Theory of Income, Fall 2008
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Problem Set 8

1 Fiscal Policy in the Neoclassical Growth Model

1.1 Steady states

Feasibility is given by

$$g_t + c_t + k_{t+1} = F(k_t, n_t) + (1-\delta) k_t,$$

where F is a neoclassical constant returns to scale production function, k_t is capital, c_t is consumption, n_t is labor, and δ is the depreciation rate of capital. There is an endowment of time of 1 per period, so that leisure is $l_t = 1 - n_t$. Preferences are given by

$$\sum_{t=0}^{\infty} \beta^t v(c_t, 1-n_t).$$

The consumer has a budget constraint given by

$$\sum_{t=0}^{\infty} p_t [c_t + x_t + \tau_t] = \sum_{t=0}^{\infty} p_t [(1-\tau_{lt}) w_t n_t + k_t v_t - k_t (v_t - \delta) \tau_{kt}],$$

and the law of motion of capital

$$k_{t+1} = x_t + (1-\delta) k_t,$$

where p_t is the Arrow-Debreu price of consumption goods at time t in terms of time zero consumption good, and w_t and v_t are the before tax real wage and rental rate of capital in terms of consumption t goods. There are two tax rates, τ_{lt} is the tax rate on labor income and τ_{kt} is that tax rate on capital and a lump sum tax τ_t, denominated in time t units of the consumption good. Government purchases are denoted by g_t. Notice that the capital income taxes are levied on the net income of capital, i.e. there is an allowance for depreciation, thus the time t income from renting capital equals

$$v_t k_t - \tau_{kt} (v_t - \delta) k_t = v_t k_t (1-\tau_{kt}) + \tau_{kt} \delta k_t.$$

The firm's problem is

$$\max_{k_t, n_t} \ F(k_t, n_t) - w_t n_t - v_t k_t.$$

The government budget constraint is

$$\sum_{t=0}^{\infty} p_t [\tau_t + \tau_{lt} w_t n_t + k_t (v_t - \delta) \tau_{kt}] = \sum_{t=0}^{\infty} p_t g_t.$$

We will analyze the steady state as a function of the assumed constant g, τ_k and τ_l. We assume that the lump sum taxes τ adjust to satisfy the government budget constraint.

We will assume that v is C^2, strictly concave and strictly increasing in (c, l), and that satisfies the following Inada conditions:

$$\begin{aligned}
\lim_{c \to 0} \frac{v_c(c,l)}{v_l(c,l)} &= \infty \text{ for all } l \in (0,1). \\
\lim_{l \to 0} \frac{v_l(c,l)}{v_c(c,l)} &= \infty \text{ for all } c > 0. \\
\lim_{l \to 1} \frac{v_l(c,l)}{v_c(c,l)} &= 0 \text{ for all } c > 0.
\end{aligned}$$

so that we don't have to worry about corner solutions.

1. Using λ for the multiplier of the budget constraint of the agent, write down the FOC w.r.t. c_t and n_t. Use these FOCs to obtain an expression for the marginal rate of substitution of c_{t+1} and c_t and for the marginal rate of substitution of c_t and n_t.

2. Write down the FOCs for the firm's maximization problem.

3. Using the households budget constraint, show that

$$v_{t+1} = \delta + \frac{r_t}{1 - \tau_{kt+1}},$$

must hold if agents will find optimal to choose $0 < k_{t+1} < \infty$. [Hint: Collect the terms in the budget constraint with k_{t+1}

$$... - p_t k_{t+1} + ... + p_{t+1} k_{t+1} \left[(1-\delta) + v_{t+1} (1 - \tau_{kt+1}) + \tau_{kt+1} \delta \right] + ...,$$

use the definition of r_t, impose that the optimal k_{t+1} is interior, and go through the algebra].

4. Assume that the tax rates are constant, i.e., $\tau_{kt} = \tau_k$ and $\tau_{lt} = \tau_l$ for all t. Show that in a steady state r, $\kappa \equiv k/n$, w, x, c, n and k solve the following

system of 7 equations:

$$\begin{aligned} r &= \rho, \\ F_k(\kappa, 1) &= \frac{\rho}{1-\tau_k} + \delta, \\ w &= F_n(\kappa, 1), \\ \frac{v_l(c,l)}{v_c(c,l)} &= w(1-\tau_l), \\ c &= (F(\kappa,1) - \delta\kappa)(1-l) - g, \\ x &= \delta\kappa \ (1-l), \\ k &= \kappa \ (1-l), \end{aligned}$$

where $\beta = 1/(1+\rho)$. [Hint: Modify the equations used for the steady state with no taxation in the class notes].

The point of this problem is to realize that the introduction of distorting taxes precludes us from solving the consumer problem subject to the feasibility constraint (i.e., the Pareto problem), since in this case there is a wedge between the marginal rate of substitution and the marginal rate of transformation due to the presence of distortionary taxation.

5. i) Show that if c and l are normal goods, then

$$\begin{aligned} \frac{\partial}{\partial c} \frac{v_l(c,l)}{v_c(c,l)} &> 0 \\ \frac{\partial}{\partial l} \frac{v_l(c,l)}{v_c(c,l)} &< 0 \end{aligned}$$

respectively (i.e., the first condition is necessary for normality of leisure and the second for normality of consumption).

ii) Show that if c and l are normal goods, one can define a function $\phi(l,\omega)$ such that

$$\frac{v_l(\phi(l,\omega), l)}{v_c(\phi(l,\omega), l)} = \omega.$$

Moreover ϕ is increasing in ω and l.

iii) Suppose that $v(c,l) = (1-\alpha)\log c + \alpha \log l$. Show that in this case the function $\phi(l,\omega)$ is given by

$$\phi(l,\omega) = \omega \frac{1-\alpha}{\alpha} l.$$

6. Consider the system of two equations

$$\begin{aligned} \frac{v_l(c,l)}{v_c(c,l)} &= F_n(\kappa,1)(1-\tau_l), \\ c &= (F(\kappa,1)-\delta\kappa)(1-l)-g, \end{aligned}$$

in two unknowns, c and l, for given κ, g, and τ_l. Assume that $v(c,l)$ is such that c and l are both normal goods. Using the previous results write this system as

$$\begin{aligned} c &= \phi(l\ ,\ F_n(\kappa,1)(1-\tau_l)\ ), \\ c &= (F(\kappa,1)-\delta\kappa)\ (1-l)-g. \end{aligned}$$

i) Show that, given κ, g, and τ_l, there is a unique c and l that solve this equation. Define $C(\kappa,g,\tau_l)$ and $L(\kappa,g,\tau_l)$ as the solution. [Hint: Plot the RHS of the two expressions with c in the vertical axis and l in the horizontal axis. The intersection of the two functions define the solutions $C(\kappa,g,\tau_l)$ and $L(\kappa,g,\tau_l)$].

