Theory of Income, Fall 2008
Fernando Alvarez, U of C

Problem Set 6

1 Adjustment cost model

Consider the following discrete-time dynamic programming problem in sequence formulation;

$$\max_{\{x_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t \left[h(x_t) - a(x_{t+1} - x_t) \right]$$

where x_0 given. Assume that $h''(x) \leq 0$ all $x \in R$, that there is a unique x^* such that $h'(x^*) = 0$, and that $a''(z) \geq 0$ for all $z \in R$ and that $a'(z) = a(z) = 0$ only for $z = 0$.

(1) Write down the Bellman equation for this problem. Write down explicitly the period return function in terms of h and a. Use x for the current state and y for next period state. Use v for the value function.

Ans: The Bellman equation associated with the above problem can be written as

$$v(x) = \max_{y \in R} \{h(x) - a(y - x) + \beta v(y)\}, \tag{1}$$

where

$$F(x, y) = h(x) - a(y - x),$$

is the period return function.

(2) Let $F(x, y)$ the period return function used in (1). Answer true or false, and give a short proof or counter-example.

(2.1) Is $F(x, y)$ increasing in x for all the values of y?

Ans: False. Counterexample: Let $h(x) = -x^2$ and $a(z) = z^2$, which clearly satisfy the required assumptions. Then,

$$F(x, y) = -x^2 - (y - x)^2 = -x^2 - y^2 - x^2 + 2xy,$$

so that $F_x(x, y) = -4x + 2y$. Hence, $F(x, y)$ is decreasing for $x > y/2$.

(2.2) Is $F(x, y)$ concave in (x, y)?

Ans: True. We must show that $F_{xx} \leq 0$, $F_{yy} \leq 0$ and $F_{xx}F_{yy} - F_{xy}^2 \geq 0$. To see this, note that

$$F_{xx}(x,y) = h''(x) - a''(y-x) \leq 0 \quad \text{since } h'' \leq 0 \text{ and } a'' \geq 0,$$

$$F_{yy}(x,y) = -a''(y-x) \leq 0,$$

and, since $F_{xy}(x,y) = a''(y-x)$,

$$\begin{aligned} F_{xx}F_{yy} - F_{xy}^2 &= -\left(h''(x) - a''(y-x)\right)a''(y-x) - a''(y-x)^2 \\ &= -h''(x)\,a''(y-x) \geq 0, \end{aligned}$$

so $F(x,y)$ is concave in (x,y).

(3) Let v the value function. Just answer true or false, and give a short proof or counter-example.

(3.1) Is $v(x)$ concave?

Ans: True. We will show that the sufficient conditions for concavity of the value function are satisfied: 1) $F(x,y)$ is concave in (x,y) (shown above); 2) the constraint set $\Gamma(x) = R$ is clearly convex. Therefore, we can apply the theorem stated in the class notes (or Theorem 4.8 in RMED) to conclude that $v(x)$ is concave.

(3.2) Is $v(x)$ increasing?

Ans: False. Counterexample: Let $h(x) = -x^2$ and $a(z) = z^2$, so that $F(x,y) = -2x^2 + 2xy - y^2$. We will solve for the value function $v(x)$ using the Guess and Verify method. This method works in this case because the value function is unique (since the return function is concave and the constraint set is convex). The method works as follows:

1. Guess a value function v^g with unknown parameter c;

2. Introduce this guess in the Bellman equation and optimize over y;

3. Find the parameter c for which the maximized Bellman equation equals the initial guess for the value function (i.e., a c such that $Tv^g = v^g$).

Our guess for the value function will be $v(x) = cx^2$, where $c < 0$ (since $v(x)$ is concave) is a number to be determined. In this case, the Bellman equation is

$$cx^2 = \max_{y \in R}\left\{-2x^2 - y^2 + 2xy + \beta c y^2\right\}.$$

The FOC associated with this problem is

$$-2y + 2x + 2\beta c y = 0 \Rightarrow y = \frac{x}{1-\beta c}.$$

Inserting this result into the candidate value function we obtain

$$\begin{aligned} cx^2 &= -2x^2 - \left(\frac{x}{1-\beta c}\right)^2 (1-\beta c) + 2x\left(\frac{x}{1-\beta c}\right), \\ &= \left(-2 - \frac{1}{1-\beta c} + \frac{2}{1-\beta c}\right) x^2, \end{aligned}$$

or

$$c = \frac{2\beta c - 1}{1-\beta c},$$

which gives the quadratic equation

$$Q(c) \equiv \beta c^2 + (2\beta - 1) c - 1 = 0.$$

Now, since $\beta > 0$ and $Q(0) = -1$, then the above equation has two real roots, one positive and one negative. Since $v(x)$ is concave, we must pick the negative root of $Q(c) = 0$, which we denote by c^*. Thus, $v(x) = c^* x^2$, which is clearly decreasing for $x > 0$.

(4) Assuming differentiability of v, write down the first order conditions for the problem. Use $y = g(x)$ for the optimal decision rule.

Ans: The FOC of (1) is

$$-a'(g(x) - x) + \beta v'(g(x)) = 0,$$

or

$$a'(g(x) - x) = \beta v'(g(x)). \tag{2}$$

(5) Use the envelope to write down an expression for the derivative of the value function v. Use $y = g(x)$ for the optimal decision rule.

Ans: The envelope condition is

$$v'(x) = h'(x) + a'(g(x) - x). \tag{3}$$

(6) Let $\bar{x} = g(\bar{x})$ denote a steady state. Use your answer to (5) and (6) to show that $\bar{x} = x^*$ is the unique steady state (2 lines maximum).

Ans: Evaluate (2) and (3) at the steady state (i.e., $\bar{x} = g(\bar{x})$) to obtain

$$a'(\bar{x} - \bar{x}) = \beta \left(h'(\bar{x}) + a'(\bar{x} - \bar{x}) \right),$$

or, using that $a'(0) = 0$,

$$\beta h'(\bar{x}) = 0 \Rightarrow h'(\bar{x}) = 0.$$

Then, $\bar{x} = x^*$ since by assumption $h'(\bar{x}) = 0$ iff $\bar{x} = x^*$.

(7) First order conditions and shape of the optimal decision rule g.

(7.1) Let $F(x, y)$ be the period return function for this problem. Plot the function $-F_y(x, y)$ (in terms of derivatives of h and a) for a fixed value of x with $x < x^*$, and $\beta v'(y)$ with y in the horizontal axis. Indicate in the horizontal axis the value of x and the value of y that corresponds to $g(x)$.

Ans: We have that

$$-F_y(x, y) = a'(y - x).$$

Since $a''(z) \geq 0$, $a'(z)$ is increasing z (and therefore, in y). Thus, $-F_y(x, y)$ is increasing in y. Moreover, $v(y)$ is concave, so $\beta v'(y)$ is decreasing in y.

Also, at the steady state $\bar{x}(= x^*)$ we know that $\bar{x} = g(\bar{x})$, then from the FOC (2) and the assumption $a'(0) = 0$ we obtain

$$a'(g(\bar{x}) - \bar{x}) = v'(\bar{x}) = 0.$$

See Figure 1 for a plot.

(7.2) In the same plot used for (7.1) draw the function $-F_y(x', y)$ for a higher value of x, i.e. for $x < x' < x^*$. Make sure to identify the new value of $g(x')$ in your plot.

Ans: Figure 1 also shows the optimal choice for a higher value of x, $x < x' < \bar{x}$. Since $x < x'$, the curve $a'(y - x)$ shifts down for all values of y. Since the curve $\beta v'(y)$ is unchanged, it follows that

$$g\left(x'\right) \geq g(x), \qquad \text{all } x' > x.$$

Alternatively, assume by way of contradiction that $x' > x$ implies $g(x) > g(x')$. Then,

$$\begin{aligned}
\beta v'\left(g\left(x'\right)\right) &= a'\left(g\left(x'\right) - x'\right), \qquad \text{(from [2])}, \\
&\leq a'(g(x) - x), \qquad \text{(since } a''(z) \geq 0\text{)}, \\
&= \beta v'(g(x)), \qquad \text{(from [2])},
\end{aligned}$$

which implies that $g(x') \geq g(x)$ since $v(y)$ is concave. Thus, we have reached a contradiction, so we conclude that $g(x)$ is increasing.

(7.3) Given your answer to (7.1) and (7.2) is $g(x)$ increasing or decreasing in x? [one word].

Ans: Increasing.

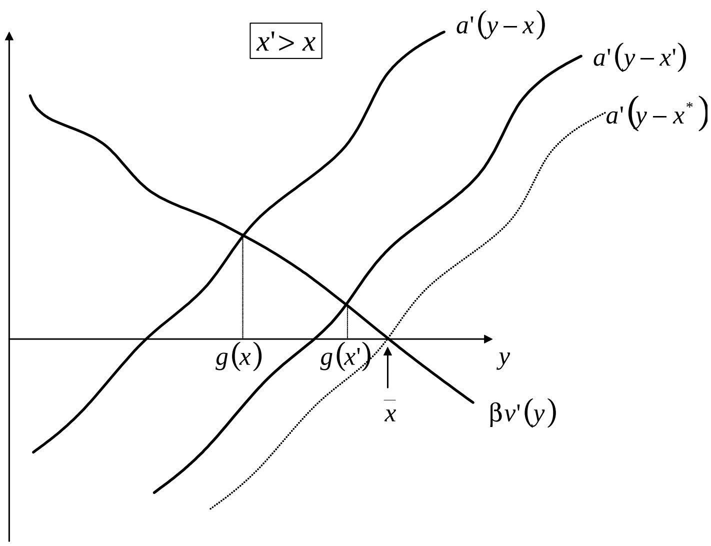

Figure 1: Policy rule, $g(x)$.

(8) Roots of Euler Equation

Let $F_y(x, g(x)) + \beta F_x[g(x), g(g(x))] = 0$ be the Euler equation for an arbitrary dynamic programing problem with period return $F(x, y)$ concave and differentiable and optimal policy $y = g(x)$. Assume that the state x is one dimensional and that g is differentiable.

(8.1) Differentiate the expression for the Euler equation w.r.t. x. Make sure that each of the derivatives is evaluated at the correct arguments [one line].

Ans: Differentiating w.r.t. to x and rearranging

$$F_{yx}(x, g(x)) + \left(\frac{F_{yy}(x, g(x))}{\beta} + F_{xx}[g(x), g(g(x))] + F_{xy}[g(x), g(g(x))] g'(g(x)) \right) \beta g'(x) = 0$$

(8.2) Evaluate the expression in (8.1) at the steady state values $\bar{x} = g(\bar{x})$ [one line].

Ans:

$$F_{yx}(\bar{x}, \bar{x}) + \left[\frac{F_{yy}(\bar{x}, \bar{x})}{\beta} + F_{xx}(\bar{x}, \bar{x}) + F_{xy}(\bar{x}, \bar{x}) g'(\bar{x}) \right] \beta g'(\bar{x}) = 0.$$

(8.3) Assuming that $F_{yx}(\bar{x},\bar{x}) \neq 0$, *divide the resulting expression in (8.2) by this quantity* and write down a quadratic function on the variable λ whose zeros are solved by $g'(\bar{x})$. Denote this quadratic expression by $Q(\lambda)$ where

$$Q(\lambda) = 1 \; + \; b\,\lambda \; + \; \beta\,\lambda^2,$$

and

$$b \equiv \frac{F_{yy}(\bar{x},\bar{x}) + \beta F_{xx}(\bar{x},\bar{x})}{F_{yx}(\bar{x},\bar{x})}.$$

Ans: Dividing by $F_{yx}(\bar{x},\bar{x})$, rearranging and letting $\lambda \equiv g'(\bar{x})$,

$$Q(\lambda) = 1 + \frac{F_{yy}(\bar{x},\bar{x}) + \beta F_{xx}(\bar{x},\bar{x})}{F_{xy}(\bar{x},\bar{x})}\lambda + \beta\lambda^2 = 0$$

or

$$Q(\lambda) = 1 + b\lambda + \beta\lambda^2 = 0,$$

where

$$b = \frac{F_{yy}(\bar{x},\bar{x}) + \beta F_{xx}(\bar{x},\bar{x})}{F_{xy}(\bar{x},\bar{x})}.$$

(9) Speed of Convergence:

(9.1) Write down an expression for each of the following derivatives: $F_{xx}(\bar{x},\bar{x})$, $F_{xy}(\bar{x},\bar{x})$, and $F_{yy}(\bar{x},\bar{x})$ for the problem with adjustment cost using the functions h and a as well as $\bar{x} = x^*$. Write down an expression for b in terms of the derivatives of h and a evaluated at the steady state values.

Ans:

$$\begin{aligned} F(x,y) &= h(x) - a(y-x), \\ F_x(x,y) &= h'(x) + a'(y-x), \\ F_y(x,y) &= -a'(y-x), \end{aligned}$$

so that the second derivatives are

$$\begin{aligned} F_{xx}(x,y) &= h''(x) - a''(y-x), \\ F_{yy}(x,y) &= -a''(y-x), \\ F_y(x,y) &= a''(y-x). \end{aligned}$$

At the steady state, $\bar{x} = g(\bar{x})$, we obtain

$$\begin{aligned} F_{xx}(\bar{x}, \bar{x}) &= h''(\bar{x}) - a''(0), \\ F_{yy}(\bar{x}, \bar{x}) &= -a''(0), \\ F_{yx}(\bar{x}, \bar{x}) &= a''(0). \end{aligned}$$

(9.2) Write down the quadratic equation $Q(\lambda)$ derived in (8.3) for the problem of adjustment cost [this should be a function of λ and parameters $a'(0), \beta$ and $h''(x^*)$ only]

Ans:

$$Q(\lambda) = 1 + \frac{-a''(0) + \beta(h''(\bar{x}) - a''(0))}{a''(0)}\lambda + \beta\lambda^2,$$

or

$$Q(\lambda) = 1 - \left(1 + \beta - \beta\frac{h''(\bar{x})}{a''(0)}\right)\lambda + \beta\lambda^2.$$

(9.3) Compute the values of $Q(\lambda)$ for $\lambda = 0$, $\lambda = 1$, $\lambda = 1/\beta$ and λ^*, where λ^* is such that $Q'(\lambda^*) = 0$. Is $\lambda^* > 1$? What is the sign of $Q'(\lambda)$ in $\lambda \in (0, 1)$? What is the limit $\lim_{\lambda\to\infty} Q(\lambda)$?

Ans:

$$\begin{aligned} Q(0) &= 1, \\ Q(1) &= \beta\frac{h''(x^*)}{a''(0)} < 0, \\ Q(1/\beta) &= \frac{h''(x^*)}{a''(0)} < 0. \end{aligned}$$

Moreover,

$$Q'(\lambda^*) = -\left(1 + \beta - \beta\frac{h''(\bar{x})}{a''(0)}\right) + 2\beta\lambda^* = 0,$$

so that

$$\lambda^* = \frac{1}{2}\left(1 + \frac{1}{\beta} - \frac{h''(\bar{x})}{a''(0)}\right) > 1.$$

Now, since $Q(0) > 1$, $Q(1) < 0$ and the (unique) minimum is obtained at $\lambda^* > 1$, we conclude that $Q'(\lambda) < 0$ for $\lambda \in (0, 1)$. Finally, the same reasoning can be used to argue that $\lim_{\lambda\to\infty} Q(\lambda) = \infty$.

