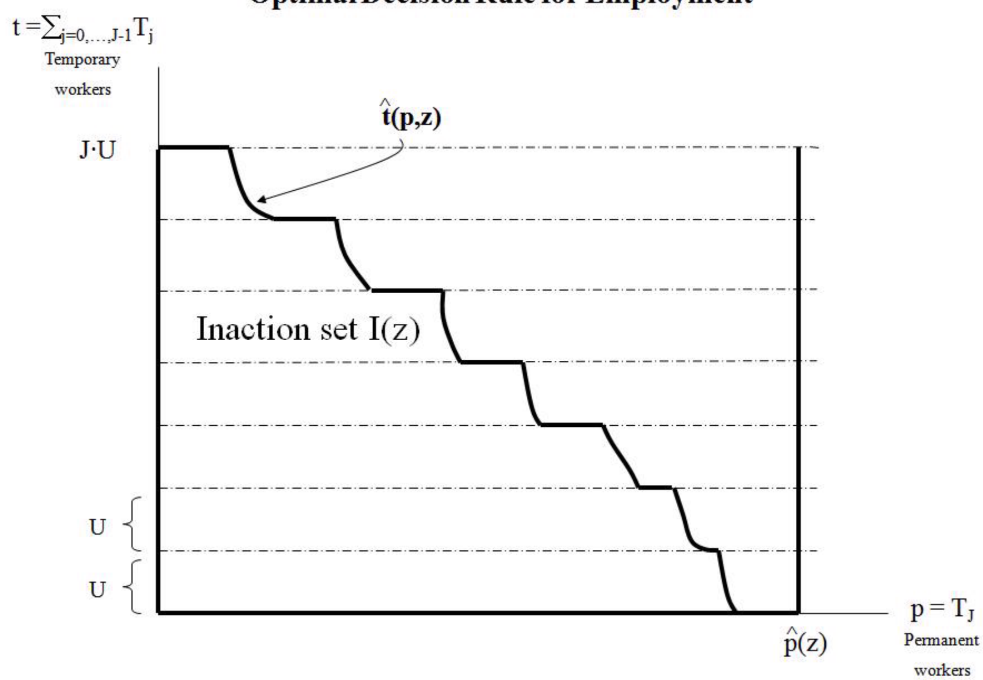
Figure 1 illustrates the typical shape of the set of Inaction (for a given value of $z$).

The threshold $\hat{p}(z)$ solves

$$\theta - \tau = f(\hat{p}(z), z) + \beta \int \frac{\partial v}{\partial p}(U, \hat{p}(z))\, Q(z, dz')$$

That is, $\hat{p}$ is the lowest number of permanent workers for which the marginal value of a permanent worker is equal to $\theta - \tau$ and, thus, any additional permanent worker would be returned to the central location.

**Figure 1**
**Optimal Decision Rule for Employment**



Given $(p, z)$, the function $\hat{t}(p, z)$ is defined as the lowest number of temporary workers $t$ for which the marginal value of a temporary worker is equal to $\theta$ and, thus, any additional temporary worker would be returned to the central location. The function $\hat{t}(p, z)$ solves

$$\theta = f\left(\hat{t}(p, z) + p, z\right) + \beta \int \frac{\partial v}{\partial t}\left(\hat{t}(p, z) + U,\ p\right) Q\left(z, dz'\right)$$

for $\hat{t}(p, z) \leq (J-1)U$ and

$$\theta = f\left(\hat{t}(p, z) + p, z\right) + \beta \int \frac{\partial v}{\partial p}\left(JU,\ p + \hat{t}(p, z) - (J-1)U\ \right) Q\left(z, dz'\right)$$

for $\hat{t}(p, z) \in ((J-1)U, JU]$. To simplify the exposition we have written these expressions assuming that $v$ is differentiable. If $v$ is evaluated at integers multiples of $U$ (where $v$ is not differentiable), the expressions would have to be rewritten in terms of the subgradients of $v$.