ii) Show that $C(\kappa,g,\tau_l)$ is decreasing in τ_l and $L(\kappa,g,\tau_l)$ is increasing in τ_l. [Hint: Use the plot described above and shift the RHS of the top equation].

iii) Show that $C(\kappa,g,\tau_l)$ is increasing in κ but $L(\kappa,g,\tau_l)$ can be decreasing or increasing in κ. [Hint: Use the plot described above and shift the RHS of the both equations].

iv) Show that $C(\kappa,g,\tau_l)$ is decreasing in g and $L(\kappa,g,\tau_l)$ is decreasing in g. [Hint: Use the plot described above and shift the RHS of the bottom equation].

v) Define $C^*(\tau_k,\tau_l,g)$ and $L^*(\tau_k,\tau_l,g)$ as

$$\begin{aligned} C^*(\tau_k,\tau_l,g) &= C(\kappa^*(\tau_k),g,\tau_l), \\ L^*(\tau_k,\tau_l,g) &= L(\kappa^*(\tau_k),g,\tau_l), \end{aligned}$$

where $\kappa^*(\tau_k)$ solves

$$F_k(\kappa^*(\tau_k),1) = \delta + \frac{\rho}{1-\tau_k}.$$

7. Based on the previous results, complete the following table. In each entry write $+$, $-$, $=$ or ? if the quantity increases, decreases, stays constant, or its effect cannot be determined. That is, examine what happens with the steady states values C^*, L^*, κ^* as well as other steady state objects (r, w, $w(1-\tau_l)$, v, $v-\delta$, $(v-\delta)(1-\tau_k)$, n, and x) as a function of the fiscal policy parameters τ_l, τ_k and g. Assume that c and l are normal goods. [Hint: Use the plot described above and shift the RHS of both equations, also taking into account the effect of τ_k in $\kappa^*(\tau_k)$].

1.2 Productivity growth

Now we analyze the effect of labor augmenting labor productivity and an increase in steady state allocation. Let A be an index of labor augmenting productivity, so that the production function is written as

$$y = F(k, n\ A).$$

Notice that using κ for the capital-labor ratio (in natural, as opposed to efficiency, units), $\kappa = k/n$ we have:

$$\begin{aligned}\frac{\partial}{\partial k}F(k,n\ A) &= F_k(k,nA) = F_k\left(\frac{\kappa}{A},1\right),\\ \frac{\partial}{\partial n}F(k,n\ A) &= A\ F_n(k,nA) = A\ F_k\left(\frac{\kappa}{A},1\right).\end{aligned}$$

1. Show that steady states are given by the solution $\kappa^*(A) = k(A)/n(A)$ to:

$$F_k\left(\kappa\frac{1}{A},1\right) = \delta + \frac{\rho}{1-\tau_k},$$

so that

$$\kappa^*(A) = \kappa^*(1)\ A,$$

for all $A > 0$. Also show that

$$C^*(A,\tau_k,\tau_l,g) = C(\kappa,A,\tau_k,\tau_l,g),$$

and

$$L^*(A,\tau_k,\tau_l,g) = L(\kappa,A,\tau_k,\tau_l,g),$$

where $C(A,\kappa,\tau_k,\tau_l,g)$ and $L(\kappa,A,\tau_k,\tau_l,g)$ are the values of c and l that, given κ,τ_k,τ_l,g solve:

$$\begin{aligned}c &= \phi\left(l\ ,\ AF_n\left(\frac{\kappa}{A},1\right)(1-\tau_l)\right),\\ c &= A\left(F\left(\frac{\kappa}{A},1\right)-\delta\frac{\kappa}{A}\right)\ (1-l)-g.\end{aligned}$$

2. Assume that $v(c,l)$ are normal goods. Show that $C^*(A)$ is increasing in A. Is $L^*(A)$ increasing in A? Discuss your answer in terms of income and substitution effects.

3. Let

$$v(c,l) = \frac{c^{1-\sigma}h(l)}{1-\sigma},$$

for $\sigma > 0$, $\sigma \neq 1$. Assume that

$$\frac{h'(l)}{h(l)} \frac{1}{1-\sigma},$$

is a decreasing function of l. i) Show that c and l are normal goods. ii) Show that the function ϕ defined above takes the form

$$\phi(l, \omega) = \omega \; \varphi(l),$$

for an increasing function φ. iii) Assume that $g = A\hat{g}$, so government purchases are proportional to A. Use ii) to argue that in this case

$$\begin{aligned} C^*(A) &= A \; C^*(1), \\ L^*(A) &= L(1), \end{aligned}$$

for all $A > 0$.

We have shown that L^* can be increasing or decreasing in A, depending on the income and substitution effects, while L^* is decreasing in τ_l. Why is that for the case of labor taxes the substitution effect dominates the income effect? The next exercise is designed to explain this difference. The key to it, is that, in equilibrium, for labor taxes there is NO income effect, since it is assumed that the extra revenue produced by the income taxes is rebated back to the households. To simplify the exposition in the next exercise we set $A = 1$, $g = 0$ and $\tau_k = 0$, and just consider the comparative static w.r.t. τ_l.