(9.4) Plot $Q(\lambda)$ with λ in the horizontal axis. Make sure to identify $\lambda = 0$, $\lambda = 1$, $\lambda = \lambda^*$ and $\lambda = 1/\beta$ as well as the corresponding values of $Q(\lambda)$. Make sure you label the smallest root of Q, and denote it by λ_1. How does λ_1 depend on b?

Ans: λ_1 is the smallest root that solves $Q(\lambda) = 1 + b\lambda + \beta\lambda^2$, or $\lambda_1 = -\left(b + \sqrt{b^2 - 4\beta}\right)/2\beta$. Thus,

$$\begin{aligned}\frac{\partial \lambda_1}{\partial b} &= -\frac{1 + b/\sqrt{b^2 - 4\beta}}{2\beta}, \\ &= -\frac{b + \sqrt{b^2 - 4\beta}}{2\beta}\frac{1}{\sqrt{b^2 - 4\beta}}, \\ &= \frac{\lambda_1}{\sqrt{b^2 - 4\beta}} > 0.\end{aligned}$$

See Figure 2.

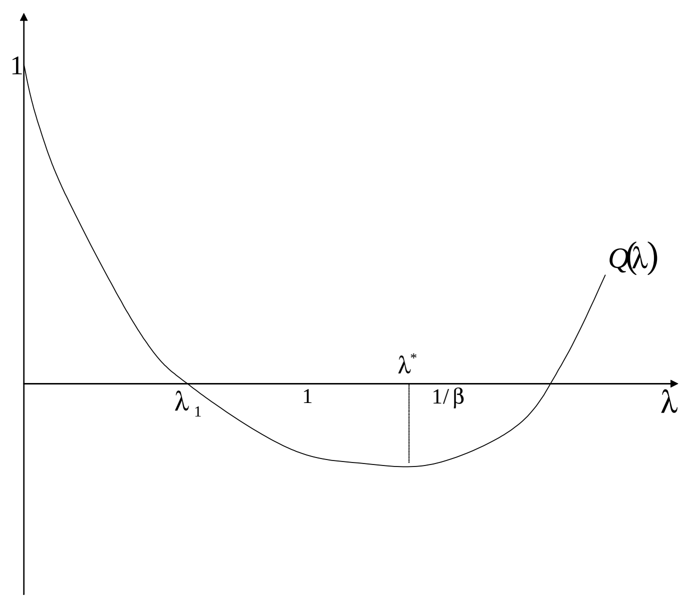

Figure 2: Determination of λ_1.

(9.5) Draw a second quadratic function, denoted as $\hat{Q}(\lambda)$, that corresponds to a problem with a larger value of $-h''(x^*)/a''(0) = |h''(x^*)/a''(0)|$. How does $|h''(x^*)/a''(0)|$ relate to b? Denote the smallest root of this equation by $\hat{\lambda}_1$. How is $\hat{\lambda}_1$ compared with λ_1?

Ans: See Figure 3. Note that

$$b = -\left(1 + \beta - \beta\frac{h''(\bar{x})}{a''(0)}\right) = -\left(1 + \beta + \beta\left|\frac{h''(\bar{x})}{a''(0)}\right|\right),$$

so that b decreases with $|h''(\bar{x})/a''(0)|$. This implies that $\hat{\lambda}_1 < \lambda_1$.

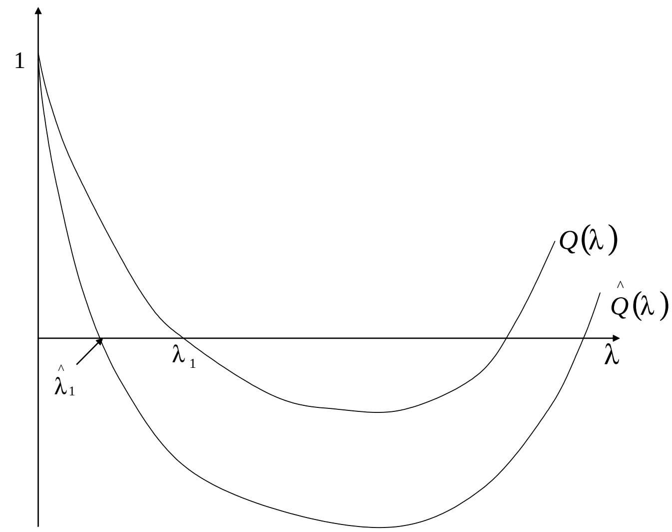

Figure 3: $\hat{\lambda}_1$ versus λ_1.

(9.6) Recall that if $|\lambda_1| < 1$, then $g'(\bar{x}) = \lambda_1$. How does the speed of convergence of $\{x_t\}$ depend on $-h''(x^*)/a''(0) = |h''(x^*)/a''(0)|$? What is the economic intuition for this dependence? (Explain the intuition for each of the parameters: $|h''(x^*)|$ and $|a''(0)|$).

Ans: As shown above, λ_1 decreases with $|h''(\bar{x})/a''(0)|$, which in turn means that the system converges faster to the steady state. The intuition goes as follows: $a''(0)$ measures the rate at which the adjustment cost increases on the margin. A decrease in $|a''(0)|$ means that the marginal cost of adjusting the state x is lower. Similarly, $h''(\bar{x})$ measures the rate at which the period return increases on the margin. An increase in $|h''(\bar{x})|$ means that the marginal benefits of adjusting the state x are higher. Both effects should speed up convergence to the steady state, as it actually happens.

2 Speed of convergence and intertemporal substitution

Consider the following dynamic programing problem where the state x is one-dimensional, i.e. $x \in R$.

Program 1:

$$V(x) = \max_{y \in R} \{F(x, y) + \beta V(y)\}.$$

Denote by $y = g(x)$ its optimal policy rule.

Assumptions. F is strictly concave, twice differentiable, strictly increasing in x and strictly decreasing in y, so that $F_x > 0,\ F_y < 0$. The cross derivative of F is positive, i.e. $F_{xy} > 0$.

Background. The Euler equation is given by

$$F_y\left(x, g\left(x\right)\right) + \beta F_x\left(g\left(x\right), g\left(g\left(x\right)\right)\right) = 0,$$

for all x. Let $\bar{x}$ be an steady state, so that $\bar{x} = g\left(\bar{x}\right)$ which implies that

$$F_y\left(\bar{x}, \bar{x}\right) + \beta F_x\left(\bar{x}, \bar{x}\right) = 0.$$

Recall that differentiating the Euler equation w.r.t. x and evaluating it at a steady state $\bar{x}$ we obtain the following quadratic expression for $g'\left(\bar{x}\right)$:

$$F_{yx}\left(\bar{x}, \bar{x}\right) + \left[F_{yy}\left(\bar{x}, \bar{x}\right) + \beta F_{xx}\left(\bar{x}, \bar{x}\right)\right] g'\left(\bar{x}\right) + \beta F_{xy}\left(\bar{x}, \bar{x}\right)\left[g'\left(\bar{x}\right)\right]^2 = 0,$$

so that, if $F_{xy}\left(\bar{x}, \bar{x}\right) > 0$, then $g'\left(\bar{x}\right)$ solves $Q\left(\lambda\right) = 0$, where

$$Q\left(\lambda\right) = 1 \ + \ b\,\lambda \ + \ \beta\,\lambda^2,$$

and

$$b \equiv \frac{F_{yy}\left(\bar{x}, \bar{x}\right) + \beta F_{xx}\left(\bar{x}, \bar{x}\right)}{F_{yx}\left(\bar{x}, \bar{x}\right)}.$$

As discussed elsewhere, the two roots of Q, denoted by λ_1 and λ_2 come in almost reciprocal pairs, i.e., they satisfy

$$\lambda_1 \lambda_2 = \frac{1}{\beta}.$$

Letting λ_1 be the root with smallest absolute value, if $|\lambda_1| < 1$ then the steady state is locally stable and the speed of convergence is determined by $g'\left(\bar{x}\right)$.

Questions 1-4:

1) Show that, if $F_{xy}\left(\bar{x}, \bar{x}\right) > 0$, then $b < 0$ and $\lambda_1, \lambda_2 > 0$.

Ans: By the strict concavity of F we know that $F_{xx}\left(\bar{x}, \bar{x}\right) < 0$ and $F_{yy}\left(\bar{x}, \bar{x}\right) < 0$. Thus,

$$b = \frac{F_{yy}\left(\bar{x}, \bar{x}\right) + \beta F_{xx}\left(\bar{x}, \bar{x}\right)}{F_{yx}\left(\bar{x}, \bar{x}\right)} < 0,$$

whenever $F_{yx}\left(\bar{x}, \bar{x}\right) > 0$. Since λ_1 is a root of Q, it satisfies

$$1 + b\lambda_1 + \beta\lambda_1^2 = 0,$$

or, multiplying by λ_2,

$$\lambda_2 + \frac{b}{\beta} + \lambda_1 = 0,$$

where we use that $\lambda_1\lambda_2 = 1/\beta$. Since $b < 0$, this implies that $\lambda_1 + \lambda_2 > 0$. But we also have that $\lambda_1\lambda_2 = 1/\beta > 0$, so it follows that $\lambda_1, \lambda_2 > 0$.

Another way to show this result comes from realizing that

$$Q(\lambda) = 1 + b\lambda + \beta\lambda^2 > 0,$$

for $\lambda \leq 0$ (since $b < 0$), so that λ_1 and λ_2 must be strictly positive.

2) Show that $\lambda_1 = g'(\bar{x}) \in (0, 1)$ if and only if $1 + b + \beta < 0$.

Ans: Notice that

$$\begin{aligned} Q'(\lambda) &= b + 2\beta\lambda, \\ Q''(\lambda) &= 2\beta > 0, \end{aligned}$$

so Q is strictly convex. Now, assume that $Q(1) = 1 + b + \beta < 0$. Then, since Q is strictly convex and $Q(0) = 1 > 0$, the smallest root is $\lambda_1 = g'(\bar{x}) \in (0, 1)$. Similarly, assume that $\lambda_1 = g'(\bar{x}) \in (0, 1)$. Then, since $\lambda_1\lambda_2 = \frac{1}{\beta}$, Q is strictly convex and $Q(0) = 1 > 0$, it follows that $Q(1) = 1 + b + \beta < 0$.

3) Show that

$$\lambda_1 = g'(\bar{x}) = \frac{-b - \sqrt{b^2 - 4\beta}}{2\beta}.$$

[Hint: Use the formula for the roots of a quadratic function and argue that λ_1 is the smallest root].

Ans:

$$\lambda_i = \frac{-b \pm \sqrt{b^2 - 4\beta}}{2\beta}.$$

Thus,

$$\lambda_1 = g'(\bar{x}) = \frac{-b - \sqrt{b^2 - 4\beta}}{2\beta} < \frac{-b + \sqrt{b^2 - 4\beta}}{2\beta} = \lambda_2,$$

since λ_1 is the smallest root.

4) Use the answer to 3) to show that $g'(\bar{x})$ is an increasing function of b. [Hint: Differentiate the expression for λ_1].

Ans:

$$\begin{aligned} \frac{\partial \lambda_1}{\partial b} &= \frac{-1 - b/\sqrt{b^2 - 4\beta}}{2\beta}, \\ &= \frac{-b - \sqrt{b^2 - 4\beta}}{2\beta} \frac{1}{\sqrt{b^2 - 4\beta}}, \\ &= \frac{\lambda_1}{\sqrt{b^2 - 4\beta}} > 0. \end{aligned}$$

Consider the following related problem.

Program 2:

$$\tilde{V}(x) = \max_{y \in R} \left\{ \tilde{F}(x, y) + \beta \tilde{V}(y) \right\},$$

with

$$\tilde{F}(x, y) = U[F(x, y)],$$

where $U' > 0$ and $U'' < 0$. Thus $\tilde{F}$ is a strictly increasing and concave transformation of F.

Denote the optimal policy of program 2 by $\tilde{g}(x)$.

Questions 5-10

5) For future reference, compute the first derivatives of $\tilde{F}(x, y)$ w.r.t. x and y, i.e., $\tilde{F}_y(x, y)$ and $\tilde{F}_x(x, y)$, in terms of the derivatives U and F.

Ans:

$$\begin{aligned} \tilde{F}_x(x, y) &= U'[F(x, y)] F_x(x, y), \\ \tilde{F}_y(x, y) &= U'[F(x, y)] F_y(x, y). \end{aligned}$$

6) Show that $\bar{x}$ is a steady state of program 1 if and only if it is an steady state of program 2. [Hint: Use your answer to 5) and the definition of a steady state for programs 1 and 2].

Ans: The steady state for program 1 solves

$$F_y(\bar{x}, \bar{x}) + \beta F_x(\bar{x}, \bar{x}) = 0.$$

In turn, the steady state for program 2 solves

$$\tilde{F}_y(\bar{x}, \bar{x}) + \beta \tilde{F}_x(\bar{x}, \bar{x}) = 0,$$

or, using 5),

$$U'[F(\bar{x}, \bar{x})] [F_y(\bar{x}, \bar{x}) + \beta F_x(\bar{x}, \bar{x})] = 0.$$

Since $U' > 0$, the result follows immediately.

7) For future reference, compute the second derivatives of $\tilde{F}$, i.e., $\tilde{F}_{yy}(x,y)$, $\tilde{F}_{xx}(x,y)$, and $\tilde{F}_{xy}(x,y)$ as a function of the first and second derivatives of U and F.

Ans:

$$\begin{aligned}
\tilde{F}_{xx}(x,y) &= U''[F(x,y)][F_x(x,y)]^2 + U'[F(x,y)]F_{xx}(x,y), \\
\tilde{F}_{yy}(x,y) &= U''[F(x,y)][F_y(x,y)]^2 + U'[F(x,y)]F_{yy}(x,y), \\
\tilde{F}_{xy}(x,y) &= U''[F(x,y)]F_x(x,y)F_y(x,y) + U'[F(x,y)]F_{xy}(x,y).
\end{aligned}$$

8) Assume that $g'(\bar{x}) \in (0,1)$. Show that

$$g'(\bar{x}) < \tilde{g}'(\bar{x}) < 1.$$

[Hint: The main idea is to express the coefficient $\tilde{b}$ corresponding to the quadratic equations of $\tilde{g}'(\bar{x})$ in terms of b, $-U''/U$, and other derivatives of F and then use the answer to 4)]. Here is a more detailed set of hints:

i) Use the definition of $\tilde{b}$ as a function of β, $\tilde{F}_{xx}, \tilde{F}_{yy}$ and $\tilde{F}_{xy}$ evaluated at $(\bar{x},\bar{x})$.

