Theory of Income, Fall 2008
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Problem Set 5

1 Convergence after a permanent productivity shock in the Neoclassical Growth Model

Let $f(k) = \varepsilon\ k^{\alpha}$ be a neoclassical production function with productivity parameter $\varepsilon > 0$ and capital share parameter $\alpha \in (0,1)$. The law of motion of capital is given by $dk/dt = f(k) - c$ (we set depreciation rate to zero). Preferences are $\int_0^{\infty} e^{-\rho t} U(c(t))\, dt$ for $U(c) = c^{1-\gamma}/(1-\gamma)$, with $\gamma > 0$.

Exercise 1. Continuous time Euler Equation . Write down the differential equation that c must satisfy in an optimal path. Your solution should contain, $\rho,\ f'(k),\ dc/dt,\ c$ and γ.

Ans: The current-value Hamiltonian is

$$\mathcal{H} = \frac{c^{1-\gamma}}{1-\gamma} + \lambda\left[f(k) - c\right],$$

where λ is the co-state variable. The optimality conditions are, as usual, $\mathcal{H}_c = 0$, and $\dot{\lambda} = \rho\lambda - \mathcal{H}_k$. That is,

$$\begin{aligned} c &: \ c(t)^{-\gamma} = \lambda(t) \\ k &: \ \frac{\dot{\lambda}}{\lambda} = \left(\rho - f'(k)\right). \end{aligned}$$

Taking logs of the first equation and differentiating w.r.t. time we obtain

$$\frac{\dot{c}}{c} = -\frac{1}{\gamma}\frac{\dot{\lambda}}{\lambda} = \frac{1}{\gamma}\left[f'(k) - \rho\right],$$

or, using the definition of f,

$$\frac{\dot{c}}{c} = \frac{1}{\gamma}\left[\alpha\varepsilon k^{\alpha-1} - \rho\right]. \tag{1}$$

Exercise 2. Steady states . Write down two equations in two unknowns (c^* and k^*) that determine the steady state values of consumption and capital.

Ans: The dynamic system that characterizes the solution of the problem is given by (1) and the capital accumulation equation

$$\dot{k} = \varepsilon k^{\alpha} - c. \tag{2}$$

In steady state, $\dot{c} = \dot{k} = 0$. The first equation becomes

$$\alpha\varepsilon k^{*(\alpha-1)} = \rho, \text{ (i.e. } f'(k^*) = \rho),$$

and the second

$$\varepsilon k^{*\alpha} = c^*, \text{ (i.e. } f(k^*) = c^*),$$

where (c^*, k^*) denotes steady state values. Solving for k^* and c^* we get

$$k^*(\varepsilon) = \left(\frac{\alpha\varepsilon}{\rho}\right)^{\frac{1}{1-\alpha}}, \tag{3}$$

and

$$c^*(\varepsilon) = \varepsilon^{\frac{1}{1-\alpha}} \left(\frac{\alpha}{\rho}\right)^{\frac{\alpha}{1-\alpha}}, \tag{4}$$

(making the dependence on ε explicit).

Exercise 3. Steady state consumption-capital ratio . Write an expression for the steady state value of consumption over capital, i.e. c^*/k^* as a function of ρ and α.

Ans: Dividing (4) by (3) we find the steady state consumption to capital ratio:

$$\frac{c^*}{k^*} = \frac{\rho}{\alpha}. \tag{5}$$

Note that this ratio is independent of the technology level ε.

Exercise 4. Slope of the saddle path or optimal consumption function . Let $c(k)$ be the value of consumption that belong to the saddle path for a given capital k. In this exercise you need to find an expression for the slope of the saddle path evaluated at (c^*, k^*), i.e. you need to find $c'(k^*)$. Notice that $c(k)$ is also the optimal decision rule for consumption for a given k.

4.1) Derive the following quadratic equation:

$$c'(k^*)\left[\rho - c'(k^*)\right] = -\frac{\rho^2}{\gamma}\frac{1-\alpha}{\alpha},$$

whose positive solution is $c'(k^*)$.