4. i) Show that a steady state is the solution of the following 6 equations in 6 unknowns:

$$\begin{aligned} \rho + \delta &= F_k(\kappa, 1), \\ \kappa &= k/n, \\ w &= F_n(\kappa, 1), \\ l &= 1 - n, \\ c + wl &= w + k\rho, \\ \frac{v_l(c,l)}{v_c(c,l)} &= w(1 - \tau_l). \end{aligned}$$

Why is it that there are no taxes in the equation $c + wl = w + k\rho$ (that is neither τ_l, τ_k or the lump sum tax τ show up there)?

ii) Let κ^* be the solution of $\rho + \delta = F_k(\kappa^*, 1)$, let w^* be the solution of $w^* = F_n(\kappa^*, 1)$. Draw a diagram with c in the vertical axis and l in the horizontal axis. iia) Draw a piecewise linear set of points for which $c = w^*(1-l) + k\rho$ for l in $[0, 1]$ for a given k. This is like a budget line for the points $l > 0$ and then has a vertical part for $l = 1$, so we will refer to it as the "economy-wide budget set".

iib) Draw a set of indifference curves for $v(c,l)$ in the same graph. iic) Identify the pair $\left(\bar{c}(k), \bar{l}(k)\right)$ where the indifference curve is tangent to the economy-wide budget set. iid) Identify the pair $(C(k;\tau_l), L(k;\tau_l))$ on the economy-wide budget set whose slope is $(1-\tau_l)$ times the slope of the economy-wide budget set, i.e., the pair

$$(C(k;\tau_l), L(k;\tau_l)) = (c,l),$$

that solves:

$$\begin{aligned} c &= w^*(1-l) + k\rho, \\ \frac{v_l(c,l)}{v_c(c,l)} &= w^*(1-\tau_l). \end{aligned}$$

iie) Argue that $L(k;\tau_l) > \bar{l}(k)$. Explain why the relative strength of the income and substitution effects are not relevant to establish this inequality (i.e., why labor income taxes do not have, in equilibrium, an income effect). iif) Argue that the steady state l is given by $L^*(\tau_l) = L(k^*(\tau_l), \tau_l)$, where $k = k^*(\tau_l)$ solves

$$(1 - L(k;\tau_l)) = (1/\kappa^*)\ k.$$

iig) Use the function $N(k;\tau_l) \equiv 1 - L(k;\tau_l)$ and the equation

$$N(k^*(\tau_l);\tau_l) = (1/\kappa^*)\ k^*(\tau_l),$$

to argue that the steady state labor supply is decreasing in τ_l. [Hint: In a diagram with k in the horizontal axis and N in the vertical axis, use the the solution to iid)-iie) to compare the intersection of the functions $N(k;\tau_l')$ and $N(k;\tau_l)$ with $\tau_l' > \tau_l$ with the the function $(1/\kappa^*)\ k$].

1.3 Population and productivity growth

Consider a version of the neoclassical growth model with population growth, at the rate of λ per year, and labor augmenting productivity, at the rate of π per year.

Assume that preferences are given by

$$\sum_{t=0}^{\infty} \Lambda_t \beta^t v(c_t, 1-n_t) = \sum_{t=0}^{\infty} \Lambda_t \beta^t v(c_t, l_t),$$

where c_t is per-capita consumption, and n_t is per-capita labor supply at time t. The size of the population at time t is $\Lambda_t = \Lambda_0 (1+\lambda)^t$. The time t total production if per-capita capital is k_t and labor per person n_t are used is

$$F\left(\Lambda_t k_t, (1+\pi)^t \Lambda_t n_t\right).$$

Thus, feasibility is given by

$$\Lambda_t [c_t + x_t + g_t] = F\left(\Lambda_t k_t, \Lambda_t (1+\pi)^t n_t\right),$$

where x_t and g_t denote per-capita investment and government purchases. The law of motion of capital is given by

$$\Lambda_{t+1}k_{t+1} = \Lambda_t x_t + \Lambda_t k_t (1-\delta).$$

We define the * allocation by letting

$$\begin{aligned} g_t^* &= \frac{g_t}{(1+\pi)^t}, \\ c_t^* &= \frac{c_t}{(1+\pi)^t}, \\ x_t^* &= \frac{x_t}{(1+\pi)^t}, \\ k_t^* &= \frac{k_t}{(1+\pi)^t}, \\ l_t^* &= l_t, \\ n_t^* &= n_t, \end{aligned}$$

and prices

$$\begin{aligned} r_t^* &= \frac{p_t}{p_{t+1}} - 1, \\ w_t^* &= \frac{w_t}{(1+\pi)^t}, \\ \tau_t^* &= \frac{\tau_t}{(1+\pi)^t}. \end{aligned}$$

5. Show that feasibility and law of motion equations are equivalent to

$$c_t^* + x_t^* + g_t^* = F(k_t^*, n_t^*),$$

and

$$(1+\lambda)(1+\pi)k_{t+1}^* = x_t^* + k_t^*(1-\delta).$$

6. Show that the following budget constraint for the family

$$\begin{aligned} & \sum_{t=0}^{\infty} p_t \Lambda_t [c_t + x_t + (1-\tau_{tl}) w_t l_t + \tau_t] \\ = & \sum_{t=0}^{\infty} p_t \Lambda_t [w_t (1-\tau_{tl}) + k_t v_t (1-\tau_{kt}) + k_t \delta \tau_{kt}], \end{aligned}$$

is equivalent to

$$\begin{aligned} & \sum_{t=0}^{\infty} p_t \Lambda_t (1+\pi)^t [c_t^* + x_t^* + (1-\tau_{tl}) w_t^* l_t^* + \tau_t^*] \\ = & \sum_{t=0}^{\infty} p_t \Lambda_t (1+\pi)^t [w_t^* (1-\tau_{tl}) + k_t^* (v_t (1-\tau_{kt}) + \delta \tau_{kt})]. \end{aligned}$$

7. Show, by an arbitrage argument, that collecting all the terms involving k_{t+1}^* for $t \geq 0$ in the budget constraint, such as

$$-p_t \Lambda_t (1+\lambda) + p_{t+1} \Lambda_{t+1} \left[(1-\delta) + (v_{t+1} (1-\tau_{kt+1}) + \delta \tau_{kt+1})\right],$$

then in equilibrium if $0 < k_{t+1}^* < \infty$ it must be that

$$v_{t+1} = \delta + \frac{r_t}{1-\tau_{kt+1}}.$$

Use this result to show that the budget constraint of the household, in equilibrium, must be

$$\begin{aligned} & \sum_{t=0}^{\infty} p_t \Lambda_t (1+\pi)^t \left[c_t^* + (1-\tau_{tl}) w_t^* l_t^* + \tau_t^*\right] \\ = \quad & \Lambda_0 k_0^* (1 + (v_0 - \delta)(1-\tau_{k0})) + \sum_{t=0}^{\infty} p_t \Lambda_t (1+\pi)^t \left[w_t^* (1-\tau_{tl})\right]. \end{aligned}$$