Ans:

$$\tilde{b} = \frac{\tilde{F}_{yy}(\bar{x},\bar{x}) + \beta \tilde{F}_{xx}(\bar{x},\bar{x})}{\tilde{F}_{yx}(\bar{x},\bar{x})}.$$

ii) Use your answer to 7) to rewrite $\tilde{b}$ as a function of U', U'', $F(x,y)$, β, F_{xx}, F_{yy} and F_{xy} evaluated at $(\bar{x},\bar{x})$.

Ans:

$$\tilde{b} = \frac{U''[F_y(\bar{x},\bar{x})]^2 + U'F_{yy}(\bar{x},\bar{x}) + \beta\left[U''[F_x(\bar{x},\bar{x})]^2 + U'F_{xx}(\bar{x},\bar{x})\right]}{U''F_x(\bar{x},\bar{x})F_y(\bar{x},\bar{x}) + U'F_{xy}(\bar{x},\bar{x})}.$$

iii) Reorder the terms in $\tilde{b}$ to obtain an expression that (apart from other terms) depends on b and γ given by

$$\gamma \equiv -\frac{U''(F(\bar{x},\bar{x}))}{U'(F(\bar{x},\bar{x}))}.$$

Ans: Omitting arguments to save on notation,

$$\begin{aligned}\tilde{b} &= \frac{U''\left(F_y^2+\beta F_x^2\right)+U'\left(F_{yy}+\beta F_{xx}\right)}{U''F_xF_y+U'F_{xy}}, \\ &= \frac{-\left(-U''/U'\right)\left(F_y^2+\beta F_x^2\right)/F_{xy}+\left(F_{yy}+\beta F_{xx}\right)/F_{xy}}{-\left(-U''/U'\right)F_xF_y/F_{xy}+1}, \\ &= \frac{-\gamma\left(F_y^2+\beta F_x^2\right)/F_{xy}+b}{1-\gamma F_xF_y/F_{xy}}.\end{aligned}$$

iv) Use the steady state Euler equation for program 1 to eliminate $F_y\left(\bar{x},\bar{x}\right)$ from your expression.

Ans: From the Euler equation we know that $F_y\left(\bar{x},\bar{x}\right)=-\beta F_x\left(\bar{x},\bar{x}\right)$. Thus,

$$\begin{aligned}\tilde{b} &= \frac{-\gamma\left(\left(-\beta F_x\right)^2+\beta F_x^2\right)/F_{xy}+b}{1-\gamma F_x\left(-\beta F_x\right)/F_{xy}}, \\ &= \frac{-\gamma\beta\left(1+\beta\right)F_x^2/F_{xy}+b}{1+\gamma\beta F_x^2/F_{xy}}.\end{aligned}$$

v) Rearrange $\tilde{b}$ to obtain

$$\tilde{b}=-\left(1+\beta\right)\frac{\alpha}{1+\alpha}+b\frac{1}{1+\alpha},$$

for

$$\alpha=\gamma\beta\left[F_x\left(\bar{x},\bar{x}\right)\right]^2/F_{xy}\left(\bar{x},\bar{x}\right)>0.$$

Ans:

$$\begin{aligned}\tilde{b} &= -\left(1+\beta\right)\frac{\gamma\beta F_x^2/F_{xy}}{1+\gamma\beta F_x^2/F_{xy}}+\frac{b}{1+\gamma\beta F_x^2/F_{xy}}, \\ &= -\left(1+\beta\right)\frac{\alpha}{1+\alpha}+\frac{b}{1+\alpha},\end{aligned}$$

where

$$\alpha\equiv\gamma\beta\left[F_x\left(\bar{x},\bar{x}\right)\right]^2/F_{xy}\left(\bar{x},\bar{x}\right)>0.$$

vi) Use v) to show that $\tilde{b}$ satisfies

$$\tilde{b}-b=-\frac{\alpha}{1+\alpha}\left(1+\beta+b\right).$$

Ans:

$$\begin{aligned}\tilde{b}-b &= -(1+\beta)\frac{\alpha}{1+\alpha}+\frac{b}{1+\alpha}-b \\ &= -\frac{\alpha}{1+\alpha}(1+\beta+b).\end{aligned}$$

vii) Use that $g'(\bar{x}) \in (0,1)$ implies $(1+\beta+b) < 0$ and v) to show that $\tilde{b} > b$.

Ans:

$$\tilde{b}-b = -\frac{\alpha}{1+\alpha}(1+\beta+b) > 0,$$

since $(1+\beta+b) < 0$ and $\alpha > 0$.

viii) Use that v) implies $1+\tilde{b}+\beta < 0$, to show that $\tilde{g}'(\bar{x}) \in (0,1)$.

Ans:

$$\begin{aligned}1+\beta+\tilde{b} &= 1+\beta-(1+\beta)\frac{\alpha}{1+\alpha}+b\frac{1}{1+\alpha} \\ &= \frac{1}{1+\alpha}(1+\beta+b) < 0,\end{aligned}$$

since $(1+\beta+b) < 0$ and $\alpha > 0$. Thus, from 2) we know that $\tilde{g}'(\bar{x}) \in (0,1)$.

ix) Use 4) and vi) to show that $\tilde{g}'(\bar{x}) > g'(\bar{x})$.

Ans: From 4) we know that $g'(\bar{x})$ is an increasing function of b and from vi) we know that $\tilde{b} > b$. Hence, $\tilde{g}'(\bar{x}) > g'(\bar{x})$.

9) Explain, intuitively, why the speed of convergence is slower for the solution to program 2 than for the solution to program 1.

Ans: Since $\tilde{F}$ is a concave transformation of F, agents dislike relatively more having variations in F through time, so that the optimal policy entails approaching $\bar{x}$ in a slower fashion.

10) Consider the following two versions of the neoclassical growth model. Version 1:

$$\begin{aligned}&\max_{c_t,i_t}\sum_{t\geq 0}\beta^t v(c_t), \\ c_t &= f(k_t)-i_t, \text{ and } k_{t+1}=i_t+(1-\delta)k_t,\end{aligned}$$

for strictly concave v and f, satisfying Inada conditions, and version 2:

$$\begin{aligned} & \max_{c_t, i_t} \sum_{t \geq 0} \beta^t U \left[v \left(c_t \right) \right], \\ c_t &= f \left(k_t \right) - i_t, \text{ and } k_{t+1} = i_t + \left(1 - \delta \right) k_t, \end{aligned}$$

where U is increasing and concave. What do you conclude about the speed of convergence of the solutions of these two programs?

In particular map the first version into the $F(x, y)$ type of notation, argue that the hypothesis of 8) are satisfied.

Ans: The period return function is

$$F(x, y) = v(f(x) + (1 - \delta) x - y).$$

We must also argue that F is strictly concave (i.e., $F_{xx} < 0$, $F_{yy} < 0$ and $F_{xx}F_{yy} - F_{xy}^2 > 0$), strictly increasing in x, strictly decreasing in y and with a positive cross-partial derivative, that is $F_x > 0$, $F_y < 0$ and $F_{xy} > 0$. It is straightforward to show that.

11) Consider the following two versions of the neoclassical growth model. Version 1:

$$\begin{aligned} & \max_{c_t, i_t} \sum_{t \geq 0} \beta^t c_t, \\ c_t &= f(k_t) - i_t - \Phi(k_{t+1} - k_t), \\ k_{t+1} &= i_t + (1 - \delta) k_t, \end{aligned}$$

for strictly concave f satisfying Inada conditions, and with Φ satisfying

$$\Phi \geq 0, \ \Phi(0) = \Phi'(0) = 0, \quad \Phi' \geq 0, \ \Phi'' > 0;$$

and version 2:

$$\begin{aligned} & \max_{c_t, i_t} \sum_{t \geq 0} \beta^t U[c_t], \\ c_t &= f(k_t) - i_t - \Phi(k_{t+1} - k_t), \\ k_{t+1} &= i_t + (1 - \delta) k_t, \end{aligned}$$

for strictly increasing and strictly concave U.

In particular, map the first version into the $F(x, y)$ type of notation, argue that the hypothesis of 8) are satisfied.

Ans: The period return function is

$$F(x,y) = f(x) + (1-\delta)x - y - \Phi(y-x).$$

We must also argue that F is strictly concave (i.e., $F_{xx} < 0$, $F_{yy} < 0$ and $F_{xx}F_{yy} - F_{xy}^2 > 0$), strictly increasing in x, strictly decreasing in y and with a positive cross-partial derivative, that is $F_x > 0$, $F_y < 0$ and $F_{xy} > 0$. It is straightforward to show that.

3 Speed of Convergence versus Slope of the Saddle Path

Program i.

Consider the continuous time dynamic problem in the control-state notation described by the period return h and the law of motion g as:

$$V(x_0) = \max_{u(t)\in U,\ t\geq 0} \int_0^\infty e^{-\rho t} h(x(t), u(t))\, dt,$$

$$\dot{x}(t) = g(x(t), u(t)),$$

with $x(0) = x$ given. Denote the optimal path $x^*(t)$.

Program ii.

Consider the continuous time dynamic problem in the control-state notation described by period return $\hat{h}$ and law of motion $\hat{g}$ as,

$$\hat{V}(x_0) = \max_{u(t)\in U,\ t\geq 0} \int_0^\infty e^{-\hat{\rho} t} \hat{h}(x(t), u(t))\, dt,$$

$$\dot{x}(t) = \hat{g}(x(t), u(t)),$$

with $x(0) = x$ given.

Assume that, for some strictly positive number κ,

$$\begin{aligned} \hat{h}(x,u) &= \kappa\ h(x,u), \\ \hat{g}(x,u) &= \kappa\ g(x,u), \\ \hat{\rho} &= \kappa\rho. \end{aligned}$$

In this question we will show that two problems can have exactly the same saddle path, and different speed of convergence to the steady state. Clearly the fact that $\hat{h}$ is multiplied by a strictly positive number κ does not affect the results, since it defines a monotone transformation of the objective function (i.e., of the discounted integral), but it does turn out to be convenient for this problem.

a) Write down the Hamiltonian $H(x, u, \lambda)$ for program i, where λ denotes the co-state.

Ans:

$$H(x, u, \lambda) = h(x, u) + \lambda g(x, u).$$

b) Write down the optimality conditions for the control of program i.

Ans:

$$u : H_u(x, u, \lambda) = 0 \Leftrightarrow h_u(x, u) + \lambda g_u(x, u) = 0.$$

c) Write down a system of differential equations for the law of motion of the co-state λ and for the state of program i. Write this system in terms of the derivatives of H.

Ans:

$$x : H_x = \rho\lambda - \dot{\lambda} \Leftrightarrow \dot{\lambda} = \rho\lambda - H_x(x, u, \lambda).$$

Thus,

$$\begin{aligned} \dot{\lambda} &= \rho\lambda - H_x(x, u, \lambda), \\ \dot{x} &= H_\lambda(x, u, \lambda). \end{aligned}$$

d) Define $u = \mu(x, \lambda)$ as the solution of u in *b*). Display an expression for the derivatives of μ in terms of the derivatives of H.

Ans:

$$\begin{aligned} H_{ux} + H_{uu}\mu_x(x, \lambda) &= 0 \Rightarrow \mu_x(x, \lambda) = -H_{uu}^{-1}H_{ux}, \\ H_{u\lambda} + H_{uu}\mu_\lambda(x, \lambda) &= 0 \Rightarrow \mu_\lambda(x, \lambda) = -H_{uu}^{-1}H_{u\lambda}. \end{aligned}$$

e) Use the function μ to write a system of differential equations for x and λ in terms of (x, λ). Write this system in terms of the derivatives of H and the function μ.

Ans:

$$\begin{aligned} \dot{\lambda} &= \rho\lambda - H_x(x, \mu(x, \lambda), \lambda), \\ \dot{x} &= H_\lambda(x, \mu(x, \lambda), \lambda). \end{aligned}$$

f) Denote by $(\bar{x}, \bar{\lambda})$ a stationary solution of e). Write the equations that define $(\bar{x}, \bar{\lambda})$.

Ans:

$$\begin{aligned} \dot{\lambda} &= 0 \Leftrightarrow \rho\bar{\lambda} = H_x\left(\bar{x}, \mu\left(\bar{x}, \bar{\lambda}\right), \bar{\lambda}\right), \\ \dot{x} &= 0 \Leftrightarrow 0 = H_\lambda\left(\bar{x}, \mu\left(\bar{x}, \bar{\lambda}\right), \bar{\lambda}\right). \end{aligned}$$

g) Linearize the system obtained in e). The system should be of the form

$$\begin{bmatrix} \dot{\lambda} \\ \dot{x} \end{bmatrix} = A \begin{bmatrix} \lambda - \bar{\lambda} \\ x - \bar{x} \end{bmatrix}.$$

Display the expressions for the entries of the matrix A in terms of the derivatives of H and μ.

Ans:

$$\begin{bmatrix} \dot{\lambda} \\ \dot{x} \end{bmatrix} = \begin{bmatrix} \rho - H_{x\lambda} - H_{xu}\mu_\lambda & -H_{xx} - H_{xu}\mu_x \\ H_{\lambda\lambda} + H_{\lambda u}\mu_\lambda & H_{\lambda x} + H_{\lambda u}\mu_x \end{bmatrix} \begin{bmatrix} \lambda - \bar{\lambda} \\ x - \bar{x} \end{bmatrix}.$$

Diagonalize the matrix A, as

$$A = P^{-1}\ \Phi\ P,$$

with diagonal $\Phi = diag\{\theta_i\}$. Define

$$z(t) = P \begin{bmatrix} \lambda - \bar{\lambda} \\ x - \bar{x} \end{bmatrix},$$

so that the linear system is, for each variable i,

$$\dot{z}_i(t) = \theta_i\ z(t).$$

Assume that half of the eigenvalues of Φ are strictly negative, and the other half strictly positive. Without loss of generality, assume that the first half are the positive ones. Write the matrix P with the eigenvectors in four squared blocks as follows

$$P = \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix},$$

where the $[P_{11}, P_{12}]$ correspond to the strictly positive eigenvalues.

h) Derive an expression for the slope of the saddle path. We can represent the saddle path as a function s, setting $\lambda = s(x)$. When the free initial condition for λ is set using the given initial condition for x, $\lambda(0) = s(x(0))$, then the dynamic system in $e)$ converges to the steady state $(\bar{\lambda}, \bar{x})$. You should find a linear approximation of s around the steady state. The expression for $s(x)$ should be function of $\bar{\lambda}$, $\bar{x}$, P_{11} and P_{12}. Show that $\partial s(\bar{x})/\partial x = -[P_{11}]^{-1} P_{12}$.

Ans: Let $z_i(t)$ for $i = 1, 2$ denote the first and second half of the vector z. For the first half of the variables, corresponding to the unstable eigenvalues, we have

$$\begin{bmatrix} z_1(t) \\ z_2(t) \end{bmatrix} = \begin{bmatrix} 0 \\ z_2(t) \end{bmatrix} = \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix} \begin{bmatrix} \lambda - \bar{\lambda} \\ x - \bar{x} \end{bmatrix}.$$

Thus,

$$\begin{aligned} 0 &= P_{11}\left(\lambda - \bar{\lambda}\right) + P_{12}(x - \bar{x}), \\ \left(\lambda - \bar{\lambda}\right) &= -[P_{11}]^{-1} P_{12}(x - \bar{x}), \end{aligned}$$

or

$$\lambda = \bar{\lambda} - [P_{11}]^{-1} P_{12}(x - \bar{x}),$$

so that the slope of the saddle path is

$$\frac{\partial s(\bar{x})}{\partial x} = -[P_{11}]^{-1} P_{12}.$$

We essentially repeat steps $a)$ to $h)$ for program ii, and relate its results to those of $a)$-$h)$.