4.2) Give an intuitive explanation of why $c'(k^*)$ is decreasing in γ. Your explanation should contain, *at most,* three lines.

[Hint for 4.1: To obtain the quadratic equation, evaluate

$$\frac{dc(k^*)}{dk} = \lim_{k \to k^*} \frac{dc/dt}{dk/dt},$$

where dc/dt is the expression obtained in 1) and dk/dt is the law of motion of capital, and where c is written as a function of k, i.e. $c = c(k)$ in both expressions. You need to use L'Hôpital's rule to find an expression for the right hand side, since at the steady state $dc/dt = dk/dt = 0$. To use L'Hôpital, differentiate the denominator and the numerator with respect to k, and evaluate them at the steady state value k^*].

Ans: Note that

$$c'(k^*) \equiv \frac{dc(k^*)}{dk} = \lim_{k\to k^*} \frac{\dot{c}(t)}{\dot{k}(t)} = \lim_{k\to k^*} \frac{\frac{1}{\gamma}(f'(k(t)) - \rho)\,c(k(t))}{f(k(t)) - c(k(t))} = \frac{0}{0}.$$

Thus, using L'Hôpital's rule we obtain

$$\begin{aligned}
\lim_{k\to k^*} \frac{\frac{1}{\gamma}(f'(k(t)) - \rho)\,c(k(t))}{f(k(t)) - c(k(t))} &= \lim_{k\to k^*} \frac{\frac{1}{\gamma} f''(k(t))\,c(k(t)) + \frac{1}{\gamma}(f'(k(t)) - \rho)\,c'(k(t))}{f'(k(t)) - c'(k(t))} \\
&= \frac{\frac{1}{\gamma} f''(k^*)\,c(k^*) + \frac{1}{\gamma}\left[f'(k^*) - \rho\right] c'(k^*)}{f'(k^*) - c'(k^*)}.
\end{aligned}$$

Given that $f'(k^*) = \rho$, from the above expression we can write

$$\begin{aligned}
c'(k^*)\left[\rho - c'(k^*)\right] &= \frac{1}{\gamma} f''(k^*)\,c(k^*) \\
&= \frac{1}{\gamma}\left[\alpha(\alpha-1)\varepsilon\ (k^*)^{\alpha-2}\right]\varepsilon(k^*)^{\alpha} \\
&= \frac{1}{\gamma}\alpha(\alpha-1)\left[\varepsilon(k^*)^{\alpha-1}\right]^2 \\
&= \frac{1}{\gamma}\alpha(\alpha-1)\left[\varepsilon\left(\frac{\rho}{\alpha\varepsilon}\right)\right]^2 \\
c'(k^*)\left[\rho - c'(k^*)\right] &= -\frac{\rho^2}{\gamma}\frac{1-\alpha}{\alpha},
\end{aligned}$$

as desired. Note that $c'(k^*)$ is independent of the technology level ε.

The quadratic equation $x[\rho - x] = -\frac{\rho^2}{\gamma}\frac{(1-\alpha)}{\alpha}$ has solutions

$$x = \frac{\rho}{2}\left[1 \pm \sqrt{1 + \frac{4}{\gamma}\left(\frac{1-\alpha}{\alpha}\right)}\right].$$

Since we also know that $c'(k^*) > 0$, and $\sqrt{1 + \frac{4}{\gamma}\left(\frac{1-\alpha}{\alpha}\right)} > 1$, then the solution is

$$c'(k^*) = \frac{\rho}{2}\left[1 + \sqrt{1 + \frac{4}{\gamma}\left(\frac{1-\alpha}{\alpha}\right)}\right].$$