The intuition for why the frontier of the set of inaction, given by $\hat{t}$, is decreasing in $p$ is that temporary and permanent workers are perfect substitutes in production. Indeed, it can be shown that $\hat{t}$ is strictly decreasing for values of $p$ such that $\hat{t}(p, z)$ is not an integer multiple of $U$. At the points on which $\hat{t}$ is an integer multiple of $U$, this function can be flat: At these points the function $v$ may not be differentiable, as explained above. While all these properties are quite intuitive, the proofs are involved because of the non-differentiability of $v$ (Appendix B provides more details).

The optimal decision rule for the island planning problem generates considerable churning of temporary workers. To see that this is the case, consider an island in a state $(t, p, z)$ satisfying that $p \leq \hat{p}(z)$ and $t \geq \hat{t}(p, z)$. From equation (19) we know that such an island employs $\hat{t}(p, z)$ temporary workers and $p$ permanent workers. Since the island receives $U$ new workers every period it follows that, as long as its productivity level remains at $z$, its state will be given by $\left(\hat{t}(p, z) + U, p, z\right)$ and it will employ $\hat{t}(p, z)$ temporary workers and $p$ permanent workers. Given that the first temporary workers to be sent back to the central location are those with the longest tenure, as long as the productivity level remains at $z$, the island's planner keeps replacing $U$ temporary workers with the longest tenure with the new arrivals. Observe that the undirected search assumption is crucial for obtaining this result: If search was directed all new arrivals would be at islands that expand their employment levels and, therefore, no churning of temporary workers would take place.

## 8.2 Wage Profiles in a Spot Labor Market Equilibrium

Since in a spot labor market equilibrium (Section 6) workers and firms take competitive wages as given, the equilibrium pattern of wages across tenure levels must induce firms and workers to follow the employment adjustments described in Figure 1. Proposition 4 provides a partial characterization of equilibrium wages.

It shows that there are three equilibrium levels of wages in a given location: one level for temporary workers with tenures $j = 0, ..., J-2$, a second level for workers that are about to become permanent, i.e., those with tenure $J-1$, and a third level for permanent workers, i.e., those with tenure $J$ or higher. Temporary workers with tenures $j = 0$ to $j = J-2$ are paid their marginal productivity. Wages of workers with tenure $J-1$, i.e., those that would become permanent if they were to work during the current period, are (weakly) smaller than their marginal productivity. Wages of permanent workers are (weakly) higher than those with tenure $J-1$.

**Proposition 4** *Suppose that separation costs satisfy equations (14)-(15). Let* $\{U, \theta, w, G, B, m, W, \mu\}$ *be a spot labor market equilibrium (SLM). Without loss of generality, the equilibrium wage* $w$ *can be chosen to satisfy*

*a) for all* $j = 0, 1, ..., J-2$

$$w_j(T, z) = f\left(\sum_{i=0}^{J} G_i(T, z), z\right),$$

*b) for all* $j = 0, 1, ..., J-2$

$$\begin{aligned} w_j(T, z) - \beta\tau &\leq w_{J-1}(T, z) \leq w_j(T, z) \\ w_j(T, z) - \beta\tau &\leq w_J(T, z) \leq w_j(T, z) + \tau \\ w_{J-1}(T, z) &\leq w_J(T, z) \leq w_{J-1}(T, z) + \tau \end{aligned}$$

*and if* $E_J(T, z) < T_J$ *:*

$$w_{J-1}(T, z) \leq w_j(T, z) < w_J(T, z),$$

*c) and the equilibrium value function* $W$ *for workers can be chosen so that they satisfy:*

$$\begin{aligned} W_0(T, z) &\geq W_1(T, z) \geq \cdots \geq W_{J-1}(T, z) \\ W_J(T, z) &\geq W_{J-1}(T, z) \ . \end{aligned}$$

The proof of Proposition 4 follows, essentially, from the analysis of the first order conditions of the firms problem (see Alvarez and Veracierto, 2010, for details).