$\widehat{a})$ Write down the Hamiltonian for program ii, denote it by $\hat{H}(x, u, \lambda)$ and relate it to the one in for i denoted by $H(x, u, \lambda)$ and the number κ. Show that $\hat{H} = \kappa H$.

Ans:

$$\begin{aligned} \hat{H}(x, u, \lambda) &= \hat{h}(x, u) + \lambda \hat{g}(x, u), \\ &= \kappa\left[h(x, u) + \lambda\ g(x, u)\right], \\ &= \kappa H(x, u, \lambda). \end{aligned}$$

$\widehat{b})$ Write down the optimality conditions for the control of program ii, in terms of $\hat{H}$. Relate it to the one in $b)$ in terms of the derivatives of H and the number κ.

Ans:

$$u : \hat{H}_u(x, u, \lambda) = 0 \Rightarrow \kappa H_u(x, u, \lambda) = 0.$$

$\widehat{c}$) Write down a system of differential equations for the law of motion for the co-state λ and for the state for program ii. Write this system in terms of derivatives of $\hat{H}$. Relate it to the one in c) in terms of the derivatives of H and the number κ.

Ans:

$$x : \hat{H}_x = \hat{\rho}\lambda - \dot{\lambda} \Leftrightarrow \dot{\lambda} = \hat{\rho}\lambda - \hat{H}_x (x, u, \lambda).$$

Thus,

$$\begin{aligned} \dot{\lambda} &= \hat{\rho}\lambda - \hat{H}_x (x, u, \lambda), \\ \dot{x} &= \hat{H}_\lambda (x, u, \lambda), \end{aligned}$$

or

$$\begin{aligned} \dot{\lambda} &= \kappa [\rho\lambda - H_x (x, u, \lambda)], \\ \dot{x} &= \kappa H_\lambda (x, u, \lambda). \end{aligned}$$

$\widehat{d}$) Define $u = \hat{\mu}(x, \lambda)$ as the solution of u of $\widehat{b}$). Relate it to the one in b) in terms of the derivatives of H and κ. Show that $\hat{\mu} = \mu$. Display an expression for the derivatives of $\hat{\mu}$ in terms of the derivatives of $\hat{H}$.

Ans:

$$\hat{H}_u (x, \hat{\mu}(x, \lambda), \lambda) = 0 \Rightarrow \kappa H_u (x, \hat{\mu}(x, \lambda), \lambda) = 0,$$

which implies that $\hat{\mu}(x, \lambda) = \mu(x, \lambda)$. Thus,

$$\begin{aligned} \hat{\mu}_x (x, \lambda) &= -\hat{H}_{uu}^{-1} \hat{H}_{ux} = -H_{uu}^{-1} H_{ux} = \mu_x (x, \lambda), \\ \hat{\mu}_\lambda (x, \lambda) &= -\hat{H}_{uu}^{-1} \hat{H}_{u\lambda} = -H_{uu}^{-1} H_{u\lambda} = \mu_\lambda (x, \lambda). \end{aligned}$$

$\widehat{e}$) Use the function $\hat{\mu}$ to write a system of differential equations for x and λ in terms of (x, λ). Write this system in terms of the derivatives of $\hat{H}$ and the function $\hat{\mu}$. Relate it to the one found in e) in terms of the derivatives of H, the function μ and the number κ.

Ans:

$$\begin{aligned} \dot{\lambda} &= \hat{\rho}\lambda - \hat{H}_x (x, \hat{\mu}(x, \lambda), \lambda), \\ \dot{x} &= \hat{H}_\lambda (x, \hat{\mu}(x, \lambda), \lambda), \end{aligned}$$

or

$$
\begin{aligned}
\dot{\lambda} &= \kappa\left[\rho\lambda - H_x\left(x, \mu\left(x, \lambda\right), \lambda\right)\right], \\
\dot{x} &= \kappa H_\lambda\left(x, \mu\left(x, \lambda\right), \lambda\right).
\end{aligned}
$$

$\widehat{f}$) Denote by $\left(\bar{x}, \bar{\lambda}\right)$ a stationary solution of $\widehat{e}$). Write the equations that define $\left(\bar{x}, \bar{\lambda}\right)$. Show that the steady state for the program i, found in f) also solve these equations, and hence it is the steady state for program ii.

Ans:

$$
\begin{aligned}
\dot{\lambda} &= 0 \Leftrightarrow \hat{\rho}\bar{\lambda} = \hat{H}_x\left(\bar{x}, \hat{\mu}\left(\bar{x}, \bar{\lambda}\right), \bar{\lambda}\right) \Leftrightarrow \rho\bar{\lambda} = H_x\left(\bar{x}, \mu\left(\bar{x}, \bar{\lambda}\right), \bar{\lambda}\right), \\
\dot{x} &= 0 \Leftrightarrow 0 = \hat{H}_\lambda\left(\bar{x}, \hat{\mu}\left(\bar{x}, \bar{\lambda}\right), \bar{\lambda}\right) \Leftrightarrow 0 = H_\lambda\left(\bar{x}, \mu\left(\bar{x}, \bar{\lambda}\right), \bar{\lambda}\right).
\end{aligned}
$$

$\widehat{g}$) Linearize the system of differential equations obtained in $\widehat{e}$). The system should be of the form

$$
\begin{bmatrix} \dot{\lambda} \\ \dot{x} \end{bmatrix} = \hat{A} \begin{bmatrix} \lambda - \bar{\lambda} \\ x - \bar{x} \end{bmatrix}.
$$

Display the expressions for the entries of the matrix $\hat{A}$ in terms of the derivatives of $\hat{H}$ and $\hat{\mu}$. Relate it to the one in g) in terms of the derivatives of H and the number κ. Show that $\hat{A} = \kappa\ A$.

Ans:

$$
\begin{bmatrix} \dot{\lambda} \\ \dot{x} \end{bmatrix} = \begin{bmatrix} \hat{\rho} - \hat{H}_{x\lambda} - \hat{H}_{xu}\hat{\mu}_\lambda & -\hat{H}_{xx} - \hat{H}_{xu}\hat{\mu}_x \\ \hat{H}_{\lambda\lambda} + \hat{H}_{\lambda u}\hat{\mu}_\lambda & \hat{H}_{\lambda x} + \hat{H}_{\lambda u}\hat{\mu}_x \end{bmatrix} \begin{bmatrix} \lambda - \bar{\lambda} \\ x - \bar{x} \end{bmatrix}.
$$

Thus,

$$
\hat{A} = \kappa \begin{bmatrix} \rho - H_{x\lambda} - H_{xu}\mu_\lambda & -H_{xx} - H_{xu}\mu_x \\ H_{\lambda\lambda} + H_{\lambda u}\mu_\lambda & H_{\lambda x} + H_{\lambda u}\mu_x \end{bmatrix} = \kappa A.
$$

As done above, diagonalize the matrix $\hat{A}$, as

$$
\hat{A} = \hat{P}^{-1}\ \hat{\Phi}\ \hat{P},
$$

with diagonal $\hat{\Phi} = diag\left\{\hat{\theta}_i\right\}$. Define

$$
z\left(t\right) = \hat{P} \begin{bmatrix} \lambda - \bar{\lambda} \\ x - \bar{x} \end{bmatrix},
$$

so that the linear system is

$$\dot{z}_i(t) = \hat{\theta}_i \; z_i(t).$$

Write the matrix $\hat{P}$ with the eigenvectors with four squared blocks as follows:

$$\hat{P} = \begin{bmatrix} \hat{P}_{11} & \hat{P}_{12} \\ \hat{P}_{21} & \hat{P}_{22} \end{bmatrix},$$

where the $\left[\widehat{P}_{11}, \widehat{P}_{12}\right]$ corresponds to the strictly positive eigenvalues.

$\widehat{h}$) Use that $\hat{A} = \kappa A$ to show that $\hat{P} = P$ and $\hat{\Phi} = \kappa \Phi$. Then, denoting by $\lambda = \hat{s}(x)$ the saddle path of the solution of program ii, show that

$$\frac{\partial s}{\partial x}(\bar{x}) = \frac{\partial \hat{s}}{\partial x}(\bar{x}),$$

and that the speed of convergence of the solution to program ii is κ times the speed of convergence of the solution to program i.

Ans: Since

$$\hat{A} = \kappa A = \kappa P^{-1} \Phi P,$$

then

$$\begin{aligned} \hat{\Phi} &= \kappa \Phi, \\ \hat{P} &= P, \\ \hat{P}^{-1} &= P^{-1}, \end{aligned}$$

solves

$$\hat{A} = \kappa A = P^{-1} \hat{\Phi} P = P^{-1} [\kappa \Phi] P.$$

Thus,

$$\frac{\partial \hat{s}(\bar{x})}{\partial x} = -\left[\hat{P}_{11}\right]^{-1} \hat{P}_{12} = -[P_{11}]^{-1} P_{12} = \frac{\partial s(\bar{x})}{\partial x}.$$

That is, the slope of the saddle path is the same but the eigenvalues are multiplied by κ.

$\widehat{i}$) Give an interpretation of the difference in program i and program ii.

Ans: Program ii is one where we measure time in different units. Say, if $\kappa > 1$, time is running at a faster rate.

$\widehat{j}$) Show that $V = \hat{V}$. *Hint*: Write the continuous time Bellman equation for V in program i, and the one with $\kappa > 0$, for $\hat{V}$ in program ii. Show that these two functional equations have the same solution.

Ans: Program i: V solves

$$\rho V(x) = \max_{u \in U} \left\{ h(x,u) + g(x,u) V'(x) \right\}.$$

Program ii: $\hat{V}$ solves

$$\hat{\rho}\hat{V}(x) = \max_{u \in U} \left\{ \hat{h}(x,u) + \hat{g}(x,u) \hat{V}'(x) \right\}.$$

By definition of $\hat{\rho}$, $\hat{h}$ and $\hat{g}$ we have

$$\rho\kappa\hat{V}(x) = \max_{u \in U} \left\{ \kappa h(x,u) + \kappa g(x,u) \hat{V}'(x) \right\},$$

or since $\kappa > 0$,

$$\rho\kappa\hat{V}(x) = \kappa \max_{u \in U} \left\{ h(x,u) + g(x,u) \hat{V}'(x) \right\},$$

or dividing by κ,

$$\rho\hat{V}(x) = \max_{u \in U} \left\{ h(x,u) + g(x,u) \hat{V}'(x) \right\}.$$

Hence,

$$\hat{V} = V.$$

4 Non-renewable resource extraction (extended version of core question summer 2008)

In this question we analyze the optimal depletion, and associated shadow price, of a non-renewable resource, which for concreteness we refer to as oil. There are two final non-durable goods: energy and the 'general' good. Energy is produced using oil reserves R and/or general good as inputs. The period utility is $u(e) + \alpha z$ where e is the consumption rate of energy, z is the consumption rate of the general good, and u is a strictly increasing and strictly concave function satisfying Inada conditions. The planner preferences are

$$\int_0^\infty \exp(-\rho t)[u(e(t)) + \alpha z(t)]dt$$

where $\rho > 0$ is the discount factor. The marginal value (price) of energy at t in terms of the general good, $v(t)$, equals the marginal rate of substitution, i.e. $v(t) = u'(e(t))/\alpha$. We use $p(t)$ for the shadow value of an unit of oil reserves.

There are two technologies available for the production of energy. One is an extractive technology: to produce one unit of energy per period requires one unit of oil and κ_1 units of the general good. The alternative technology produces one unit energy using κ_2 units of the general good, and uses no oil. We assume that:

$$0 \leq \kappa_1 < \kappa_2.$$

Let $E = (e_1, e_2)$ denote the control: a vector containing the quantity of energy produced by the extractive and alternative technologies respectively. Let $h(E, R)$ be the per period return, where the oil reserves R is the state with law of motion $g(E, R)$ so:

$$h(E, R) \equiv u(e_1 + e_2) - \alpha(\kappa_1\ e_1 + \kappa_2\ e_2)$$

and $\dot{R} \equiv g(E, R) = -e_1$.

1. (40 points). i) Write the Hamiltonian for this problem using p for the co-state. ii) Write the first order conditions with respect to the controls e_1 and e_2. Write each of them as a pair of weak inequality-equality to take into account whether the the non-negativity of each of the e_i binds. iii) Write the o.d.e. for the co-state p. iv) Write the transversality condition involving the state and costate.

   Answer:

   $$H(E, R, p) \equiv h(E, R) + p\ g(E, R) = u(e_1 + e_2) - \alpha(\kappa_1\ e_1 + \kappa_2\ e_2) - pe_1$$

   The Hamiltonian has a intuitive interpretation: the planner wants to maximize the utility of energy, net of the labor cost of producing it using any combination of the two technologies, and net of the shadow cost of the oil used in the extractive technology.

   $$u'(e_1 + e_2) - \alpha\kappa_1 \leq p \text{ with } = \text{ if } e_1 > 0$$

   $$u'(e_1 + e_2) - \alpha\kappa_2 \leq 0 \text{ with } = \text{ if } e_2 > 0$$

   The previous equations have a clear interpretation: the use of either technology has the same marginal benefit, but the marginal cost differ: the extractive technology also uses oil.

   $$\dot{p} = \rho\ p$$

   The intuition for the previous equation is clear: oil is an asset that does not depreciate, nor it gives any net return, hence its shadow value must increase at the rate of return of all the other investment at which agent discount utility - ρ.

   $$lim_{t\to\infty} \exp(-\rho t) p(t) R(t) = 0.$$

We will show that there is a $0 < T < \infty$ such that the extractive technology is used for

$t \in [0, T]$ and the alternative technology is used for $t \in [T, \infty)$. Moreover in the solution the oil reserves hit zero at T, i.e. $R(T) = 0$.

1.

2. (10 points) Use iii) of the previous question to solve explicitly for $p(t)$ as a function of $p(0), \rho$ and t (one line).

   Answer:

   $p(t) = p(0) \exp(\rho t)$

3. (20 points) Show that whenever the alternative technology is used, then $e_1(t) + e_2(t)$ must be constant. Find an expression for this constant, and denote it by $\bar{e}$.

   Answer:

   From the foc:

   $$u'(\bar{e}) = \alpha \kappa_2$$

   Since alternative technology has a constant -static- marginal cost, and a strictly decreasing marginal benefit -$u'(\cdot)$ is strictly decreasing-, then if the technology is used, its level is given by $\bar{e}$.

   Since , then there is a unique solution.

4. (20 points) Show that if there is an interval $[t_1, t_2]$ with $t_1 < t_2$ for which the extractive technology is used, then $e_1(t) + e_2(t)$ must be strictly decreasing in this interval. Display an (implicit) equation for this sum, your expression should be a function of $\alpha, \kappa_1, p(0), \rho, t, p(0)$, and the function $u'(\cdot)$.