Note that

$$\begin{aligned}\left\{\left[\rho - c'\left(k^*\right)\right] - c'\left(k^*\right)\right\} dc'\left(k^*\right) &= \frac{\rho^2}{\gamma^2}\frac{1-\alpha}{\alpha}d\gamma \\ \frac{dc'\left(k^*\right)}{d\gamma} &= \frac{\rho^2}{\gamma^2}\frac{1-\alpha}{\alpha}\frac{1}{\rho - 2c'\left(k^*\right)}.\end{aligned}$$

Thus,

$$\operatorname{sgn}\frac{dc'\left(k^*\right)}{d\gamma} = \operatorname{sgn}\left[\rho - 2c'\left(k^*\right)\right] = \operatorname{sgn}\left[\rho - 2\frac{\rho}{\alpha}\right] = \operatorname{sgn}\left[\alpha - 2\right] < 0,$$

which implies that the slope of the saddle path is decreasing in γ (which is the inverse of the elasticity of substitution); that is, the lower is γ the higher is the speed of convergence towards the steady. The intuition for this results is as follows: the lower is γ, the higher is the elasticity of substitution, that is, the more willing people are to accept low consumption early on in exchange for higher consumption in the future. Thus, as γ falls, capital accumulates more rapidly and the economy converges quicker to the steady state.

Exercise 5. Impact effect on consumption of a permanent change in productivity . Suppose that at time $t = 0$ the economy is in a steady state corresponding to productivity ε. Suppose that at $t = 0$ we learn that productivity will immediately and permanently increase by a very small amount from ε to $\varepsilon' \left(> \varepsilon\right)$. We let $c^*\left(\cdot\right)$ and $k^*\left(\cdot\right)$ be the steady state consumption and capital levels and $c\left(k, \cdot\right)$ be the optimal consumption decision rules -as a function of capital- that corresponding to each productivity level ε and ε'.

It turns out that consumption at time $t = 0$ may increase, decrease or stay the same value relative to the old steady state, i.e. that $c\left(k^*\left(\varepsilon\right), \varepsilon'\right) \geq (\leq)\ c^*\left(\varepsilon\right) = c\left(k^*\left(\varepsilon\right), \varepsilon\right).$

5.1. Draw a phase diagram, labelling the saddle paths and steady states for both values of productivity (ε and ε') that is consistent with $c\left(k^*\left(\varepsilon\right), \varepsilon'\right) > c^*\left(\varepsilon\right) = c\left(k^*\left(\varepsilon\right), \varepsilon\right).$ Indicate $c\left(k^*\left(\varepsilon\right), \varepsilon'\right)$ in your diagram. Make sure that your phase diagram respects the qualitative properties shown in 3) and 4) for the consumption-capital ratios and the slopes of the saddle paths. Draw the saddle path as if it has constant slope.

5.2. Draw a phase diagram, labelling the saddle paths and steady states for both values of productivity (ε and ε') that is consistent with $c\left(k^*\left(\varepsilon\right), \varepsilon'\right) < c^*\left(\varepsilon\right) = c\left(k^*\left(\varepsilon\right), \varepsilon\right).$ Indicate $c\left(k^*\left(\varepsilon\right), \varepsilon'\right)$ in your diagram. Make sure that your phase diagram respects the qualitative properties shown in 3) and 4) for the consumption-capital ratios and the slopes of the saddle paths. Draw the saddle path as if it has constant slope.

5.3. Explain why it may be the case that consumption does not increase in impact (i.e. explain case 5.2). Make sure to mention the income effect and intertemporal substitution effects of the increase in productivity in your explanation.

5.4. Given your previous explanation, for which values of γ do you think that case 5.2 will occur ? Explain.

Hints. Use 3), 4) to argue that $c'(k^*(\varepsilon), \varepsilon) = c'(k^*(\varepsilon'), \varepsilon')$. Use a phase diagram to argue that whether $c(k^*(\varepsilon), \varepsilon') \geq (\leq)\ c^*(\varepsilon)$ depend on whether $c'(k^*(\varepsilon), \varepsilon) = c'(k^*(\varepsilon'), \varepsilon')$ is smaller (higher) than $c^*(\varepsilon)/k^*(\varepsilon) = c^*(\varepsilon')/k^*(\varepsilon')$.