8. Using preferences of the form

$$v(c,l) = c^{1-\gamma} h(l) / (1-\gamma),$$

for $\gamma > 0$, $\gamma \neq 1$, and

$$v(c,l) = \log c + h(l),$$

for $\gamma = 1$, show that

$$\sum_{t=0}^{\infty} \Lambda_t \beta^t v(c_t, l_t) = \sum_{t=0}^{\infty} \Lambda_t \beta^{*t} v(c_t^*, l_t),$$

where

$$\beta^* = \beta (1+\pi)^{1-\gamma},$$

for $\gamma > 0$, $\gamma \neq 1$and,

$$\sum_{t=0}^{\infty} \Lambda_t \beta^t v(c_t, l_t) = \log(1+\pi) \sum_{t=0}^{\infty} \Lambda_t \beta^t t + \sum_{t=0}^{\infty} \Lambda_t \beta^t v(c_t^*, l_t),$$

for $\gamma = 1$.

9. Show that the vector $(r, w, v, x,\ k, l, \kappa, c)$ is a steady state, that is

$$\begin{aligned} g_t^* &= g, \\ c_t^* &= c, \\ x_t^* &= x, \\ k_t^* &= k, \\ l_t^* &= l, \\ n_t^* &= n, \end{aligned}$$

and

$$\begin{aligned} r_t^* &= r, \\ w_t^* &= w, \\ \tau_t^* &= \tau, \end{aligned}$$

if and only if it satisfies the following equations:

$$\begin{aligned} 1+r &= (1+\rho)(1+\pi)^{\gamma} \text{ or } r \cong \rho + \gamma\pi, \\ x &= [(1+\lambda)(1+\pi)-(1-\delta)]\, k \text{ or } x \cong (\lambda+\delta+\pi)\, k, \\ v &= \delta + \frac{r}{1-\tau_k}, \\ v &= F_k(\kappa, 1), \\ w &= F_l(\kappa, 1), \\ w(1-\tau_l) &= c\frac{h'(l)}{(1-\gamma)\, h(l)}, \\ g+c &= F(\kappa,1)(1-l) - [(1+\lambda)(1+\pi)-(1-\delta)]\,\kappa(1-l) \text{ or} \\ g+c &\cong F(\kappa,1)(1-l) - (\delta+\lambda+\pi)\,\kappa(1-l), \\ \kappa &= \frac{k}{1-l}, \end{aligned}$$

where the symbol $\cong$ is interpreted as a first order approximation around $\rho = \pi = \lambda = 0$ of the relevant expression.

10. Assume that

$$\begin{aligned} v(c,l) &= \left(c^{1-\alpha} l^{\alpha}\right)^{1-\phi} / (1-\phi) = c^{(1-\alpha)(1-\phi)} l^{\alpha(1-\phi)} / (1-\phi), \\ F(k,n) &= k^{\theta} n^{1-\theta}, \end{aligned}$$

so that

$$\gamma = \alpha + \phi(1-\alpha).$$

Take the benchmark values

$$\begin{aligned} \rho &= 0.01, \\ \gamma &= 2, \\ \alpha &= 0.7, \\ \pi &= 0.02, \\ \lambda &= 0.01, \\ \delta &= 0.05, \\ \theta &= 0.3, \\ g/y &= 0.14, \\ \tau_k &= 0.30, \\ \tau_l &= 0.20. \end{aligned}$$

Then a steady state is given by

$$
\begin{aligned}
r &\cong \rho+\gamma\pi = 0.01+2\times 0.02 = 0.05,\\
v &\cong \delta+\frac{r}{1-\tau_k} = 0.05+\frac{0.05}{0.7} = 0.05+0.07 = 0.12,\\
\frac{k}{y} &\cong \frac{\theta}{v} = \frac{0.3}{0.12} = 2.5,\\
\frac{x}{k} &\cong (\delta+\pi+\lambda) = 0.05+0.02+0.01 = 0.08,\\
\frac{x}{y} &\cong \frac{x}{k}\frac{k}{y} = 0.08\times 2.5 = 0.2,\\
\frac{c}{y} &\cong 1-\frac{g}{y}-\frac{x}{y} = 1-0.14-0.2 = 0.66,\\
l &\cong 1/\left(1+\frac{(1-\theta)}{c/y}\left(1-\tau_l\right)\frac{1-\alpha}{\alpha}\right) = 1/\left(1+\frac{0.7}{0.66}0.8\frac{0.3}{0.7}\right),\\
l &\cong 0.73,\ n\cong .27,\\
k &= \left[\frac{v}{\theta}\right]^{\frac{1}{\theta-1}} n = \left[\frac{0.12}{0.3}\right]^{\frac{1}{-0.7}} 0.27 = [.4]^{-\frac{1}{0.7}}\,0.27.
\end{aligned}
$$

Recompute the values for r, v, k/y, x/y, $c/y, l$ and k for
i) $\tau_k = 0.2$, $\tau_l = 0.2$ and $g/y = 0.14$,
ii) $\tau_k = 0.3$, $\tau_l = 0.3$ and $g/y = 0.14$,
iii) $\tau_k = 0.3$, $\tau_l = 0.2$ and $g/y = 0.16$, and
iv) $\tau_k = 0.2$, $\tau_l = 0.3$ and $g/y = 0.16$.

2 Ricardian Equivalence does not imply that the timing of DISTORTING taxes is immaterial

So far in class we considered either lump sum taxes, constant income taxes in the case with no variable labor supply, or taxation in steady sate. With lump sum taxes, the only important feature of the tax system, as we have seen, is the present value of taxes. In this case, the timing of taxation was immaterial. To avoid that you get the incorrect notion that this is the whole story, I include here some simple analysis of distorting taxation with variable labor supply. In this case there are powerful forces that indicates that it is better, from a welfare perspective, to keep the tax rates constant trough time. To simplify the point I have excluded capital accumulation (including capital accumulation so that we can talk about capital taxation is very important, but capital accumulation brings a different set of issues so it is better to treat that important case elsewhere). The analysis here follows closely Lucas and Stokey's JME paper.