   Answer:

   The f.o.c. for the extractive technology is:

   $$u'(e_1(t) + e_2(t)) = \alpha \kappa_1 + p(0) \exp(\rho t)$$

   Since we have shown that $p(t)$ is strictly increasing, and $u'(\cdot)$ is strictly decreasing, then $e_1(t) + e_2(t)$ must be strictly decreasing.

5. (10 points) Argue that there cannot be an interval $[t_1, t_2]$ with $t_1 < t_2$ for which both technologies are used simultaneously. (two lines maximum)

   Answer:

   It follows from the previous two answers, since $e_1(t) + e_2(t)$ has to either stay constant, or it has to be (strictly) decreasing with t.

6. (20 points) Assuming that the planner is indifferent between utilizing either of the technologies at $t = T$, use the answers to the previous questions to find the value of $p(T)$. Your solution should be a function of κ_1, κ_2, and α.

   Answer:

   Using both f.o.c. we obtain

$$\alpha\kappa_2 = u'(\bar{e}) = \alpha\kappa_1 + p(T)$$

   or

$$p(T) = \alpha(\kappa_2 - \kappa_1)$$

7. (20 points) Use the answers to the previous questions to find an expression for $p(0)$ in terms of exogenous parameters and T.

   Answer:

$$p(0) = \exp(-\rho T)p(T) = \exp(-\rho T)\alpha(\kappa_2 - \kappa_1)$$

8. (20 points) Use the f.o.c. w.r.t. e_1 for $t < T$, and the expression for $p(t)$ in terms of the solution for $p(T)$ to obtain an expression for $e_1(t)$. Your expression should use $(u')^{-1}(\cdot)$, i.e. the inverse of $u'(\cdot)$, evaluated in a simple function of $\alpha, \kappa_1, \kappa_2$, and $\rho(t - T)$.

   Answer:

$$\begin{aligned} e_1(t) &= (u')^{-1}(\ \alpha\kappa_1 + p(t)) \\ &= (u')^{-1}(\ \alpha\kappa_1 + p(0)\exp(\rho(t))\ ) \\ &= (u')^{-1}(\ \alpha\kappa_1 + \alpha(\kappa_2 - \kappa_1)\exp(-\rho(T - t))\ ) \\ &= (u')^{-1}(\ \alpha\kappa_1[1 - \exp(-\rho(T - t))] + \alpha\kappa_2\exp(-\rho(T - t))\ ) \end{aligned}$$

   From here it is easy to see that $e_1(t)$ is decreasing, and that as $t \to T$ then $e_1(t) \to \bar{e}$. On the other extreme, $e_1(0)$ equals

$$e_1(0) = (u')^{-1}(\ \alpha\kappa_1(1 - \exp(-\rho T)) + \alpha\kappa_2\exp(-\rho T)\ )$$

9. (30 points) Use the answer to the previous question and the resource constraint for oil, to find an implicit equation (an integral between 0 and T) for T. Argue that there is a unique value of T that solves this equation, and that it is strictly increasing in $R(0)$.

   Answer:

We use that $R(T) - R(0) = -\int_0^T e_1(t)dt$, that $R(T) = 0$, and the expression for $e_1(t)$:

$$R(0) = \int_0^T (u')^{-1} \left(\alpha\kappa_1[1 - \exp(-\rho(T-t))] + \alpha\kappa_2 \exp(-\rho(T-t)) \right) dt$$

The right hand side of this expression is increasing in T, since the integrand is positive, and since u', and hence its inverse, is decreasing and, as shown in the previous question, $e_1(\cdot)$ is an increasing function of $\rho(T-t)$. Hence, there is unique value that equates the RHS to $R(0)$, and its immediate to see that it is then increasing in $R(0)$.

10. (10 points) Use the answer to the previous question to establish that the switching time T is increasing in $\alpha(\kappa_2 - \kappa_1)$. (three lines maximum)

    Answer:

    It follows from the previous expression determining T, since u', and hence its inverse, is decreasing. Thus the RHS of the previous expression decreases with $\alpha(\kappa_2 - \kappa_1)$, and hence T is increasing in $\alpha(\kappa_2 - \kappa_1)$.

11. (20 points) How is the time path of the price of energy $v(t)$ of the solution of this problem? Draw a figure with time t in the horizontal axis and $v(t)$ in the vertical.

    Answer:

    $$v(t) = u'(e_1(t))/\alpha = \kappa_1 + p(t)/\alpha.$$

    for $t < T$, so the price increases through time. For $t \geq T$:

    $$v(t) = u'(\bar{e})/\alpha = \kappa_2.$$

12. (30 points) Recall that a decrease in α is an increase in the demand of energy relative to the rest of the goods. Study the comparative statics of paths for $v(t)$ that solves the planner problem for two values of α. In particular draw a figure with time t in the horizontal axis and two paths for $v(t)$ corresponding to two values for α. Label one of them 'high α' and the other 'low α'. Clearly label the value of T for each α.

    Answer:

    The price path of $v(t)$ remains the same after T regardless of α since it is given by the marginal cost of the alternative technology κ_2. The length of T as well as the dynamics of $p(t)$ before T are afected by α. The value of T is higher for the lower α, as shown in the previous question. In words, if the demand for energy increases, then the economy switches sooner to the alternative technolgy, and hence it arrives sonner to the higher price κ_2. This implies that the price paths of oil is higher from the beginning in the case of higher demand, i.e. in the case of lower α.

5 Tree-cutting Problem

Consider the problem of a decision maker that owns a plot of land where there is a tree. The size of the tree at time t, i.e. its height, is denoted by x_t. Let Δ denote the length of the time period. Between t and $t+\Delta$ a planted (uncut) evolves as:

Consider the problem of a decision maker that owns a plot of land where there is a tree. The size of the tree at time t, i.e. its height, is denoted by x_t. Let Δ denote the length of the time period. Between t and $t+\Delta$ a planted (uncut) evolves as:

$$x_{t+\Delta} = x_t + \Delta g\left(x_t\right)$$

where the function $g(x)$ is weakly decreasing in x with $g(0) > 0$. If a tree is not cut the owner incurs a maintaining cost, say watering it down, given by $m(x_t)\Delta$ between t and $t+\Delta$. We assume that $m(0) = 0$, $m(x) \geq 0$, and $m(\cdot)$ is weakly increasing and convex in x. Thus a tree that is watered grows. When a tree is cut, the owner of the plot sells the tree for a price that depend on the size of the tree, given by function $P(x_t)$. We assume that $P(x) \geq 0$, and that $P(\cdot)$ is strictly increasing and weakly concave in x. We assume that $P'(0)/P(0) = \infty$.

The plot can have at most one tree at any time, so when a tree is cut a new one can be planted. If a new tree is planted, then the decision maker incurs in a cost s at time t in seeds, and next period it has a tree of size $x_{t+\Delta} = 0$. The decision maker can choose to keep the lot vacant, in which case she incurs no cost in seeds, the tree size stays at $x_t = 0$ and no maintenance cost m is incurred.

The decision maker maximizes discounted utility of her consumption, with a per period discount factor $1/(1+\Delta\rho)$, where ρ is the discount rate and Δ is the length of the time period.

For future reference we let $\bar{x}$ be the height such that the growth rate of the tree equals the rate of return $\rho : g(\bar{x})/\bar{x} = \rho$

5.1 Bellman Equations

We will use (x, l) for the state of the decision maker problem, where $x \in R_+$ denote the size of the tree and $l \in \{0, 1\}$ denotes whether the plot is vacant or with a tree.

A. Case of vacant lot.

Consider the problem of an agent with a vacant lot, so her state is $(x, l) = (0, 0)$ and her value function $V(0, 0)$. Her decision now is either to plant a new tree or to leave the lot vacant. If she plants the new tree she incurs today the seed cost s and next period her state is $(x', l') = (0, 1)$. If she decides not to plant the tree, this period there is no payoff, and next period state is $(x', l') = (0, 0)$. The Bellman equation for $(x, l) = (0, 0)$ is thus:

$$V(0,0) = \max\left\{-s + \frac{1}{1+\Delta\rho}V(0,1), \frac{1}{1+\Delta\rho}V(0,0)\right\} \quad (4)$$

B. Case of lot with a tree.

Consider the problem of an agent with a tree in her lot, so her state is $(x, l) = (x, 1)$ for $x \geq 0$. Her decision is to either leave the tree in the lot, paying the cost $m(x)\Delta$ during this period and having next period state being $(x', l') = (x + \Delta g(x), 1)$, or cut the tree. If she cuts the tree, she receives $P(x)$ this period, and she gets to decide whether to plant a new one or not. The current value of having a vacant lot is $V(0, 0)$. Hence cutting a tree gives current value $P(x) + V(0, 0)$. The Bellman equation for $(x, l) = (x, 1)$ is thus:

$$V(x,1) = \max\left\{-m(x)\Delta + \frac{1}{1+\Delta\rho}V(x+\Delta g(x), 1),\ P(x) + V(0,0)\right\} \quad (5)$$

Q1 [5 points]. Use the Bellman equation for $(x, l) = (0, 0)$ and argue that

$$V(0,0) = \begin{cases} -s + \frac{1}{1+\rho\Delta}V(0,1) & \text{if } s \leq \frac{1}{1+\rho\Delta}V(0,1) \\ \\ 0 & \text{otherwise} \end{cases}$$

and hence

$$V(0,0) = \max\left\{-s + \frac{1}{1+\rho\Delta}V(0,1)\ \ , 0\right\}$$

Hint. Write, separately, the form of the Bellman equation in the case in which it is NOT optimal to plant a tree, and the one in which is optimal to do so.

A. If it is not optimal to plant a tree

$$V(0,0) = \frac{1}{1+\Delta\rho}V(0,0)$$

which implies that

$$V(0,0) = 0.$$

If it is optimal to plant a tree:

$$V(0,0) = -s + \frac{1}{1+\rho\Delta}V(0,1)$$

Replacing the result of your previous question, and using the notation:

$$v(x) \equiv V(x,1)$$

we can write the Bellman equation:

$$v(x) = \max\left\{-m(x)\Delta + \tfrac{1}{1+\Delta\rho}v(x+\Delta g(x)),\ P(x) + \max\left\{-s + \tfrac{1}{1+\Delta\rho}v(0), 0\right\}\right\}$$

Let C the space of functions $C = \{f : R_+ \to R\}$ and let $T : C \to C$, the operator on the space of such functions defined by the right hand side of the Bellman equation so that so that $(T\ f)(x)$:

$$(Tf)(x,1) = \max\left\{-m(x)\Delta + \tfrac{1}{1+\Delta\rho}f(x+\Delta g(x)),\ P(x) + \max\left\{-s + \tfrac{1}{1+\Delta\rho}f(0), 0\right\}\right\} \tag{6}$$

Q2 [5 points]. Argue that $v(x) = V(x,1)$ is weakly increasing in x. Hint: Assume that $f(\cdot)$ is weakly decreasing for $x \geq \bar{x}$, use the corollary of the Contraction Mapping Theorem. In particular argue than in this case $(Tf)(x)$ is also weakly increasing in x. (The intuition, which perfectly parallels the math, is straightforward: a taller tree has to be more valuable because it can always be cut).

Assume that there is an $0 \leq x^* < \bar{x}$ such that if $x < x^*$the tree is not cut.

We will work out the continuos time limit, i.e. the limit as $\Delta \to 0$, under such decision rule, and that v is differentiable in all its domain. We will find out that in such case:

$$\begin{aligned} v(x)\rho &= -m(x) + v'(x)g(x) \text{ for } 0 \leq x \leq x^* \\ v(x) &= P(x) + \max\{-s + v(0), 0\} \text{ for } x \geq x^*. \end{aligned}$$

and hence that x^* satisfies

$$\rho[P(x^*) + \max\{v(0) - s, 0\}] \leq -m(x^*) + P'(x^*)g(x^*) \qquad \text{with } = \text{ if } x^* > 0.$$

Q3 [5 points]. Assume that there is an $0 < x^* < \bar{x}$ such that if $x < x^*$the tree is not cut. Write an equation for $v(x)$ in that range, i.e: for $x < x^*$. Use the discrete time formulation.

A.

$$v(x) = -m(x)\Delta + \frac{1}{1+\Delta\rho} v(x + \Delta g(x))$$

Q4. [5 points] Assume that for $x > x^*$ the tree is cut. Write an equation for v in that range, i.e. for $x > x^*$. Use the discrete time formulation.

A. For $x \geq x^*$:

$$v(x) = P(x) + \max\left\{-s + \frac{1}{1+\Delta\rho} v(0), 0\right\}$$

Q5 [5 points]. Show that the equation derived for $x < x^*$ as $\Delta \to 0$ becomes

$$v(x)\rho = -m(x) + v'(x) g(x)$$

Hint: Assume that v is differentiable, use the "usual" trick to convert discrete time flow equations into differential equation. Below we will solve this o.d.e.

A. Write $v(x + \Delta g(x))$ as $v(x) + v'(x)\Delta g(x) + o(\Delta g(x))$. We have:

$$\begin{aligned}
v(x) &= -m(x)\Delta + \frac{1}{1+\Delta\rho} v(x + \Delta g(x)) \\
v(x) &= -m(x)\Delta + \frac{1}{1+\Delta\rho}\left[v(x) + v'(x)\Delta g(x) + o(\Delta g(x))\right] \\
v(x)(1+\Delta\rho) &= -m(x)\Delta(1+\Delta\rho) + v(x) + v'(x)\Delta g(x) + o(\Delta g(x)) \\
v(x)\rho &= -m(x)(1+\Delta\rho) + v'(x) g(x) + \frac{o(\Delta g(x))}{\Delta}
\end{aligned}$$

as $\Delta \to 0$, $o(\Delta g(x)) \to 0$ faster. Thus, as $\Delta \to 0$,

$$v(x)\rho = -m(x) + v'(x) g(x)$$

Q6 [5 points]. Write down the equation derived for $x > x^*$ as $\Delta \to 0$. Hint: Simply take the limit as $\Delta \to 0$.

A. For $x \geq x^*$

$$v(x) = P(x) + \max\{-s + v(0), 0\}$$

Q7 [5 points]. Consider the values of x such that $x < x^*$ and $x + \Delta g(x) \geq x^*$. Use the discrete time formulation. Write down an inequality that states that it is not optimal to cut the tree if $x_t = x$ and that it is optimal to do so at $x_{t+\Delta} = x + \Delta g(x)$. For the inequality compare the values of cutting the tree at t with the ones of cutting it at $t + \Delta$. The inequality should involve $m(x), \Delta,\ \rho,\ P(x + \Delta g(x)),\ \max\{v(0) - s, 0\}$, and $P(x)$.

A.

$$-m(x)\Delta + \frac{1}{1+\Delta\rho}\left[P(x+\Delta g(x)) + \max\left\{-s + \frac{1}{1+\Delta\rho}v(0), 0\right\}\right] \geq P(x) + \max\left\{-s + \frac{1}{1+\Delta\rho}v(0), 0\right\}$$

Q8 [5 points]. Write down the expression you just found as equality at $x = x^*$, i.e. assume that the agent is indifferent between cutting the tree today or tomorrow if $x = x^*$.