Ans: If ε increases, both curves describing the steady state move. The curve $\dot{c} = 0$ moves to the right and the curve $\dot{k} = 0$ moves up for all levels of capital. Therefore, a higher ε means that both the steady state level of capital and consumption increase. Moreover, since the ratio c^*/k^* is independent of ε, both steady states lie in the same ray:

$$\frac{c^*(\varepsilon)}{k^*(\varepsilon)} = \frac{c^*(\varepsilon')}{k^*(\varepsilon')} = \frac{\rho}{\alpha}.$$

Also, we must have that

$$c'(k^*(\varepsilon), \varepsilon) = c'\left(k^*\left(\varepsilon'\right), \varepsilon'\right),$$

since the slope of the saddle path is invariant to changes in the productivity parameter.

5.1) Figure 1 below depicts the case where $c'[k^*(\cdot), \cdot] < c^*(\cdot)/k^*(\cdot)$. Before the permanent change in productivity, the system is in the steady state $(c^*(\varepsilon), k^*(\varepsilon))$. The slope of the associated saddle path is $c'(k^*(\varepsilon), \varepsilon)$. Just after the unexpected increase in ε occurs, consumption jumps up to $c(k^*(\varepsilon), \varepsilon')$ and the stock of capital remains at $k^*(\varepsilon)$. After that, the system starts converging to the new steady state $(c^*(\varepsilon'), k^*(\varepsilon'))$ through the saddle path with slope $c'(k^*(\varepsilon'), \varepsilon')$.

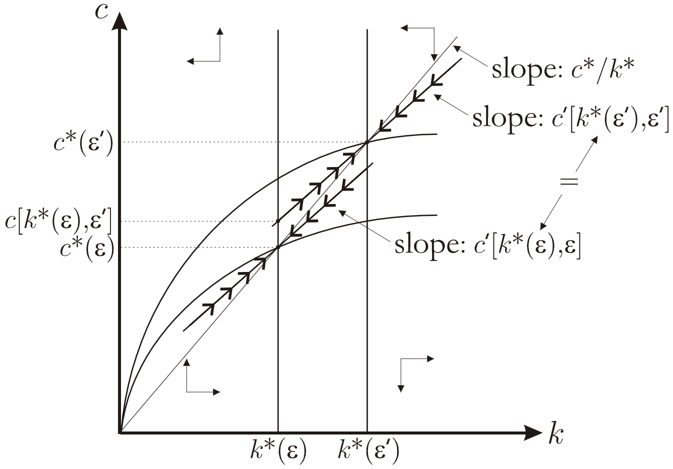

Figure 1. The slope of the saddle path is less than c^*/k^* and, thus, consumption jumps on impact.

5.2) Figure 2 depicts the case where $c'\left[k^*\left(\cdot\right),\cdot\right] > c^*\left(\cdot\right)/k^*\left(\cdot\right)$. Here consumption decreases when the productivity change takes place. The rest of the dynamics are identical to the previous case.

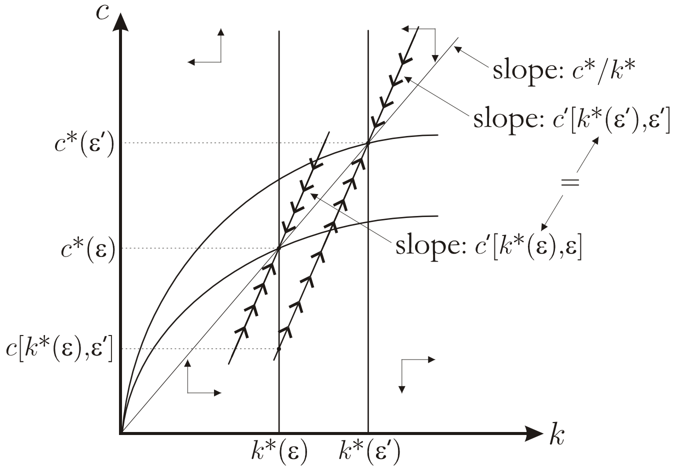

Figure 2. The slope of the saddle path is greather than c^*/k^* and, thus, consumption falls on impact.