2.1 Tax smoothing in the Ramsey model. Warming up with a simple case

Assume that all agents are identical and have utility function given by

$$\sum_{t=0}^{\infty} \beta^t u(c_t, l_t),$$

where c_t is consumption and l_t is labor supply and $\beta \in (0,1)$. There is no capital in this economy, the production technology is linear in labor and each period the government purchases a variable amount $g_t \leq 1$ of consumption goods, which must be financed using a linear income tax of rate τ_t. Thus, the resource constraint is $c_t + g_t = l_t$ for $t \geq 0$. Competitive firms pay a wage equal to marginal productivity, 1, and households receive after-tax wages of $(1-\tau_t)$. The government has no other expenditures and starts in period $t = 0$ with no assets or liabilities. The government chooses $\{\tau_t\}$ to maximize the representative agent utility, subject to being able to finance its purchases $\{g_t\}$ with taxes $\{\tau_t\}$. In this case the Arrow-Debreu budget constraint of a household and the government are

$$\sum_{t=0}^{\infty} p_t \left[c_t - l_t (1-\tau_t)\right] = 0, \text{ and } \sum_{t=0}^{\infty} p_t \left[l_t \tau_t - g_t\right] = 0,$$

where p_t is the price of a unit of consumption at time t in terms of time zero consumption goods.

a. Write down the FOCs for the household problem using θ for the Lagrange multiplier of the budget constraint.

Consider the utility function

$$u(c,l) = c - \frac{1}{1+\phi} l^{\phi+1},$$

for $\phi > 0$.

b. Use a. to argue that if the solution of the household problem is interior, $p_t = \beta^t$, after-tax wages at time t satisfy $1 - \tau_t = l_t^{\phi}$, and time t tax revenues satisfy $\tau_t l_t = l_t - l_t^{1+\phi}$.

c. Replace the expressions obtained in b. for tax revenues and $\{p_t\}$ in the budget constraint of the government. Your answer should be one equation that depend on the sequences $\{g_t, l_t\}$ and the parameters β and ϕ.

d. Replace c_t using the resource constraint in the agent's utility function. Your answer should be one expression that depends on the sequences $\{g_t, l_t\}$ and the parameters β and ϕ.

e. Using c. and d. consider the welfare consequences of two fiscal policies. Assume that the allocations are interior. In the first policy the budget is balanced period by period. In the second policy, tax rates are constant. Which one is better? Why?

d. What is the best policy among all tax policies (assume that the allocations are all interior)? How is labor supply l_t during time periods with different government purchases when the best policy is followed?

2.2 Tax smoothing in a Ramsey problem. A bit more general result

Consider the same model as in the previous question.

i. Write down (again!) the time zero Arrow-Debreu budget constraint of a household. Use p_t for price of a unit of consumption at time t in terms of zero consumption goods for arbitrary taxes τ_t and intertemporal prices p_t. Your answer should take 1 line.

ii. Write down (again!) the time zero Arrow-Debreu budget constraint of the government. Your answer should take 1 line.

iii. Write down (again!) the first order conditions for c_t and l_t all t, for the household problem, for arbitrary taxes τ_t and intertemporal prices p_t. These are two equations for each t; use θ for the Lagrange multiplier of the budget constraint of the household.

iv. Define a competitive equilibrium with taxes for this economy (list all the objects and the conditions that they must satisfy, your answer should take no more than 3 lines).

v. Use i., ii. and iii. to derive an equation that we will call "implementability constraint". The implementability constraint is the budget constraint of the agent where the prices p_t and after-tax wages $1 - \tau_t$ have been replaced by the marginal rate of substitution from the agent's problem. Thus, the implementability constraint is a function of $\{c_t, l_t\}$; no $p_t's$ or $\tau_t's$ should be in this expression. The implementability constraint is

$$\sum_{t=0}^{\infty} \beta^t \left[u_c\left(c_t, l_t\right) c_t + u_l\left(c_t, l_t\right) l_t \right] = 0.$$

[Hint: Use i. and iii. to derive this equation. Your answer should take no more than five lines].

vi. Show that if an allocation $\{c_t, l_t\}_{t=0}^{\infty}$ is resource feasible (i.e., $c_t + g_t = l_t$ for $t \geq 0$) and satisfies the implementability constraint in v., it is a competitive equilibrium with taxes. [Hint: Use the given allocation, i. and iii. to find p_t, τ_t and θ]. Your answer should take no more than five lines.

vii. The Ramsey problem, as explained above, is to choose taxes $\{\tau_t\}$ such that the corresponding competitive equilibrium maximizes the utility of the representative agent. In view of the previous results, this problem is then

$$\max_{\{c_t,l_t\}} \sum_{t=0}^{\infty} \beta^t u\left(c_t, l_t\right),$$

subject to the implementability constraint derived in v. and the resource constraint $c_t + g_t = l_t$ for all $t \geq 0$. Use iv. to derive the FOC for the Ramsey problem w.r.t. c_t and l_t. Use λ for the Lagrange multiplier of the implementability constraint and $\beta^t \mu_t$ for the Lagrange multiplier of resource constraint at time t.

viii. Consider the case where $g_t = 0$ for all $t \geq 0$, except for some date $s > 0$. Solve for the Ramsey allocation. Use the FOCs for the Ramsey problem derived in vii. to argue that this allocation has $c_t = \bar{c}$, $l_t = \bar{l}$ for all $t \neq s$. Show that then $\tau_t = \bar{\tau}$ for all dates $t \neq s$. Finally argue that $\bar{\tau} > 0$ in spite of the fact that $g_t = 0$ for $t \neq s$. What is the intuition for this result? Why taxes are positive even in periods when government purchases are zero, that is, why not just tax on the period s where $g_s > 0$? [Hint: To formally show this result, take an arbitrary period $t \neq s$. Multiply the FOC of the Ramsey problem w.r.t. c_t by c_t, and the FOC w.r.t. l_t by l_t and add them up for each t. Use the resource constraint and the fact that $g_t = 0$ to eliminate the terms containing the multiplier μ_t. Notice that this sum has a term which is a quadratic form of u, and hence it can be signed as negative. Using that the multiplier of the implementability constraint, λ, is positive, this implies that other term, $u_c\left(t\right) + u_l\left(t\right)$ is positive. Use the FOC for the agent's problem to show that this implies that $\tau_t > 0$].

ix. Consider the utility function

$$u\left(c_t, l_t\right) = \frac{1}{1-\alpha} c^{1-\alpha} - \frac{1}{1+\phi} l_t^{\phi+1},$$

for $1 \neq \alpha > 0$ and $\phi > 0$. Show that if the allocation is interior, i.e. c_t and $l_t > 0$ all t, then the Ramsey allocation has *constant income taxes* τ_t for all t. What is the intuition for this result? How do l_t and c_t move in periods where g_t is bigger? [Hint: To formally show this result, take an arbitrary period t. Add the FOC for c_t and l_t to eliminate μ_t, use the functional form of u to relate $u_{cc}c$ to u_c and $u_{ll}l$ to u_l. Finally compare the marginal rate of substitution between c_t and l_t to determine that it is the same for any period t and, hence, tax rates are the same for all time periods].