The equality involves the terms of $m(x^*), \Delta, \rho, P(x^* + \Delta g(x^*)), \max\{v(0) - s, 0\}$ and $P(x^*)$.

A.

$$-m(x^*)\Delta + \frac{1}{1+\Delta\rho}\left[P(x^* + \Delta g(x^*)) + \max\left\{\frac{1}{1+\Delta\rho}v(0) - s, 0\right\}\right] = P(x^*) + \max\left\{\frac{1}{1+\Delta\rho}v(0) - s, 0\right\}$$

Q9 [5 points]. Take the limit as $\Delta \to 0$ in the equality just derived for x^* and show that

$$\rho[P(x^*) + \max\{v(0) - s, 0\}] = -m(x^*) + P'(x^*)g(x^*)$$

A. use that $P(x^* + \Delta g(x^*)) = P(x^*) + P'(x^*)\Delta g(x^*) + o(\Delta g(x^*))$ so that

$$-m(x^*)\Delta + \frac{1}{1+\Delta\rho}\left[P(x^*) + P'(x^*)\Delta g(x^*) + o(\Delta g(x^*)) + \max\left\{\frac{1}{1+\Delta\rho}v(0) - s, 0\right\}\right] = P(x^*) + \max\left\{\frac{1}{1+}\right.$$

$$-m(x^*)\Delta(1+\Delta\rho) + \left[P(x^*) + P'(x^*)\Delta g(x^*) + o(\Delta g(x^*)) + \max\left\{\frac{1}{1+\Delta\rho}v(0) - s, 0\right\}\right] = (1+\Delta\rho)P(x^*) +$$

$$-m(x^*)\Delta(1+\Delta\rho) + P'(x^*)\Delta g(x^*) + o(\Delta g(x^*)) = \Delta\rho P(x^*) + \Delta\rho \max\left\{\frac{1}{1+\Delta\rho}v(0) - s, 0\right\}$$

$$-m(x^*)(1+\Delta\rho) + P'(x^*)g(x^*) + \frac{o(\Delta g(x^*))}{\Delta} = \rho\left[P(x^*) + \max\left\{\frac{1}{1+\Delta\rho}v(0) - s, 0\right\}\right]$$

and letting $\Delta \to 0$:

$$\rho[P(x^*) + \max\{v(0) - s, 0\}] = -m(x^*) + P'(x^*)g(x^*)$$

Q10 [10 points]. What is the interpretation of $\rho[P(x^*) + \max\{v(0) - s, 0\}]$? What is the interpretation of $-m(x^*) + P'(x^*)g(x^*)$? Why should these expressions be equal for the optimal x^*?

A. $[P(x) + \max\{v(0) - s, 0\}]$ is the payoff of cutting a tree. Hence the opportunity cost of postponing the decision for a period equals the interest earned on this amount, i.e.: $\rho[P(x) + \max\{v(0) - s, 0\}]$.

$P'(x)g(x)$ is the increase in the value of the tree per period, so $-m(x) + P'(x)g(x)$ is the net change in the value of the tree per period.

At the optimal value x^* the net gains from growth of the tree must equal the opportunity cost.

Q11 [5 points]. Suppose that $v(0) - s$ is so large that it is convenient to cut tree right away, so that $x^* = 0$. Write the corresponding condition for x^* (it should be a weak inequality version of the condition for $x^* > 0$).

Note: this configuration will not be part of the solution of the problem once we solve for x^* and $v(0)$. But at this juncture we are taking $v(0)$ as given, so it is a case we have to consider.

Answer:

$$\rho[P(x^*) + \max\{v(0) - s, 0\}] \geq P'(x^*)g(x^*) - m(x^*) \text{ if } x^* = 0$$

Define $\mu(x) \equiv \frac{m(x)}{P(x)+\max\{v(0)-s,0\}}$. $\mu(x)$ measures the flow cost of a tree of size x divided by the value of a lot where this tree is planted.

Q12 [15 points]. Show that given $m(0) = 0$, $m' > 0$, $m'' \geq 0$, $P \geq 0$, $P' > 0$, and $P'' \leq 0$ we have

$$\mu'(x) \geq 0 \text{ and } \frac{xP'(x)}{P(x) + \max\{v(0) - s, 0\}} \leq 1$$

Hint: using convexity of m and concavity of P we have:

$$m(0) \geq m(x) + m'(x)(0 - x) \quad \text{and } P(0) \leq P(x) + P'(x)(0 - x)$$

A.

$$\begin{aligned} & \frac{d}{dx}\frac{m(x)}{P(x) + \max\{v(0) - s, 0\}} \\ = \quad & \frac{m'[P + \max\{v(0) - s, 0\}] - P'm}{[P + \max\{v(0) - s, 0\}]^2} \\ = \quad & \frac{m}{x[P + \max\{v(0) - s, 0\}]}\left(x\frac{m'}{m} - x\frac{P'}{[P + \max\{v(0) - s, 0\}]}\right) \end{aligned}$$

$$m(0) \geq m(x) + m'(x)(0-x) \implies m'(x)x \geq m(x)$$
$$P(0) \leq P(x) + P'(x)(0-x) \implies P(0) + P'(x)x \leq P(x)$$

$$\frac{m'(x)x}{m(x)} \geq 1$$
$$\frac{P'(x)x}{[P(x) + \max\{v(0)-s,0\}]} \leq \frac{P'(x)x}{P(x)} \leq 1 - \frac{P(0)}{P(x)} \leq 1$$

so

$$\mu'(x) = \frac{d}{dx}\frac{m(x)}{P(x) + \max\{v(0)-s,0\}} \geq 0$$

Q13 [15 points]. Given the assumptions on m, P and g show that there exist a unique $x^* \geq 0$ that satisfies

$$\rho \geq \frac{P'(x^*)}{[P(x^*) + \max\{v(0)-s,0\}]}g(x^*) - \mu(x^*) \qquad \text{with } = \text{ if } x^* > 0.$$

A. Write this equation as (using the definition of $\mu(x)$):

$$[P(x^*) + \max\{v(0)-s,0\}]\rho = P'(x^*)g(x^*) - m(x^*)$$

The RHS is strictly decreasing. The LHS is strictly increasing, so there is at most one solution. For $x^* = 0$ we have

$$\lim_{x\to 0} P'(x)x > 0$$
$$\rho < \lim_{x\to 0}\left[\frac{P'(x)}{P(x) + \max\{v(0)-s,0\}}\right]g(x) - \frac{m(0)}{P(0) + \max\{v(0)-s,0\}}$$
$$\left[\frac{P'(x)}{P(x) + \max\{v(0)-s,0\}}\right]g(x) \leq \frac{P'(0)g(0)}{P(0) + \max\{v(0)-s,0\}}$$

For large x^* we have

$$\lim_{x^*\to\infty}\left[1 + \frac{\max\{v(0)-s,0\}}{P(x^*)}\right]\rho > \frac{P'(x^*)}{P(x^*)}\frac{g(x^*)}{x^*} > 0$$

That there is at least a solution it follows since

$$\lim_{x^*\to\infty} x^*\frac{P'(x^*)}{P(x^*)}\frac{g(x^*)}{x^*} - \mu(x^*) \leq \lim_{x^*\to\infty}\frac{g(x^*)}{x^*} = 0$$

That the solution is unique follows since $(P'/P)g$ is strictly decreasing (P is positive and convex, $P'/P > 0$ and strictly decreasing. Also $g > 0$ and decreasing)

Q14 [5 points]. Compute x^* for the following example:

$$P(x) = A\ x^{\varepsilon},\ m(x) = \max\{v(0) - s, 0\} = 0,\ g(x) = \bar{g}$$

A. $x^* = \varepsilon\ \frac{\bar{g}}{\rho}$

Q15 [5 points]. Compute x^* for the following example:

$$P(x) = A\ x,\ m(x) = \max\{v(0) - s, 0\} = 0,\ g(x) = \bar{g}\ x^{-\gamma}$$

A. $x^* =\ (\bar{g}/\rho)^{1/(1+\gamma)}$

Q16 [5 points]. Compute x^* for the following example:

$$P(x) = A\ x,\ m(x) = 0,\ v(0) - s = A > 0,\ g(x) = \rho/2$$

A. $x^* = 0$

$$\rho\,[Ax + A] \geq A\frac{\rho}{2} \text{ for all } x, \text{ so } x^* = 0.$$

Q17 [25 points]. What is the effect on x^* if: a) the function $g(\cdot)$ is replaced by other one that is higher everywhere? b) ρ is increased, c) the function $\mu(\cdot)$ is replaced by other one that is higher everywhere? d) the functions m and P, and the constant $(v(0) - s)$ are multiplied by a positive constant larger than one. e) The value of $\max\{v(0) - s, 0\}$ increases.

For these comparative statics we take $v(0)$ as a given number, i.e. not as a function of g, P, and ρ.

A. a) x^* increased, b) x^* decreased, c) x^* decreased, d) no change on x^*, e) x^* decreases.

Now we return to solve the ode for the value value of an uncut tree.

Q18 [20 points]. Show that the solution to

$$v'(z) = v(z)\,\frac{\rho}{g(z)} + \frac{m(z)}{g(z)}$$

for $z \in (x, x^*)$ is given by

$$v(x) = v(x^*)\,e^{-\int_x^{x^*} \frac{\rho}{g(z)}dz} - \int_x^{x^*} \left(e^{-\int_x^{z} \frac{\rho}{g(s)}ds}\right) \frac{m(z)}{g(z)}dz \tag{7}$$

Hint. To verify it, differentiate with respect to x both sides of the expression and replace into the ODE.

A.

$$v'(x) = \frac{\rho}{g(x)} v(x^*) e^{-\int_x^{x^*} \frac{\rho}{g(z)} dz} - \frac{\rho}{g(x)} \int_x^{x^*} \left(e^{-\int_x^z \frac{\rho}{g(s)} ds} \right) \frac{m(z)}{g(z)} dz + \frac{m(x)}{g(x)}$$

or

$$v'(x) = \frac{\rho}{g(x)} \left[v(x^*) e^{-\int_x^{x^*} \frac{\rho}{g(z)} dz} - \int_x^{x^*} \left(e^{-\int_x^z \frac{\rho}{g(s)} ds} \right) \frac{m(z)}{g(z)} dz \right] + \frac{m(x)}{g(x)}$$

and replacing back the expression for $v(x)$:

$$v'(x) = \frac{\rho}{g(x)} v(x) + \frac{m(x)}{g(x)}$$

Q19 [10 points]. To help understand (7), in particular to interpret the expression

$$e^{-\int_x^{x^*} \frac{\rho}{g(z)} dz}$$

what is the interpretation of

$$\tau(x, x^*) \equiv \int_x^{x^*} \frac{1}{g(z)} dz$$

In what units is τ measured?

Hint: notice that

$$\frac{1}{g(x)} dx = \frac{1}{dx/dt} dx = dt$$

A. $\tau(x, x^*)$ is the time that it takes the tree to growth from height x to x^*.

Q20 [15 points]. To help understand (7), in particular to interpret the expression

$$\int_x^{x^*} \left(e^{-\int_x^z \frac{\rho}{g(s)} ds} \right) \frac{m(z)}{g(z)} dz$$

which, once we use $\tau(\cdot)$ is

$$\int_x^{x^*} e^{-\rho\tau(x,z)} \frac{m(z)}{g(z)} dz \ ,$$

Give an interpretation to $m(x)/g(x)$. In what units is this measures? Hint, notice that letting $x(t)$ be the size of the tree as function of time:

$$\frac{m(x(t))}{g(x(t))} dx = \frac{m(x(t))}{dx/dt} dx = m(x(t))\, dt$$

A. $m(x)/g(x)$ are the maintainance cost that are incurred per unit of time.

Q21 [15 points]. Using the solution to the previous questions we can write (7) as

$$v(x^*)\, e^{-\rho\tau(x(t),x^*)} - \int_0^{\tau(x(t),x^*)} e^{-\rho\tau} m(x(t+s))\, ds = v(x(t))$$

where $x(r)$ solves $dx(\tau) = g(x(\tau))$ for $\tau \in (t, t+\tau(x,x^*))$ and where $x(t) = x$. What is the intepretation of this equation, i.e. what $v(x(t))$ equals? (Hint: describe $v(x(t))$ as present values)

A. $v(x)$ equals the present value of $v(x^*)$ minus the present value of the cost incurred until hitting x^*.

Based upon the previous questions, let $x(\tau)$ be the solution of

$$\frac{dx(t)}{dt} = g(x(t)) \text{ for } t \in (0,\tau)$$

with $x(0) = 0$. Consider the problem: $\max_{\tau\geq 0} F(\tau)$ where F is given by

$$F(\tau) \equiv [P(x(\tau)) + \max\{v(0) - s, 0\}]\, e^{-\rho\tau} - \int_0^{\tau} e^{-\rho t} m(x(t))\, dt$$

In this problem we are taking $v(0)$ as a parameter.

Q22.[10 points] Show that

$$v(0) = F(\tau(0, x^*))$$

Hint: use $x(\tau, 0) = x,\ \tau = \tau(0,x),\ t = \tau(0, x(t))$, and $m(x(t))\, dt = \frac{m(x(t))}{g(x(t))} dx(t)$.

A.

$$\begin{aligned}
& [P(x(\tau)) + \max\{v(0) - s, 0\}]\, e^{-\rho\tau} - \int_0^{\tau} e^{-\rho t} m(x(t))\, dt \\
= \;& [P(x) + \max\{v(0) - s, 0\}]\, e^{-\rho\tau(0,x)} - \int_0^{\tau(0,x)} e^{-\rho\tau t} m(x(t))\, \frac{dx/dt}{g(x(t))} dt \\
= \;& [P(x) + \max\{v(0) - s, 0\}]\, e^{-\rho\tau(0,x)} - \int_0^{\tau(0,x)} e^{-\rho\tau(0,z)} \frac{m(z)}{g(x)} dz
\end{aligned}$$

Q23 [10 points].Write the first order condition with respect to τ of the function $F(\tau)$.