5.3) A higher ε means that the economy is wealthier, so one might suspect that consumption should increase. It does in the long run, for sure. However it can be the case that it first decreases. The reason is that there are offsetting income and substitution effects: the income effect comes from the fact that the agent is wealthier, the present value of earnings is higher, so that is a force towards increasing current consumption. However a higher ε means that for a given level of k, the marginal productivity of capital (and therefore the interest rate) is higher. And a higher interest rate means that future consumption is cheaper relative to today's consumption. So, that's a force toward reducing current consumption in favor of future consumption. If the substitution effect is stronger than the income effect, we will see that current consumption will decline.

5.4) Of course, which effect dominates depends not only on the change in the relative price, but also on the willingness to substitute intertemporally. A higher γ means that the agent is less willing to substitute intertemporally, in this case the income effect will tend to dominate and consumption will increase. As we saw in the figures, for flatter slopes of the saddle path, the income effect dominates and consumption increases, and for steeper slopes, the substitution effect dominates and consumption decreases.

Notice that if the saddle paths have constant slopes, and using that in both steady states the consumption to capital ratio is constant, we have an easy way to see for what values of γ

which effect dominates:

$$\begin{aligned}
\text{if } c'(k^*) &< \frac{c^*}{k^*} = \frac{\rho}{\alpha} \rightarrow \text{ income effect dominates} \rightarrow c\left(k^*\left(\varepsilon_0\right), \varepsilon_1\right) > c^*\left(\varepsilon_0\right), \\
\text{if } c'(k^*) &> \frac{c^*}{k^*} = \frac{\rho}{\alpha} \rightarrow \text{ substitution effect dominates} \rightarrow c\left(k^*\left(\varepsilon_0\right), \varepsilon_1\right) < c^*\left(\varepsilon_0\right), \\
\text{if } c'(k^*) &= \frac{c^*}{k^*} = \frac{\rho}{\alpha} \rightarrow \text{ income and subst. effects cancel out} \rightarrow c\left(k^*\left(\varepsilon_0\right), \varepsilon_1\right) = c^*\left(\varepsilon_0\right).
\end{aligned}$$

Here is when the assumption that the productivity change is small becomes crucial. For infinitesimal changes in ε, we can assume that the slope of the saddle path is, indeed, constant, and the above analysis goes through. For larger changes in ε, the above analysis turns out to be 'approximately' right.

But we can go a little bit further. We know the slope of the saddle path, and we can solve for the γ that satisfies the above conditions. Since $c'(k^*) = \frac{\rho}{2}\left[1 + \sqrt{1 + \frac{4}{\gamma}\left(\frac{1-\alpha}{\alpha}\right)}\right]$, the conditions are

$$\begin{aligned}
\text{if } \gamma &> \alpha \rightarrow \text{ income effect dominates} \rightarrow c\left(k^*\left(\varepsilon_0\right), \varepsilon_1\right) > c^*\left(\varepsilon_0\right), \\
\text{if } \gamma &< \alpha \rightarrow \text{ substitution effect dominates} \rightarrow c\left(k^*\left(\varepsilon_0\right), \varepsilon_1\right) < c^*\left(\varepsilon_0\right), \\
\text{if } \gamma &= \alpha \rightarrow \text{ income and subst. effects cancel out} \rightarrow c\left(k^*\left(\varepsilon_0\right), \varepsilon_1\right) = c^*\left(\varepsilon_0\right).
\end{aligned}$$

Note that there is a trade-off between the concavity of the utility function, as described by γ, and the concavity of the production function, as described by α.