3 Ramsey Taxation in the Lucas-Uzawa Endogenous Growth Model

We consider here a version of the Lucas-Uzawa model of section 1 where we introduce distorting taxation and leisure. Our aim is to analyze the optimal Ramsey taxation in this set-up in light of the result of zero capital taxation for the standard neoclassical growth model. In this model labor income is proportional to human capital, which itself is accumulated, so even labor income has features similar to capital in the standard neoclassical growth model.

Let preferences be given by

$$\sum_{t=0}^{\infty} \beta^t U\left(c_t, l_{1t} + l_{2t}\right), \tag{1}$$

where c_t is consumption, l_{1t} time working and l_{2t} time learning. We assume that U is strictly concave in its two arguments $U(c, l)$, strictly increasing in c, strictly decreasing in l, and satisfy standard Inada conditions.

The budget constraint of the household is

$$\sum_{t=0}^{\infty} p_t \left[c_t + k_{t+1}\right] = \sum_{t=0}^{\infty} p_t \left[\left(1 - \tau_t\right) w_t h_t l_{1t} + k_t R_{k,t}\right], \tag{2}$$

and

$$R_{kt} = 1 + \left(1 - \theta_t\right)\left(r_t - \delta\right) \quad \text{for all } t \geq 0, \tag{3}$$

where p_t is the Arrow-Debreu price of a consumption good at time t, w_t the real wage, τ_t the tax rate on wages, h_t the human capital of the household, so $l_{1t}h_t$ gives the total labor services offered by the household, and $w_t\left(1 - \tau_t\right)$ the after-tax wage for each unit of labor services. We use k_t for the physical capital, R_{kt} for the gross after-tax return on capital, r_t the rental rate of capital, δ the depreciation rate of capital, and θ_t the tax rate on the net rental rate of capital.

The human capital accumulation technology of the households is described by

$$h_{t+1} = G\left(l_{2t}\right) h_t + \left(1 - \delta_h\right) h_t, \tag{4}$$

for all $t \geq 0$, so that $\left[G\left(l_{2t}\right) + 1 - \delta_h\right]$ gives the (gross) growth rate of human capital as a function of l_{2t}, the time devoted to learning, and δ_h, the depreciation rate of human capital. We assume that G is positive, and unless otherwise indicated, we assume that G is strictly increasing, strictly concave, and satisfies standard Inada conditions.

The household problem is to maximize (1) subject to (2), (3) and (4), given h_0 and k_0.

The firm's problem is

$$\max_{k_t, n_t} F\left(n_t, k_t\right) - k_t r_t - w_t n_t, \tag{5}$$

where n_t are the labor services demanded and k_t the capital services demanded. We assume that F has constant returns to scale and it is neoclassical (strictly quasi-concave, satisfying Inada conditions).

Feasibility is given by

$$g_t + c_t + k_{t+1} = F\left(l_{1t}h_t, k_t\right) + \left(1 - \delta\right) k_t, \tag{6}$$

for all $t \geq 0$, where g_t are the government purchases at time t.

The government budget constraint is given by

$$\sum_{t=0}^{\infty} p_t g_t = \sum_{t=0}^{\infty} p_t \left[\tau_t w_t h_t l_{1t} + \theta_t k_t \left(r_t - \delta \right) \right]. \tag{7}$$

Definition: A competitive equilibrium with taxes where the government finances the purchases $\{g_t\}$ is an allocation $\{c_t, l_{1t}, l_{2t}, h_{t+1}, k_{t+1}\}_{t=0}^{\infty}$, a price system $\{p_t, w_t, R_{kt}, r_t\}_{t=0}^{\infty}$, and taxes $\{\tau_t, \theta_t\}_{t=0}^{\infty}$, such that i) $\{c_t, l_{1t}, l_{2t}, h_{t+1}, k_{t+1}\}_{t=0}^{\infty}$ is feasible for $\{g_t\}$ and initial conditions k_0, h_0, ii) $\{c_t, l_{1t}, l_{2t}, h_{t+1}, k_{t+1}\}_{t=0}^{\infty}$ maximizes the household utility given prices $\{p_t, w_t, R_{kt}, r_t\}_{t=0}^{\infty}$, taxes $\{\tau_t, \theta_t\}_{t=0}^{\infty}$, and initial conditions k_0, h_0, iii) firms maximize profits given prices $\{w_t, r_t\}$, and iv) the government budget constraint holds.

A balanced-growth path is an equilibrium where l_{1t} and l_{2t} are constant and where h_t, c_t, k_t grow at a common constant rate, say γ. To allow the possibility of a balanced-growth path we specialize preferences to

$$\begin{aligned} U(c,l) &= \frac{\left[c\varphi(l)\right]^{1-\sigma}}{1-\sigma} \text{ for } \sigma \neq 1, \\ &= \log c + \varphi(l) \text{ for } \sigma = 1, \end{aligned} \tag{8}$$

for some function $\varphi(l)$ and where $\sigma > 0$.