A.

$$\begin{aligned} 0 &= F'(\tau^*) \\ &= -\rho\left[P(x(\tau^*)) + \max\{v(0)-s,0\}\right]e^{-\rho\tau^*} + P'(x(\tau^*))\frac{dx(\tau^*)}{d\tau}e^{-\rho\tau^*} - e^{-\rho\tau^*}m(x(\tau^*)) \end{aligned}$$

Q24 [10 points]. Show that $F'(\tau^*) = 0$ is equivalent to

$$\rho\left[P(x^*) + \max\{v(0)-s,0\}\right] = P'(x^*)\,g(x^*) - m(x^*)$$

when

$$\begin{aligned} x^* &= x(\tau^*) \\ \tau^* &= \tau(0, x^*) \end{aligned}$$

A.

$$\begin{aligned} \rho\left[P(x(\tau^*)) + \max\{v(0)-s,0\}\right]e^{-\rho\tau^*} &= P'(x(\tau^*))\frac{dx(\tau^*)}{d\tau}e^{-\rho\tau^*} - e^{-\rho\tau^*}m(x(\tau^*)) \\ \rho\left[P(x(\tau^*)) + \max\{v(0)-s,0\}\right] &= P'(x(\tau^*))\frac{dx(\tau^*)}{d\tau} - m(x(\tau^*)) \\ \rho\left[P(x^*) + \max\{v(0)-s,0\}\right] &= P'(x^*)\,g(x^*) - m(x^*) \end{aligned}$$

Now we can state the problem and solve it. We can think of characterizing the value function by solving a system of 3 equations in 3 unknowns. The unknowns are the values of $(x^*, v(0), v(x^*))$, the optimal cut-off point, the value of a newly planted tree, and the value of a tree just before cutting it. The equations are the following:

1) The optimality of the planting decision:

$$v(x^*) = P(x^*) + \max\{v(0)-s,0\},$$

2) The value of a newly planted tree:

$$v(0) = v(x^*)\,e^{-\int_0^{x^*}\frac{\rho}{g(z)}dz} - \int_0^{x^*}\left(e^{-\int_0^{z}\frac{\rho}{g(s)}ds}\right)\frac{m(z)}{g(z)}dz,$$

3) The optimality of the threshold x^* :

$$\rho\left[P(x^*) + \max\{v(0)-s,0\}\right] \geq P'(x^*)\,g(x^*) - m(x^*) \quad \text{with} \ = \ \text{if } x^* > 0$$

We can analyze the problem, by considering two cases.

Q 25 [15 points]. First, let's consider the case where it is not optimal to cut replant a tree, so that $v(0) \leq s$. Specialize 3) to this case, and taking the resulting x^* use 1) and 2) to write a lower bound for s for which is optimal not to replant a tree.

A. 3) can be use to solve for x^* :

$$\rho P(x^*) = P'(x^*) g(x^*) - m(x^*)$$

Now we can verify if $v(0) - s \leq 0$ by checking if

$$v(0) = P(x^*) e^{-\int_0^{x^*} \frac{\rho}{g(z)} dz} - \int_0^{x^*} \left(e^{-\int_0^z \frac{\rho}{g(s)} ds} \right) \frac{m(z)}{g(z)} dz \leq s$$

so that $v(0)$ is the lower bound for s.

Q 26 [20 points]. Now consider the case where its optimal to replant a tree, so that $v(0) \geq s$. Substitute $v(x^*)$ into 2) and write an equation for $v(0)$ as a function of x^*. Call the resulting function $v(0; x^*, s)$.

A.

$$v(0; x^*, s) = \frac{[P(x^*) - s] e^{-\int_0^{x^*} \frac{\rho}{g(z)} dz} - \int_0^{x^*} \left(e^{-\int_0^z \frac{\rho}{g(s)} ds} \right) \frac{m(z)}{g(z)} dz}{1 - e^{-\int_0^{x^*} \frac{\rho}{g(z)} dz}}$$

Q27 [20 points]. Using the definition of $\tau(x^*, 0)$ given above and the expression it is immediate to obtain that

$$v(0; x^*, s) = \frac{[P(x^*) - s] e^{-\rho\tau(0,x^*)} - \int_0^{\tau(0,x^*)} e^{-\rho\tau(0,x(t))} m(x(t)) dt}{1 - e^{-\rho\tau(0,x^*)}}$$

where $x(t)$ is the solution of $dx(t)/dt = g(x(t))$ and $x(0) = 0$. What is the interpretation of this? (Give an interpretation to $[P(x^*) - s] e^{-\rho\tau(0,x^*)} - \int_0^{\tau(0,x^*)} e^{-\rho\tau(0,x(t))} m(x(t)) dt$ and to $1/\left(1 - e^{-\rho\tau(0,x^*)}\right)$ separately). Is the RHS increasing in x^*?

A. The expression $[P(x^*) - s] e^{-\rho\tau(0,x^*)} - \int_0^{\tau(0,x^*)} e^{-\rho\tau(0,x(t))} m(x(t)) dt$ is the present value of cutting a tree in τ^* periods net watering cost and the cost of planting a new one. The expression

$$\frac{1}{1 - e^{-\rho\tau(0,x^*)}}$$

is the present value of receiving 1 every $\tau(0, x^*)$ periods.

Q 28 [25 points]. Using the expression for $v(0; x^*, s)$ in terms of x^* derive the condition for

$$\frac{d}{dx^*} v(0; x^*, s) = 0$$

and show that it is equivalent to 3). Hint (you may have to substitute back $v(0;x^*,s)$.

A. $\frac{d}{dx^*}v(0;x^*,s)=0$ gives

$$\begin{aligned}&\left(e^{-\int_0^{x^*}\frac{\rho}{g(z)}dz}\right)\left(P'(x^*)-\frac{\rho}{g(x^*)}[P(x^*)-s]-\frac{m(x^*)}{g(x^*)}\right)\left(1-e^{-\int_0^{x^*}\frac{\rho}{g(z)}dz}\right)\\ =\;&\frac{\rho}{g(x^*)}e^{-\int_0^{x^*}\frac{\rho}{g(z)}dz}\left([P(x^*)-s]\,e^{-\int_0^{x^*}\frac{\rho}{g(z)}dz}-\int_0^{x^*}\left(e^{-\int_0^{z}\frac{\rho}{g(s)}ds}\right)\frac{m(z)}{g(z)}dz\right)\end{aligned}$$

or

$$\begin{aligned}&\left(e^{-\int_0^{x^*}\frac{\rho}{g(z)}dz}\right)\left(P'(x^*)-\frac{\rho}{g(x^*)}[P(x^*)-s]-\frac{m(x^*)}{g(x^*)}\right)\left(1-e^{-\int_0^{x^*}\frac{\rho}{g(z)}dz}\right)\\ =\;&\frac{\rho}{g(x^*)}e^{-\int_0^{x^*}\frac{\rho}{g(z)}dz}\left(v(0;x^*,s)\left(1-e^{-\int_0^{x^*}\frac{\rho}{g(z)}dz}\right)\right)\end{aligned}$$

or multiplying by $g(x^*)/\left[\left(e^{-\int_0^{x^*}\frac{\rho}{g(z)}dz}\right)\left(1-e^{-\int_0^{x^*}\frac{\rho}{g(z)}dz}\right)\right]$ and hence:

$$g(x^*)\,P'(x^*)+\;m(x^*)=\rho\,[P(x^*)+v(0;x^*,s)-s]$$

6 Computation of the Neoclassical Growth Model: Brute Force

Preferences are

$$\sum_{t=0}^{\infty}\beta^t\frac{c_t^{1-\sigma}}{1-\sigma};\quad \sigma>0. \tag{8}$$

The stock of capital evolves as

$$K_{t+1}=(1-\delta)\,K_t+I_t, \tag{9}$$

where I_t is gross investment. To produce one unit of capital at the beginning of period $t+1$, p_k units of consumption have to be invested in period t. Thus aggregate feasibility is given by

$$c_t+p_kI_t=AK_t^{\alpha}. \tag{10}$$

Introducing (9) into (10) we obtain

$$c_t+p_kK_{t+1}=AK_t^{\alpha}+p_k\,(1-\delta)\,K_t. \tag{11}$$

The Bellman equation of the planner's problem is

$$V(K) = \max_{K' \in \Gamma(K)} \left\{ \frac{1}{1-\sigma} \left[AK^{\alpha} + p_k (1-\delta) K - p_k K' \right]^{1-\sigma} + \beta V(K') \right\}, \tag{12}$$

where the constraint set is $\Gamma(K) = \left[0; \frac{AK^{\alpha}}{p_k} + (1-\delta) K \right]$.

Algorithm:

There are several numerical procedures available to solve this model. In this note we discretize the state space. That is, even though the state variable (i.e., the stock of capital) is allowed to take any positive real number, we will constraint it to belong to a finite grid of points. The algorithm works as follows:

i) *Find the steady state level of capital*: The first order condition of (12) is

$$p_k \left[AK^{\alpha} + p_k (1-\delta) K - p_k K' \right]^{-\sigma} = \beta V' (K').$$

The envelope condition is

$$V'(K) = \left[AK^{\alpha} + p_k (1-\delta) K - p_k K' \right]^{-\sigma} \left[\alpha A K^{\alpha-1} + p_k (1-\delta) \right].$$

Replacing the envelope condition into the FOC and evaluating it at the steady state (i.e. $K = K' = K^*$) we obtain

$$p_k = \beta \left[A\alpha (K^*)^{\alpha-1} + p_k (1-\delta) \right].$$

Solving for K^*:

$$K^* = \left[\frac{A\alpha}{p_k (\rho + \delta)} \right]^{1/(1-\alpha)},$$

where ρ is defined by $\beta \equiv \frac{1}{1+\rho}$.

ii) *Construct the grid for capital:* Let N be the number of grid points and let $K_{\min}$ and $K_{\max}$ be the lower and upper bounds of the grid respectively. Construct a vector *kgrid* as follows: for $i = 1, 2, ..., N$ let $kgrid(i) = K_{\min} + \frac{(i-1)}{N-1} K_{\max}$. The Matlab command $linspace(K_{\min}, K_{\max}, N)$ does the trick. One possibility is $N = 2000$, $K_{\min} = 0.99K^*$ and $K_{\max} = 1.01K^*$. That is, we are computing the approximate solution for a stock of capital between 99% and 101% of the steady state level of capital.

iii) *Initial Guess:* We will approximate the value function with a vector of size N. Start with an initial guess $V^0 \in R^N$, for example a vector of zeros. There are better initial guesses: one possibility is to iterate on an arbitrary (but reasonable) policy function, for example $K'(K) = K$. This procedure will produce an initial guess with approximately the same shape and level of the true value function.

iv) <u>*Main loop*</u>: Here we construct an operator that takes a vector $V \in R^N$ and returns another vector $\tilde{V} \in R^N$. An approximate solution is a fixed point of that operator.

a) Pick a convergence criterion: a positive but small number ε (e.g. $\varepsilon = 10^{-6}$).

b) Start the loop with the initial guess V^0. To compute the approximate value function, denote by V_j^s the j^{th} element of the vector V^s, $s = 0, 1, ...,$ where $V_j^s = V^s(K_j)$ and K_j is the j^{th} element of the capital grid. Let Θ be an $N \times N$ matrix (independent of s) defined by

$$\Theta(i, j) = \frac{1}{1-\sigma}\left[AK_i^{\alpha} + p_k(1-\delta)K_i - p_k K_j\right]^{1-\sigma},$$

and let the matrix $\Omega^s(i, j)$ be

$$\Omega^s(i, j) = \Theta(i, j) + \beta V_j^s.$$

$\Omega^s(i, j)$ is the present value of utility if the value function is V^s, the current stock of capital today is $K_i \in kgrid$ and the agents chooses $K_j \in kgrid$ for next period stock of capital (the matrix Ω^s changes from iteration to iteration). Given the matrix Ω^s and the vector V^s, update the value function as

$$V_i^{s+1} = \max_{j \in [1,2,...,N]} \Omega^s(i, j) \text{ for } i = 1, 2, ..., N.$$

In Matlab we can perform this maximization for all i as follows: use the following command

$$\left[V^{s+1}, J\right] = \max\left(\Omega^s, [], 2\right).$$

V^{s+1} is the vector with the maximized value function and J is a vector with the indexes where the maximum is attained. Thus the policy function is $K'(K_i) = kgrid(J(i))$. The "2" in the max command is needed because without it Matlab maximizes over columns and the way we constructed the matrix Ω^s requires to maximize over rows.

c) Compute the distance between the value functions as

$$dist = \max_{j \in [1,2,...,N]} \left[\left|(V_{s+1} - V_s)\right|\right],$$

and check if convergence is met (i.e. if $dist \leq \varepsilon$). If not, continue iterating, if yes, exit the loop.

d) Construct the policy functions: Following the last step, the (capital) policy function is

$K'(K_i) = kgrid(J(i))$. Once we know this, we can construct the investment and consumption policy function:

$$\begin{aligned} I(K_i) &= K'(K_i) - (1-\delta) K_i, \\ &= kgrid(J(i)) - (1-\delta) kgrid(i), \end{aligned}$$

and

$$\begin{aligned} C(K_i) &= AK_i^{\alpha} + p_k (1-\delta) K_i - p_k K'(K_i), \\ &= AK_i^{\alpha} - p_k [kgrid(J(i)) - (1-\delta) kgrid(i)]. \end{aligned}$$

v) *Interpolate the policy functions:* Once the loop is finished, we will have the policy functions for capital, consumption and investment only at the grid points. To create time series we will need to interpolate the policy functions. To see why, if K_t does not belong to the grid of capital then we can't compute K_{t+1}. To interpolate the policy functions use the Matlab command interp1. For example, suppose that the policy function for capital is stored in the vector $kpol$. The component $kpol(i)$ $i = 1, 2, ..., N$ gives the stock of capital next period if today's level of capital is $kgrid(i)$. To interpolate the policy function to an arbitrary level of capital k_t, we use the command $interp1(kgrid, kpol, k_t)$. Furthermore, you can specify the interpolation method (linear, cubic spline, etc.). Type help interp1 in Matlab for more information.

vi) *Construct time series:* Create the time series for the stock of capital, consumption, investment and GDP using the interpolated policy functions. For example, if $\tilde{K}(k)$ and $\tilde{C}(k)$ denote the interpolated capital and consumption policy functions respectively, then we create the times series as follows: first, given k_0 and a fixed horizon $T > 1$ construct the series for capital:

$$k_{t+1} = \tilde{K}(k_t); \quad t = 0, 1, ..., T.$$

Then construct the consumption time series:

$$c_t = \tilde{C}(k_t); \quad t = 0, 1, ..., T.$$

In your program use the following names:

σ (risk aversion):`sigma`

β (discount):`beta`

ρ (discount rate defined from $\beta = 1/(1+\rho)$) :`rho`

A (technology parameter) :`A`

α (share of capital) `:alpha`

δ (depreciation rate) `:delta`

p_k (price of investment) `:pk`

K^* (s.s. capital) `:kstar`

N (size of the capital grid) `:N`

T (number of periods to run the simulation) `:T`

K_0 (initial level of capital) `:k0`

$K_{\min}$ (lower bound on grid for capital) `:kmin`

$K_{\max}$ (upper bound on grid for capital) `:kmax`

V_0 (initial guess of the value function) `:V0`

V^s (s^{th} iterate on the value function) `:V` (this changes through the iterations)

V^{s+1} (update of the value function) `:Vp` (this changes through the iterations)

ε (convergence criterion) `:conv=10^-6`

$dist$ (distance between V^s and V^{s+1})`:dist=max(abs(Vp-V));`

Θ (N by N) (where $\Theta(i,j) = \frac{1}{1-\sigma}\left[AK_i^\alpha + p_k(1-\delta)K_i - p_kK_j\right]^{1-\sigma}$) `:theta` (same through iterations)

Ω^s (N by N) (where $\Omega^s(i,j) = \Theta(i,j) + \beta V_i^s$)`:omega` (this changes through iterations)

Policy functions (vectors of size N):

$\tilde{K}(k)$ (next period's capital policy function) `:kpol`

$\tilde{C}(k)$ (consumption policy function) `:cpol`

$\tilde{I}(k)$ (gross investment policy function, i.e. $\tilde{I}(k) = \tilde{K}(k) - (1-\delta)k$) `:ipol`

Now that you have your numerical version of the neoclassical growth model, let us revisit the question using analytical approximations.