2 Investment in the Neoclassical Growth Model and Business Cycles

This exercise examines the dynamic behavior of gross investment in the neoclassical growth model. Specifically, it considers the adjustment path to the steady state, starting with a capital stock below its steady state value. Since capital is the state variable and since the dynamics of the neoclassical growth model are stable, net investment (i.e. the change in the stock of capital) must be decreasing in its adjustment path. Nevertheless, the behavior of gross investment (i.e. the change in capital plus the value of depreciation) can be different. In particular, below it is shown that gross investment could either increase or decrease in the adjustment path to steady state. Moreover, the investment to GDP ratio could be increasing.

This question is interesting because the neoclassical growth model is used as a simple business cycles model where unanticipated permanent productivity shocks are the source of fluctuations. In particular, we will like to see if the adjustment path after a permanent unexpected increase in productivity is consistent with the increase in investment typical of an expansion.

The planner's problem is to maximize

$$\int_0^\infty e^{-\rho t} \frac{c\left(t\right)^{1-\gamma}}{1-\gamma} dt,$$

subject to

$$\dot{k} = Ak^\alpha - c - \delta k.$$

That is, the production function is Cobb-Douglas, preferences are of the CRRA form with relative risk aversion γ, ρ is the discount rate and δ is the depreciation rate of capital. The solution to this problem is given by the Euler Equation and the law of motion of capital:

$$\begin{aligned} \dot{c} &= \frac{c}{\gamma}\left(\alpha A k^{\alpha-1} - \left(\rho+\delta\right)\right), \\ \dot{k} &= Ak^\alpha - c - \delta k. \end{aligned}$$

Exercise 1. Letting x be gross investment, i.e. $x = \dot{k} + \delta k$, show that the linear approximation of $x\left(t\right)$ around the steady state satisfies

$$x\left(t\right) - x^* = \left(k\left(0\right) - k^*\right) \exp\left(\lambda_1 t\right)\left[\lambda_1 + \delta\right],$$

for all $t \geq 0$,where λ_1 solves

$$\lambda_1 = \frac{\rho - \sqrt{\rho^2 + 4\left(1-\alpha\right)\frac{\left(\rho+\delta\right)}{\gamma}\left(\frac{\rho+\delta\left(1-\alpha\right)}{\alpha}\right)}}{2},$$

and where x^* and k^* are steady state values. [**Hint**: the following steps will be useful:

Step 1 : Differentiate the equation for $\dot{k}$ with respect to t and insert the $\dot{c}$ equation into the result you obtain. This gives a second order differential equation in k (the Euler equation!). Linearize the resulting equation, that is, obtain

$$\ddot{k} = g\left(\dot{k}, k\right) \cong g\left(0, k^*\right) + g_{\dot{k}}\left(0, k^*\right)\dot{k} + g_k\left(0, k^*\right)\left(k - k^*\right),$$

and compute the coefficients. You should obtain $\ddot{k} = \rho\dot{k} - f''\left(k^*\right)\left(c^*/\gamma\right)\left(k - k^*\right)$. Moreover, denoting $z\left(t\right) = k\left(t\right) - k^*$ the last equation becomes $\ddot{z} = \rho\dot{z} - f''\left(k^*\right)\left(c^*/\gamma\right)z$

Ans: Differentiating $\dot{k}$ w.r.t. t and inserting the $\dot{c}$ equation:

$$\begin{aligned} \ddot{k} &= \alpha A k^{\alpha-1}\dot{k} - \dot{c} - \delta\dot{k}, \\ \ddot{k} &= \left(\alpha A k^{\alpha-1} - \delta\right)\dot{k} - \left[Ak^\alpha - \delta k - \dot{k}\right]\left(1/\gamma\right)\left(\alpha A k^{\alpha-1} - \left(\rho+\delta\right)\right) \equiv g\left(\dot{k}, k\right). \end{aligned}$$