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**Appendix A: Analysis of the Island Planning Problem**

Consider the problem of the planner of an island that receives $U$ workers per period and that starts with workers $(T_1, T_2, ..., T_{J-1}, T_J)$ where $T_i$ is the number of workers with tenure $i = 1, 2, ..., J$. Define $E$ as the set of possible workers tenure profiles, $E = [0, U]^{J-1} \times R_+$. The planners value function $V : E \times Z$ solves

$$H[V](T_1, T_2, ..., T_{J-1}, T_J, z) = \max_{\{E_i\}_{i=0}^{J}} \left\{ F\left(\sum_{i=0}^{J} E_i, z\right) + \sum_{i=0}^{J} \theta [T_i - E_i] - \sum_{j=1}^{J} \tau_j [T_j - E_j] \right. \left. + \beta \int V(E_0, E_1, ..., E_{J-2}, E_{J-1} + E_J, z') Q(z, dz') \right\} \tag{20}$$

subject to

$$0 \leq E_0 \leq U, \text{ and } 0 \leq E_i \leq T_i \text{ for } i = 1, 2, ..., J.$$

**Proposition 5** *$H$ maps concave functions into concave ones.*

We use the following notation for subgradients. Let $G : X \to R$ be a concave function. We use $\partial G(x)$ to denote its subgradient at $x$ (if it is clear the value of $x$ from the context we simply use $\partial G$ ). In our case $X \subset R^n$, we use $\partial G_{x_i}(x)$ for $i = 1, 2, ..., n$ (and $\partial G_{x_i}$ when it is clear) to denote the projection of $\partial G(x)$ into the subspace of the $x_i's$. Abusing notation, we use $G_{x_i}(x)$ (and $G_i$ when it is clear) to denote a generic element of $\partial G_{x_i}(x)$, so that $G_{x_i}(x) \in \partial G_{x_i}(x)$.

The next proposition gives a useful result, ordering the subgradients of $V$

**Proposition 6** *Consider a function $V$ satisfying*

$$V_{T_1} \geq V_{T_2} \geq \cdots \geq V_{T_{J-1}} \geq V_{T_J}, \tag{21}$$

$$V_{T_1} + \tau_1 \leq V_{T_J} + \tau_J \tag{22}$$

*for all $z$ and $T > 0$, where $\left(V_{T_1}, V_{T_2}, \ldots, V_{T_{J-1}}, V_{T_J}\right) \in \partial V(T, z)$. Then,*

$$H[V]_{T_1} \geq H[V]_{T_2} \geq \cdots \geq H[V]_{T_{J-1}} \geq H[V]_{T_J}, \tag{23}$$

$$H[V]_{T_1} \leq H[V]_{T_J} + \tau_J - \tau_1 \tag{24}$$

*for all $z$ and $T > 0$, where $\left(H[V]_{T_1}, H[V]_{T_2}, \ldots, H[V]_{T_{J-1}}, H[V]_{T_J}\right) \in \partial H[V](T, z)$.*

Intuitively it follows from the assumption that workers are perfect substitutes and from the fact that the $\tau_j$'s are increasing in $j$.

The next set of results establish that the fixed point $V = H[V]$ is differentiable and its derivatives are indeed given by $V_j^*$ in equation (4). The results in the next proposition are analogous to standard manipulations of first order conditions, except for the fact that $V$ may not be differentiable.

Define the function $\hat{R}(E, z)$, as follows: $\hat{R} : R_+^{J+1} \times Z \to R$

$$\hat{R}(E, z) = F\left(\sum_{i=0}^{J} E_i, z\right) - \theta \sum_{i=0}^{J} E_i + \sum_{i=1}^{J} \tau_i E_i + \beta \int V(E_0, E_1, \ldots, E_{J-2}, E_{J-1} + E_J, z') Q(z, dz').$$