Use the following (monthly) parameters values:

σ (risk aversion): 2

ρ (discount rate defined from $\beta = 1/(1+\rho)$): 0.075/12 (yearly = 0.075)

A (technology parameter): 1/12 (yearly = 1)

α (share of capital): 0.3

δ (depreciation rate): 0.075/12 (yearly = 0.075)

p_k (price of investment): 2

$K_{\min} = 0.99K^*$

$K_{\max} = 1.01K^*$

i) *Steady states I:* Compute the steady states values for consumption, value of investment, output, and capital: C^*, p_kI^*, Y^*, p_kK^*.

ii) *Steady states II:* Show the ratios $p_k K^*/Y^*$, $p_k I^*/Y^*$, C^*/Y^* for annual frequency. What would be the corresponding (average) values for the US economy in the post WWII period?

iii) *Slope of the saddle path or optimal consumption function*: Compute numerically the slope of the optimal decision rule for consumption $\tilde{C}(k)$ around the steady state K^*. Calculate the following ratio

$$c'_{num}(K^*) = \frac{\tilde{C}(K_{\max}) - \tilde{C}(K_{\min})}{K_{\max} - K_{\min}},$$

which will be approximately the slope of the consumption policy function at the steady state.

iii. a) Consider the following (monthly) numerical values: $\alpha = 0.3$, $\delta = \rho = 0.075/12$, $p_k = 2$, $A = 1/12$ and $\sigma = 2$. Let $K_{\max} = 1.01K^*$, $K_{\min} = 0.99K^*$ and $N = 2000$. Compute the slope of the decision rule at the steady state in the continuous time version. Denote that elasticity as c', which solves the quadratic equation:

$$c'(K^*)\left[\rho - \frac{c'(K^*)}{p_k}\right] = -\frac{\varepsilon}{\sigma}\left(\frac{C^*}{p_k K^*}\right) p_k [\rho + \delta],$$

that is

$$c'(K^*) = \frac{\rho p_k}{2}\left[1 + \sqrt{1 + 4\frac{\varepsilon}{\sigma}\left(\frac{C^*}{p_k K^*}\right)\frac{[\rho + \delta]}{\rho^2}}\right],$$

where

$$\begin{aligned} \varepsilon &= 1 - \alpha, \\ \left(\frac{C^*}{p_k K^*}\right) &= \frac{\rho + \delta(1-\alpha)}{\alpha}. \end{aligned}$$

Compare the two numbers ($c'_{num}(K^*)$ versus $c'(K^*)$).

iii. b) Repeat a) for the same parameters except for $p_k = 1$. Report $c'_{num}(K^*)$ and $c'(K^*)$.

iv) *Investment:* We showed elsewhere that if

$$\delta = \left(\frac{C^*}{p_k K^*}\right)\frac{\varepsilon}{\sigma},$$

where

$$\begin{aligned} \varepsilon &= 1 - \alpha, \\ \left(\frac{C^*}{p_k K^*}\right) &= \frac{\rho + \delta(1-\alpha)}{\alpha}, \end{aligned}$$

then the continuous time version of the neoclassical growth model has, in a neighborhood of the steady state, constant gross investment (i.e. $I(k)$ is constant).

Compute the solution of the model. Use a grid of 2000 points ($N = 2000$). In particular, plot investment $I(k)$ for each of the cases below. Use values of capital in the interval:

$$\begin{aligned} K_{\min} &= 0.99\ K^*, \\ K_{\max} &= 1.01\ K^*, \end{aligned}$$

and for $\alpha = 0.3,\quad \sigma = 3.966$ and $p_k = 2$. For the two cases described below you should compute the slope and elasticity of the investment policy function, that is

$$\tilde{I}'(K^*) \equiv \frac{\tilde{I}(K_{\max}) - \tilde{I}(K_{\min})}{K_{\max} - K_{\min}},$$

$$\zeta(K^*) = \tilde{I}'(K^*)\frac{K^*}{I^*} = \tilde{I}'(K^*)/\delta,$$

where K^* and I^* are the steady state values for capital and investment.

iv. a) $\rho = \delta = 0.075,\ A = 1$ (yearly). Report $\tilde{I}'(K^*)$ and $\zeta(K^*)$, and plot $\tilde{I}(K)$ against K for $K \in [K_{\min}, K_{\max}]$.

iv. b) $\rho = \delta = 0.075/12,\ A = 1/12$ (monthly). Report $\tilde{I}'(K^*)$ and $\zeta(K^*)$, and plot $\tilde{I}(K)$ against K for $K \in [K_{\min}, K_{\max}]$.

For both a) and b) we have that

$$\delta = \left(\frac{C^*}{p_k K^*}\right)\frac{\varepsilon}{\sigma},$$

but b) is closer to the continuous time in the sense that the time period is shorter (monthly vs. yearly).

iv.c) Compute $\tilde{I}'(K^*)$ and $\zeta(K^*)$, and plot $\tilde{I}(K)$ against K for $K \in [K_{\min}, K_{\max}]$ for $\rho = \delta = 0.075/12,\ A = 1/12$ (monthly) $\alpha = 0.3,\quad \sigma = 2$ and $p_k = 2$. Compare this case with iii.b) in terms of the absolute magnitudes and signs. Explain the difference.

v) *Speed of Convergence*: Consider the following (monthly) numerical values: $\alpha = 0.3$, $\delta = \rho = 0.075$, $p_k = 2$, $A = 1$ and $\sigma = 2$. Let $K_{\max} = 1.01K^*$, $K_{\min} = 0.99K^*$ and $N = 500$. Elsewhere we showed that the continuous time version of the neoclassical growth model linearized around the steady state generates the following equilibrium evolution for capital:

$$K(t) - K^* = (K(0) - K^*)\exp(\lambda t), \tag{13}$$

where $\tilde{\lambda}$ is the negative root of

$$Q(\lambda) = \lambda^2 - \lambda\rho - \left(\frac{C^*}{p_k K^*}\right)\frac{\varepsilon}{\sigma}(\rho + \delta) = 0,$$

or

$$\tilde{\lambda} = \frac{\rho}{2}\left[1 - \sqrt{1 + 4\left(\frac{C^*}{p_k K^*}\right)\frac{\varepsilon}{\sigma}\frac{(\rho+\delta)}{\rho^2}}\right].$$

We define the speed of convergence as how long it takes to close half of the gap between a given initial stock of capital $K(0)$ and K^*. That is, we look for the $\bar{t}$ that makes $K(t) - K^* = (K(0) - K^*)/2$ in (13), which is given by

$$\bar{t} = \frac{\log(1/2)}{\tilde{\lambda}}. \tag{14}$$

In this question we will compute the speed of convergence in our previous example:

v. a) Consider the following monthly parameters: $\alpha = 0.3$, $\sigma = 2$, $\rho = \delta = 0.075/12$ and $A = 1/12$. Start with $K_0 = K_{\min} = 0.99K^*$ and compute the first period $\hat{t}$ such that K_t satisfies

$$(K_{\hat{t}} - K^*) = \frac{1}{2}(K_0 - K^*).$$

Compare the $\bar{t}$ obtained from (14) with the $\hat{t}$ obtained with the simulated model. Report $\bar{t}$ and $\hat{t}$ in years, that is, report $\bar{t}/12$ and $\hat{t}/12$.

v. b) Repeat a) for the same parameters except for $p_k = 1$. Report $\bar{t}$ and $\hat{t}$ in years.

v. c) Repeat a) for $\sigma = 4$. Report $\bar{t}$ and $\hat{t}$ in years.

v. d) How do you compare the length of these half-lives relative to booms and recession in the US for the post WWII period?

Ans: The detailed answers of this question are left as an exercise. Instead, the figures below plot several functions of interest, along with the time path of the endogenous variables. The plots are constructed with the standard annual values for the parameters.

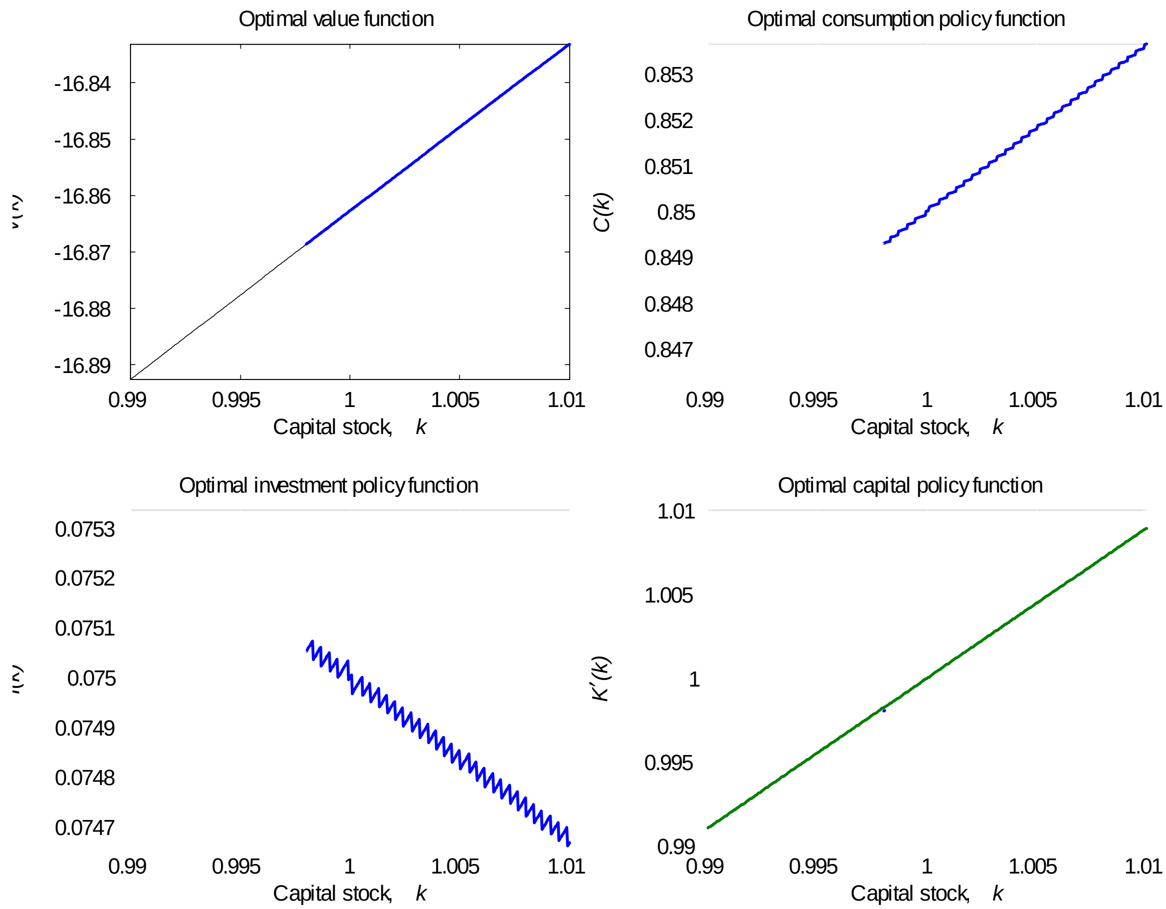

Figure 4: Optimal value and policy functions.

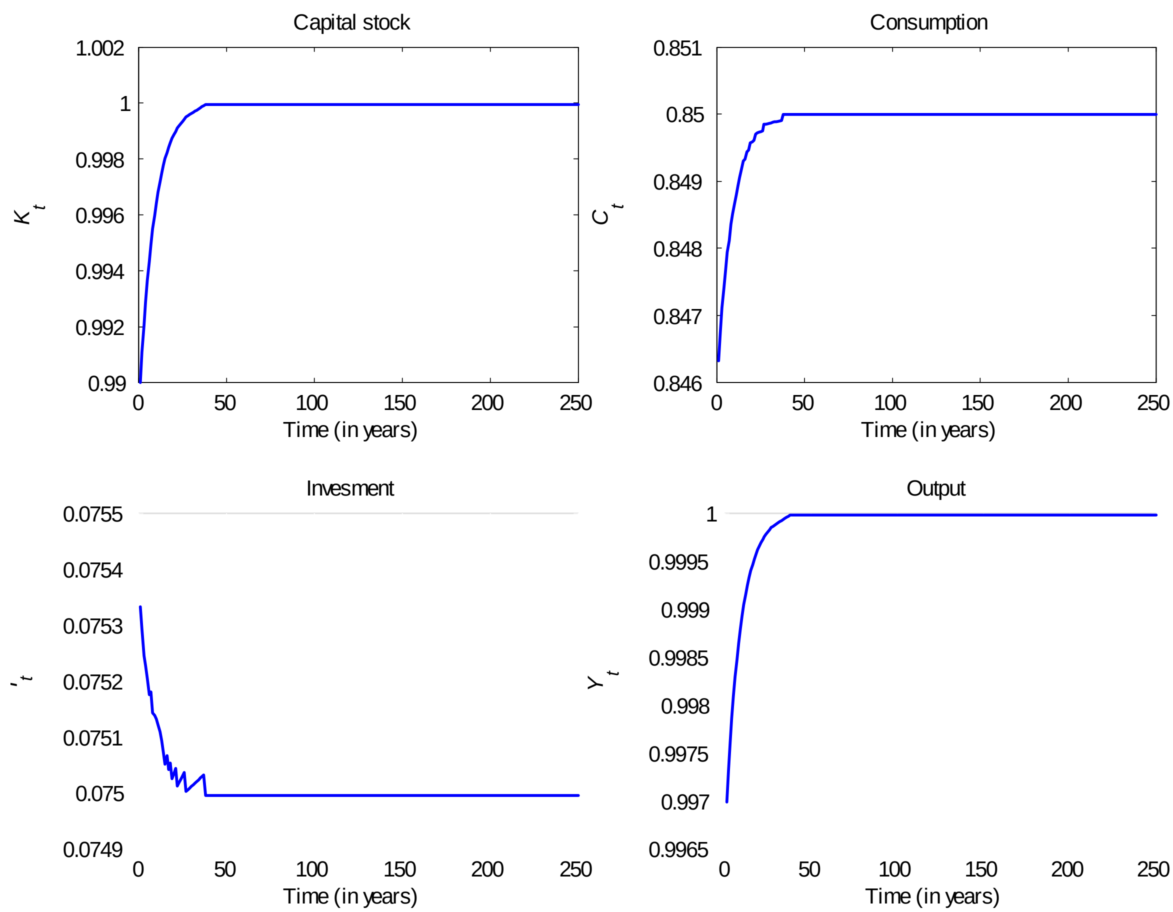

Figure 5: Time path of the endogenous variables for $k_0 = 0.99$.