Consider the linearized version:

$$\ddot{k} = g\left(\dot{k}, k\right) \cong g\left(0, k^*\right) + g_{\dot{k}}\left(0, k^*\right)\dot{k} + g_k\left(0, k^*\right)\left(k - k^*\right),$$

$$\begin{aligned} g\left(0, k^*\right) &= 0, \\ g_{\dot{k}}\left(0, k^*\right) &= \rho, \\ g_k\left(0, k^*\right) &= -f''\left(k^*\right)\left(c^*/\gamma\right). \end{aligned}$$

Thus,

$$\ddot{k} = \rho\dot{k} - f''\left(k^*\right)\left(c^*/\gamma\right)\left(k - k^*\right),$$

or denoting $z\left(t\right) = k\left(t\right) - k^*$,

$$\ddot{z} = \rho\dot{z} - f''\left(k^*\right)\left(c^*/\gamma\right)z.$$

Step 2 : Argue that the solution to the last differential equation is $z\left(t\right) = B\exp\left(\lambda t\right)$ where λ solves the quadratic equation $\lambda^2 - \rho\lambda + \left(c^*/\gamma\right)f''\left(k^*\right) = 0$. Pick the stable solution (i.e. the negative root of λ), which satisfies

$$\lambda_1 = \frac{\rho - \sqrt{\rho^2 - 4f''\left(k^*\right)\left(c^*/\gamma\right)}}{2} < 0$$

Ans: The solution to the differential equation $\ddot{z} = \rho\dot{z} - f''\left(k^*\right)\left(c^*/\gamma\right)z$ is

$$z\left(t\right) = B\exp\left(\lambda t\right),$$

since

$$\begin{aligned} \dot{z} &= B\lambda\exp\left(\lambda t\right) \\ \ddot{z} &= B\lambda^2\exp\left(\lambda t\right), \end{aligned}$$

$$B\lambda^2\exp\left(\lambda t\right) = \rho B\lambda\exp\left(\lambda t\right) - f''\left(k^*\right)\left(c^*/\gamma\right)B\exp\left(\lambda t\right),$$

or

$$\lambda^2 = \rho\lambda - f''\left(k^*\right)\left(c^*/\gamma\right),$$

or

$$\lambda^2 - \rho\lambda + \left(c^*/\gamma\right)f''\left(k^*\right) = 0.$$

This quadratic equation has solution:

$$\begin{aligned}\lambda &= \frac{-b \pm \sqrt{b^2-4ac}}{2a} \\ &= \frac{\rho \pm \sqrt{\rho^2 - 4f''(k^*)(c^*/\gamma)}}{2}\end{aligned}$$

with

$$\lambda_1 = \frac{\rho - \sqrt{\rho^2 - 4f''(k^*)(c^*/\gamma)}}{2} < 0$$

and

$$\lambda_2 = \frac{\rho + \sqrt{\rho^2 - 4f''(k^*)(c^*/\gamma)}}{2} > 0$$

Thus, focusing on the stable solution,

$$\begin{aligned} z(t) &= B\exp(\lambda_1 t) \\ k(0) - k^* &= B\exp(\lambda_1 0) = B, \end{aligned}$$

or

$$B = k(0) - k^*,$$

and

$$k(t) - k^* = (k(0) - k^*)\exp(\lambda_1 t).$$

Step 3 : Show that for the functional forms we are using:

$$f''(k^*)(c^*/\gamma) = (\alpha - 1)\frac{(\rho+\delta)}{\gamma}\left(\frac{\rho + \delta(1-\alpha)}{\alpha}\right)$$

which implies

$$\lambda_1 = \frac{\rho - \sqrt{\rho^2 + 4(1-\alpha)\frac{(\rho+\delta)}{\gamma}\left(\frac{\rho+\delta(1-\alpha)}{\alpha}\right)}}{2}$$