1) Write down the FOCs for the household problem w.r.t. c_t, l_{1t}, l_{2t}, k_{t+1} and h_{t+1} for $t \geq 0$. Use λ_h for the multiplier on the household budget constraint (2)

$$\sum_{t=0}^{\infty} p_t \left[-c_t - k_{t+1} + (1-\tau_t) w_t h_t l_{1t} + k_t R_{k,t} \right] = 0,$$

and $\beta^t \phi_t$ for the multiplier on the household human capital accumulation technology (4)

$$-h_{t+1} + G(l_{2t}) h_t + (1-\delta_h) h_t = 0.$$

2) Consider the following variational argument. Let $l'_{1t} = l_{1t} - \delta$, and $l'_{2t} = l_{2t} + \delta$, where the plan with $'$ is the variation on the original plan. In particular, analyze the effects of a small change on δ on the household budget constraint. Show that if in the original plan the household is maximizing utility, it must be that

$$p_t w_t (1-\tau_t) h_t = \frac{G'(l_{2t})}{\left[G(l_{2t}) + (1-\delta_h)\right]} \sum_{s=1}^{\infty} p_{t+s} w_{t+s} (1-\tau_{t+s}) h_{t+s} l_{1t+s}, \tag{9}$$

for all $t \geq 0$. [Hint: Derive first an expression for dh_{t+s}/dl_{2t} using (4). Perturb l_{1t} and l_{2t} in the budget constraint (2) by $-\delta$ and δ, use that the original plan was optimal, and replace your expression for dh_{t+s}/dl_{2t}]. Give a two lines economic interpretation to each side of this equation. FYI: Lucas displays a continuous time version of this equation in his supply-side paper.

3) Show that the FOCs of this problem imply equation (9). [Hints: write the FOC w.r.t. h_t as a first order linear difference equation with time varying coefficients, and solve $\beta^t \phi_t$ forward. Replace $\beta^t \phi_t$ from the FOCs w.r.t. l_{2t} and l_{1t}, use the human capital accumulation technology and rearrange].

4) Show that the FOC of the household problem derived in 1), including the multipliers λ_h and $\beta^t \phi_t$, are equivalent to the following system,

$$\frac{U_{lt}}{h_t G'(l_{2t})} = \frac{\beta U_{lt+1}}{h_{t+1} G'(l_{2t+1})} \left[G(l_{2t+1}) + (1-\delta_h) \right] + \beta U_{lt+1} l_{1t+1} / h_{t+1}, \quad (10)$$

$$U_{ct} = \beta U_{ct+1} \left[1 + (1-\theta_{t+1})(r_t - \delta) \right], \quad (11)$$

$$-\frac{U_{lt}}{U_{ct}} = (1-\tau_t) w_t h_t, \quad (12)$$

for all $t \geq 0$. FYI: the Euler equation for human capital (10) is the one displayed by Atkeson, Chari and Kehoe.

5) Show that if the implementability constraint

$$\sum_{t=0}^{\infty} \beta^t \left[U_{ct} c_t + U_{lt} l_{1t} \right] = U_c k_0 \left(1 + (1-\theta_0)(F_{k0} - \delta) \right), \quad (13)$$

the Euler equation for human capital (10), the human capital accumulation technology (4), and feasibility condition (6) hold, then there is a competitive equilibrium with taxes where the government finances the government purchases. Conversely, show that in any competitive equilibrium with taxes where the government finances the government purchases, the implementability constraint (13) and the Euler equation for human capital (10) hold. To answer this question you should use the definition of equilibrium stated above.

6) Equivalences. i) Show that the variational condition (9) and the FOC w.r.t. l_{1t} imply the following variational conditions purely in terms of allocations,

$$U_{lt} = \frac{G'(l_{2t})}{\left[G(l_{2t}) + (1-\delta_h) \right]} \sum_{s=1}^{\infty} \beta^s U_{lt+s} l_{1t+s}, \quad (14)$$

for all $t \geq 0$. [Hint: use the FOC for l_{t+s} from 1)]. ii) Show that the Euler equation for human capital (11) also implies this variational conditions purely in terms of allocations (14). [Hint: Treat the Euler equation for human capital as a linear difference equation, solve $U_{lt}/h_t G(l_{2t})$ forward and use the human capital accumulation technology].

7) Show that the implementability constraint (13) and the variational equations in terms of allocations (14) are equivalent to the system given by the implementability constraint

$$\sum_{t=0}^{\infty} \beta^t U_{ct} c_t = U_{c0} \left[1 + (1-\theta_0)(F_{k0} - \delta) \right] - U_{l10} \left[l_{10} + \frac{G(l_{20}) + (1-\delta_h)}{G'(l_{20})} \right], \quad (15)$$

and the variational equations in terms of allocations (14). [Hint: Use the variational equation in terms of allocations for $t = 0$ into the implementability constraint (13) and rearrange].

8) Using the preferences described in (8) show that the Lagrangian of the Ramsey problem can be written as

$$\max_{\{c_t, l_{1t}, l_{2t}, k_{t+1}, h_{t+1}\}} \sum_{t=0}^{\infty} \beta^t U\left(c_t, l_{1t} + l_{2t}\right)\left[1 + \lambda\left(1 - \sigma\right)\right]$$
$$-\lambda U_{c0}\left[1 + \left(1 - \theta_0\right)\left(F_{k0} - \delta\right)\right] + \lambda U_{l10}\left[l_{10} + \frac{G\left(l_{20}\right) + \left(1 - \delta_h\right)}{G'\left(l_{20}\right)}\right],$$

subject to feasibility (6), the variational equation using allocations (14), the human capital accumulation technology (4), and the implementability constraint (15) for some λ, the Lagrange multiplier of the implementability constraint (15). [Hint: Use the form of the preferences to collect $U_c c$ with U and use the answer to 7)].

9) Consider the problem described in 8). i) Derive the FOCs for k_{t+1} for $t \geq 1$. [Hint: Replace consumption in $U\left(c_t, l_{1t} + l_{2t}\right)$ using feasibility and differentiate w.r.t. k_{t+1}]. ii) Use i) and the Euler equation for consumers to establish whether capital taxes equal zero for all $t \geq 2$. Give a proof or a counter-example. Give a two lines intuitive explanation of your answer.

10. Consider the problem described in 8). Assume that $\lambda > 0$, and that θ_0 is restricted to be on $[0, 1]$. What will be the optimal value of θ_0? Give a two lines intuitive explanation of your answer.