**Proposition 7** *Let $V$ be concave. Fix $T, z$. Then, $0 \leq E^* \leq T$ is an optimal choice given $T, z$ if and only if for all $\{H[V]_i(T,z)\}_{i=0}^{J} \in \partial H[V](T,z)$, there is a $\left\{\hat{R}_i\right\}_{i=0}^{J} \in \partial \hat{R}(E^*, z)$ such that for every $i = 0, ..., J$,*

$$
\begin{aligned}
H[V]_i(T,z) &= \hat{R}_i(E^*,z) + \theta - \tau_i \\
\hat{R}_i(E^*,z) &\geq f\left(\sum_{i=0}^{J} E_i^*, z\right) - (\theta - \tau_i) + \beta \int V_{\min(J,i+1)}\left(E_0^*, \ldots, E_{J-2}^*, E_{J-1}^* + E_J^*, z'\right) Q(z, dz'), \\
\textit{with} &= \textit{ if } E_i^* > 0 \\
\hat{R}_i(E^*,z) &\geq 0, \\
0 &= \left(H[V]_i(T,z) - (\theta - \tau_i)\right)(T_i - E_i^*),
\end{aligned}
$$

where we let $U = T_0$ and $\tau_0 = 0$.

The next lemma shows that employment is bounded below, and hence marginal productivity is bounded above.

**Lemma 8** *There is an $e > 0$ such that for all $T, z : \sum_{i=0}^{J} E_i(T,z) \geq e > 0$.*

By this lemma, the solution for $V_j^*$ in equation (4) is well defined because $f\left(\sum_{i=0}^{J} E_{i,s}^*, z_s\right)$ is uniformly bounded.

**Proposition 9** *Let $V$ be the fixed point of $H$. Assume that $U > 0$. Then $V$ is differentiable with respect to $T_i$ when $T_i > 0$.*

The following proposition and corollaries are important to characterize the solution of the problem and reduce its dimensionality when the separation costs satisfy equations (14)-(15).

**Proposition 10** *Suppose that the separation costs $\tau_j$ satisfy equations (14)-(15). Let $V$ satisfy (21). Then the policies for $H[V]$ satisfy the following. Let $E = (E_0, E_1, ..., E_{J-1}, E_J) \in [0, U]^J \times R_+$ be feasible given $T$. Consider an alternative $\tilde{E} = \left(\tilde{E}_0, \tilde{E}_1, ..., \tilde{E}_{J-1}, \tilde{E}_J\right)$ such that: i) it is feasible for $T$, ii)*

$$\sum_{j=0}^{J-1} E_j = \sum_{j=0}^{J-1} \tilde{E}_j \textit{ and } E_J = \tilde{E}_J,$$

*and iii) there is a $j'$ such that $\tilde{E}_j \geq E_j$ for all $j \leq j' \leq J-1$ and that $\tilde{E}_j = 0$ for all $j$, $j' < j \leq J-1$. Then $\tilde{E}$ is weakly preferred to $E$.*

**Corollary 11** *The optimal policy can be chosen with the following property:*
*(*) If $E_j < T_j$ for some $j$, $1 \leq j \leq J-1$, then $E_{j'} = 0$ for all $j' : j < j' \leq J-1$.*

The following corollary states that when the separation costs satisfy equations (14)-(15).the ergodic set is a subset of $\mathcal{E}$, which is given by equation (16).

**Corollary 12** *If $T \in \mathcal{E}$ and $T'$ is given by the optimal policy*

$$T' = (T_1', T_2', ..., T_J') = (E_0, E_1, ..., E_{J-2}, E_{J-1} + E_J)$$

*then $T' \in \mathcal{E}$.*

The following proposition also applies to the case when the separation costs satisfy equations (14)-(15). It shows that if some permanent workers are fired then all temporary workers must have been fired.

**Proposition 13** *Suppose that the separation costs $\tau_j$ satisfy equations (14)-(15). Then, $E_J^* < T_J \Rightarrow E_i^* = 0$ for every $i \neq J$.*

**Appendix B: Analysis of the Simplified Island Planning Problem**

Throughout this appendix separation costs are assumed to satisfy equations (14)-(15).

The planner's value function $v : [0, J \cdot U] \times R_+ \times Z$ has to satisfy the functional equation $h$:

$$h[v](t,p,z) = \max_{e_t, e_p} \left\{ F(e_t + e_p, z) + \theta[t - e_t] + (\theta - \tau)[p - e_p] + \beta \int v(t', p', z') Q(z, dz') \right\} \tag{25}$$

subject to

$$0 \le e_t \le t, \text{ and } 0 \le e_p \le p,$$

where the law of motion is given by

$$t' = \min\{U + e_t, JU\} \quad \text{and} \quad p' = e_p + \max\{U + e_t - JU, 0\}$$

**Proposition 14** *Consider $V$ and $v$ such that*

$$v(T_1 + T_2 + ... + T_{J-1}, T_J, z) = V(T_1, T_2, ..., T_{J-1}, T_J, z) \tag{26}$$

*for all $(T_1, T_2, ..., T_{J-1}, T_J) \in \mathcal{E}$. Then*

$$h[v](T_1 + T_2 + ... + T_{J-1}, T_J, z) = H[V](T_1, T_2, ..., T_{J-1}, T_J, z) \tag{27}$$

*for all $(T_1, T_2, ..., T_{J-1}, T_J) \in \mathcal{E}$.*

**Lemma 15** *Assume that $V$ satisfies (21). Consider $T$ and $\hat{T}$ and $V$ such that*

$$T_1 + T_2 + ... + T_{J-1} = \hat{T}_1 + \hat{T}_2 + ... + \hat{T}_{J-1} \text{ and } T_J = \hat{T}_J. \tag{28}$$

*for any $\hat{T} \in \mathcal{E}$ and $T \in E$ then*

$$H[V](T,z) \le H[V]\left(\hat{T}, z\right).$$

**Proposition 16** *Let $v$ be the function corresponding to $V$ as in (26) defined for $T \in \mathcal{E}$. Assume that $V(\cdot, z)$ is concave, and $V$ satisfies (21). Then $h[v](\cdot, z)$ is concave in $t, p$.*

**Remark 17** *The previous proposition is not obvious since the feasible set of the problem defined by the right hand side of $h[v]$ in (25) is not convex.*

We now introduce the $R$, which is the objective function being maximized in $h[v]$. The "derivatives" of $R$ are used to define the functions $\hat{t}$ and $\hat{p}$.

**Definition 18** *Given $v$, define $R(e_t, e_p, z)$ as*

$$R(e_t, e_p, z) = F(e_t + e_p, z) - \theta e_t - (\theta - \tau) e_p$$
$$+\beta \int v(U + \min\{e_t, (J-1)U\}, e_t + e_p - \min\{e_t, (J-1)U\}, z') Q(z, dz')$$

Consider an island planner with no temporary workers ($t = 0$) and a given $z$. The quantity $\hat{p}(z)$ is the number of permanent workers that leaves the island's planner indifferent between firing "one" permanent worker and keeping all $\hat{p}(z)$ of them.

**Definition 19** *Let $R$ be defined as in definition (18). For each $z$ define $\hat{p}(z)$, such that*

$$0 \in \partial R_{e_p}(0, \hat{p}(z), z).$$

Consider an island planner with $0 < p < \hat{p}(z)$, so it does it not want to fire any permanent worker for that $z$. The quantity $\hat{t}(p, z)$ is the number of temporary workers that leaves the island's planner indifferent between firing "one" transitory worker and keeping all $\hat{t}(p, z)$ of them. Formally:

**Definition 20** *Let $R$ be defined as in definition (18). For each $p, z$ define $\hat{t}(p, z)$ as follows*:
*(i) if $R_{et} > 0$ for all $R_{et} \in \partial R_{et}(U \cdot J, p, z)$, then $\hat{t}(p, z) = J \cdot U$,*
*(ii) if $R_{et} < 0$ for all $R_{et} \in \partial R_{et}(0, p, z)$, then $\hat{t}(p, z) = 0$,*
*(iii) otherwise $\hat{t}(p, z)$ solves $0 \in \partial R_{e_t}\left(\hat{t}(p, z), p, z\right)$.*

The remaining of this appendix shows that $\hat{p}, \hat{t}$ exist, they are unique, and $\hat{t}$ is decreasing in $p$. The proofs are complicated by the fact that $R$ is not differentiable.