11) Consider the following relaxed version of problem described in 8):

$$\max_{\{c_t, l_{1t}, l_{2t}, k_{t+1}, h_{t+1}\}} \sum_{t=0}^{\infty} \beta^t U\left(c_t, l_{1t} + l_{2t}\right)\left[1 + \lambda\left(1 - \sigma\right)\right]$$
$$-\lambda U_{c0}\left[1 + \left(1 - \theta_0\right)\left(F_{k0} - \delta\right)\right] + \lambda U_{l10}\left[l_{10} + \frac{G\left(l_{20}\right) + \left(1 - \delta_h\right)}{G'\left(l_{20}\right)}\right],$$

subject to feasibility (6), the human capital accumulation technology (4), and the implementability constraint (15) for some number λ. i) Show that if a balanced growth path exists for this relaxed problem, equation (14) is satisfied for the balanced growth path. ii) Use i) to show that if a balanced growth path exists for this problem, it is the same as the balanced growth path for the first best economy (the one that has access to lump sum taxes). iii) What are the implications of ii) for labor taxes τ_t in a balanced growth path of the Ramsey problem described in 8)? Give a two lines intuitive explanation of your answer to iii).

12) *Skilled and Unskilled labor* (Optional ***).

We now consider a version with two types of labor services: skilled and unskilled. The time t utility is $U(c, l_{1t} + l_{2t} + l_{3t})$ where l_{3t} is time devoted to unskilled labor. Preferences $U(c, l_{1+}l_2 + l_3)$ are restricted as in (8). The production function F is Cobb-Douglas and uses capital k_t, skilled labor $h_t l_{1t}$ and unskilled labor services l_{3t}:

$$F(k_t,\ l_{1t}h_t,\ l_{3t}) = k_t^{\alpha_1} (l_{1t}h_t)^{\alpha_2} (l_{3t})^{1-\alpha_2-\alpha_3}.$$

The household budget constraint is

$$\sum_{t=0}^{\infty} p_t [c_t + k_{t+1}] = \sum_{t=0}^{\infty} p_t [(1-\tau_t) w_t h_t l_{1t} + k_t R_{k,t} + (1-\omega_t) u_t l_{3t}],$$

where u_t is the wage for unskilled workers and ω_t is tax rate on unskilled labor services. The rest of the economy is as before (with the obvious changes in the government budget constraint and firm's problem).

i) Write down the FOCs for the household's problem. What is the difference with the answer to 1)?

ii) Show that this economy is compatible with a balanced growth path where: a) all tax rates are constant, b) $l_{1t},\ l_{2t}$ and l_{3t} are constant, c) the constant gross growth rate of k_t and c_t is γ, d) the constant gross growth rate of h_t is γ_h, with $(\gamma)^{1-\alpha_1} = (\gamma_h)^{\alpha_2}$, d) wage u_t, w_t grow at a constant rate, and e) r_t is constant. [Hint: Just show that the FOCs, feasibility and human capital technology can be satisfied with the path described above].

iii) Show that the equivalent of the implementability constraint (13) is

$$\sum_{t=0}^{\infty} \beta^t [U_{ct}c_t + U_{lt}(l_{1t} + l_{3t})] = U_{c0}k_0 [1 + (1-\theta_0)(F_{k0} - \delta)].$$

iv) Show that the variational condition (9), and its related expression (14) also hold.

v) Adapt the argument in 7) replacing (15) for the following implementability:

$$\sum_{t=0}^{\infty} \beta^t [U_{ct}c_t + U_{lt}l_{3t}] = U_{c0}[1 + (1-\theta_0)(F_{k0} - \delta)] - U_{l10}\left[l_{10} + \frac{G(l_{20}) + (1-\delta_h)}{G'(l_{20})}\right].$$

vi) Adapt the argument in 8) using the following objective function

$$\max_{\{c_t, l_{1t}, l_{2t}, k_{t+1}, h_{t+1}\}} \sum_{t=0}^{\infty} \beta^t \{U(c_t, l_{1t} + l_{2t})[1 + \lambda(1-\sigma)] + U_{lt}l_{3t}\}$$
$$-\lambda U_{c0}[1 + (1-\theta_0)(F_{k0} - \delta)] + \lambda U_{l10}\left[l_{10} + \frac{G(l_{20}) + (1-\delta_h)}{G'(l_{20})}\right].$$

vii) Adapt the argument in 11) to show that in a balanced growth path both capital and skilled labor taxes are zero, but unskilled labor taxes are, in general, not zero.

13) *Inelastic labor supply.*

Consider a simplified version of the problem analyzed so far. Assume that $l_{1t} + l_{2t} = 1$, but that labor does not enter in the utility function U. Use $l_{1t} = 1 - l_{2t}$ for all t.

i) Derive the FOCs of the household's problem.

ii) Since the variational argument used in 2 still applies, use the FOCs of the problem to show that it implies that

$$U_{ct} F_{lt} (1 - \tau_t) h_t = \frac{G'(l_{2t})}{[G(l_{2t}) + (1 - \delta_h)]} \sum_{s=1} \beta^s U_{ct+s} F_{lt+s} (1 - \tau_{t+s}) h_{t+s} (1 - l_{2t+s}), \tag{16}$$

for all $t \geq 0$.

iii) Show that by replacing the FOCs of the household's problem, the household's budget constraint can be written as

$$\sum_{t \geq 0} \beta^t U_{ct} [c_t - (1 - \tau_t) F_{lt} (1 - l_{2t}) h_t] = U_{c0} k_0 (1 + (1 - \theta_0)(F_{k0} - \delta)). \tag{17}$$

iv) Show that if the implementability constraint (17), the variational condition (16), the human capital accumulation technology (4), and feasibility (6) hold, then there is a competitive equilibrium with taxes where the government finances the government purchases. Conversely, show that in any competitive equilibrium with taxes where the government finances the government purchases, the implementability constraint (17) and the variational equation (16) hold. Notice that both (16) and (17) include the sequence of labor taxes $\{\tau_t\}_{t=0}^{\infty}$.

v) Show that in the solution of the Ramsey problem, $\theta_t = 0$ for all $t \geq 1$ and $\tau_t = \bar{\tau} > 0$ for all $t \geq 0$. Show that the resulting allocation coincides with the first best allocation, i.e., the one with lump sum taxes. [Hint: Simply check that the FOCs for the first best allocation and the budget constraint of the government can be simultaneously satisfied if taxes are set as stated above]. Give a two lines intuitive explanation of the difference between this case and the previous one.