**Proposition 21** *Let $v$ be given by $V$ as in (26). Assume that $V$ is concave and satisfies (21). The function $R(\cdot, z)$ is strictly concave.*

**Remark 22** *It is standard to show that $h[v]$ is increasing in $t$, $p$ and $z$ if $v$ has these properties.*

**Remark 23** *Assume that $V$ satisfies (21) and (22). Let $v$ be defined as in (26). Denote by $\partial h[v]$ the subgradient of $h[v](t, p, z)$ when $v$ is considered as a function of $t$ and $p$. A corollary of Proposition (14) and Proposition (6) is that*

$$h[v]_p \leq h[v]_t \leq h[v]_p + \tau,$$

*for all $\left(h[v]_t, h[v]_p\right) \in \partial h[v](t, p, z)$.*

**Proposition 24** *Fix $t, p, z$. Assume that $V$ satisfies (21), (22), and is concave. Define $v$ as in (26). Let $\left(h[v]_t, h[v]_p\right) \in \partial h[v](t, p, z)$. Then $h[v]_p \geq \theta - \tau$. Moreover, there exists a $\bar{p}(z)$ such that for all $p \geq \bar{p}(z)$ and $t$, $h[v]_p = \theta - \tau$ for any $h[v]_p \in \partial h[v]_p(t, p, z)$.*

Given $v$ define

$$b(e_t, e_p, z) \equiv \int v(U + \min\{e_t, (J-1)U\}, e_t + e_p - \min\{e_t, (J-1)U\}, z') Q(z, dz')$$

as a function of $e_t$ and $e_p$ and $z$. Let $\partial B$ be its subgradient with respect to $(e_t, e_p)$.

**Lemma 25** *Define $v$ as in (26). Assume that $v$ is concave and satisfies $v_p \leq v_t \leq v_p + \tau$, for all $t, p, z$. Fix any $z, e_t, e_p$. Let $\left(b_{e_t}, b_{e_p}\right) \in \partial b\left(e_t, e_p, z\right)$. Then, $b_{e_p} \leq b_{e_t} \leq b_{e_p} + \tau$.*

Let $\partial R\left(e_t, e_p, z\right)$ be the subgradient of $R$ when considered as a function of $\left(e_t, e_p\right)$.

**Lemma 26** *Assume that $v$ is concave and satisfies $v_p \leq v_t \leq v_p + \tau$,for all $t, p, z$. Fix any $z$, $e_t, e_t$. For all $\left(R_{e_p}, R_{e_t}\right) \in \partial R\left(e_t, e_p, z\right)$ :*

$$R_{e_p} \geq R_{e_t} + \tau\left(1 - \beta\right).$$

**Corollary 27** *Let $e_p, e_t$ be the optimal choice of employment for Problem (25). If $e_p < p$ and $t > 0$, then $e_t = 0$. If this were not true, i.e. if $e_p < p$ and $e_t > 0$, then $R_{e_p} = R_{e_t} = 0$, which contradicts Lemma 26.*

**Lemma 28** *Let $v$ be given by $V$ as in (26), assume that $V$ is concave and satisfies (21). Let $R$ be defined as in definition (18).*
*For each $z$ there is a unique $\hat{p}$ satisfying (19). Moreover, $0 < \hat{p}\left(z\right) < \bar{p}\left(z\right) < +\infty$.*

Using the concavity of $R$ and strict concavity of $F$ we define $\hat{t}$ as follows.

**Lemma 29** *Let $v$ be given by $V$ as in (26), assume that $V$ is concave and satisfies (21). Let $R$ be defined as in definition (18).*
*Then for each $\left(p, z\right)$, $0 < p < \hat{p}\left(z\right)$, there exists a unique $\hat{t}$ that satisfies (20).*

**Proposition 30** *Assume that $V$ is concave and satisfies (21) and (22). Let $v$ be given by $V$ as in (26). Assume, without loss of generality that $v$ is concave in $\left(t, p\right)$. Then,*
*i) The optimal decision rules of $h\left[v\right]$ are described by the set of Inaction for $R$ as*

$$e_t\left(t, p, z\right) = \min\left\{t, \hat{t}\left(p, z\right)\right\}, \quad \text{and} \quad e_p\left(t, p, z\right) = \min\left\{p, \hat{p}\left(z\right)\right\}$$

*for all $t, p, z$.*
*ii) $H\left[V\right]$ is concave, and satisfies (21) and (22).*
*iii) $h\left[v\right]$ and $H\left[V\right]$ satisfy (26), and $h\left[v\right]$ is concave.*

**Lemma 31** *Let $V$ be concave, and satisfy (21) and (22). Let $v$ be defined as in (26). Let $\hat{p}$, $\hat{t}$ and $I$ be defined as in (18), (19), and equation (18), respectively. Then, the subgradients of $h\left[v\right]$ are as follows:*
*If $t \neq iU$ for $i = 1, 2, .., J - 1$, then $h\left[v\right]\left(t, p, z\right)$ is differentiable with respect to $t$.*
*If $\left(t, p\right) \in Int\left(I\left(z\right)\right)$:*

$$h\left[v\right]_t\left(t, p, z\right) = f\left(t + p, z\right) + \beta \int b_{e_t}\left(t, p, z'\right) Q\left(z, dz'\right) > \theta,$$

*If $\left(t, p\right) \in Int\left(I\left(z\right)^C\right)$:*

$$h\left[v\right]_t\left(t, p, z\right) = \theta > f\left(t + p, z\right) + \beta \int b_{e_t}\left(t, p, z'\right) Q\left(z, dz'\right),$$

*If $\left(t, p\right)$: $t = \hat{t}\left(p, z\right) < JU$:*

$$\left[\underline{h}\left[v\right]_t\left(t, p, z\right), \bar{h}\left[v\right]_t\left(t, p, z\right)\right] = \left[\theta,\ f\left(t + p, z\right) + \beta \bar{b}_{et}\left(t, p, z\right)\right]$$

**Definition 32** *We say that $\partial v_t\ (t,p,z)$ is decreasing in $p$ if it satisfies the following property. If $p < p'$, define $\underline{v}'_t, \bar{v}'_t, \underline{v_t}$ and $\bar{v}_t$ satisfying $\left[\underline{v}'_t, \bar{v}'_t\right] = \partial v_t\,(t,p',z)\,,\quad$ and $\left[\underline{v_t}, \bar{v}_t\right] = \partial v_t\,(t,p,z)\,.$ Then, $\underline{v}'_t \leq \underline{v}_t$ and $\bar{v}'_t \leq \bar{v}_t$ .*

Notice that if $v$ is differentiable at $(t,p,z)$, this property simply says that $\partial v\,(t,p,z)\,/\partial t$ is decreasing in $p$.

**Lemma 33** *. Let $V$ be concave, and satisfy (21) and (22). Let $v$ be defined as in (26). Assume that the subgradient of $v_t$ is decreasing in $p$, i.e. it satisfies definition (32). Let $\hat{t}\,(p,z)$ be defined as in (19) for the optimal rule that attains the right hand side of $h\,[v]$. Then, the subgradient of $h\,[v]_t$ is decreasing in $p$ too, i.e. it satisfies definition (32) and $\hat{t}\,(p,z)$ is weakly decreasing in $p$.*

**Proposition 34** *Let $v$ be the fixed point of $h$. Let $\hat{t}$ be defined as in definition (18). Then $\hat{t}\,(p,z)$ is decreasing in $p$. Moreover, if $\hat{t}$ is not a multiple of $U$, then $\hat{t}$ is strictly decreasing in $p$.*