Ans: Notice that with the assumed functional form for $f(k)$ we have that $f(k) = kf'(k)/\alpha$ and $f''(k) = (\alpha - 1)f'(k)/k$. Thus,

$$\begin{aligned} f''(k^*)(c^*/\gamma) &= f''(k^*)\frac{1}{\gamma}(f(k^*) - \delta k^*) \\ &= (\alpha-1)f'(k^*)\frac{1}{\gamma}\left(\frac{f'(k^*)}{\alpha} - \delta\right) \\ &= (\alpha-1)\frac{(\rho+\delta)}{\gamma}\left(\frac{\rho+\delta(1-\alpha)}{\alpha}\right), \end{aligned}$$

where the last line uses that in the steady state $f'(k^*) = \rho + \delta$. Then,

$$\lambda_1 = \frac{\rho - \sqrt{\rho^2 + 4(1-\alpha)\frac{(\rho+\delta)}{\gamma}\left(\frac{\rho+\delta(1-\alpha)}{\alpha}\right)}}{2}.$$

Step 4 : Use

$$\begin{aligned} x(t) &= \dot{k}(t) + \delta k(t) \\ &= \dot{z}(t) + \delta(z(t) + k^*) \end{aligned}$$

and replace the linearized solution to obtain the result].

Ans: Using

$$x(t) = \dot{k}(t) + \delta k(t),$$

and substituting the linearized version

$$\begin{aligned} x(t) &= \dot{z}(t) + \delta(z(t) + k^*) \\ &= B\lambda_1 \exp(\lambda_1 t) + \delta(B\exp(\lambda_1 t) + k^*) \\ &= \delta k^* + B\exp(\lambda_1 t)(\lambda_1 + \delta), \end{aligned}$$

or

$$x(t) - x^* = (k(0) - k^*)\exp(\lambda_1 t)(\lambda_1 + \delta).$$

From

$$x(t) - x^* = (k(0) - k^*)\exp(\lambda_1 t)[\lambda_1 + \delta]$$

it is clear that the parameter of interest that determines the dynamics of investment is $\lambda_1 + \delta$. In particular, whether $\lambda_1 + \delta < 0$ or $\lambda_1 + \delta > 0$.

Exercise 2. Let $\lambda_1(p)$ be the solution of λ_1 as a function of the parameters $p = (\rho, \delta, \gamma, \alpha)$. Show that

$$\lambda_1(p) + \delta \geq 0,$$

if and only if

$$\gamma \geq (1-\alpha)\left[\frac{\rho/\delta + (1-\alpha)}{\alpha}\right].$$

What does this imply for the dynamics of investment x close to the steady state if:

a) γ is large enough so that $\gamma > (1-\alpha)\left[\frac{\rho/\delta+(1-\alpha)}{\alpha}\right]$, what is the intuition for this?

b) if δ is small enough so that $\gamma < (1-\alpha)\left[\frac{\rho/\delta+(1-\alpha)}{\alpha}\right]$, what is the intuition for this?

c) if α is small enough so that $\gamma < (1-\alpha)\left[\frac{\rho/\delta+(1-\alpha)}{\alpha}\right]$, what is the intuition for this? [Hint: how does the marginal productivity of capital vary with α <u>*close*</u> to the steady state?]

[Hint: Let $F(p) \equiv 2(\lambda_1(p)+\delta)$, and use the explicit expression for λ_1 derived above. Show that

$$F(p) = 2\delta + \rho - \sqrt{(2\delta+\rho)^2 + \left(4(\rho+\delta)\frac{\delta}{\gamma}\left[(1-\alpha)\left(\frac{\rho/\delta+(1-\alpha)}{\alpha}\right)-\gamma\right]\right)}$$

].

Ans: We have

$$\begin{aligned}
F(p) &= 2(\lambda_1(p)+\delta) \\
&= 2\delta+\rho-\sqrt{\rho^2+4(1-\alpha)\frac{(\rho+\delta)}{\gamma}\left(\frac{\rho+\delta(1-\alpha)}{\alpha}\right)} \\
&= 2\delta+\rho-\sqrt{\rho^2+4(\rho+\delta)\delta+\left(4(\rho+\delta)\frac{\delta}{\gamma}\left[(1-\alpha)\left(\frac{\rho/\delta+(1-\alpha)}{\alpha}\right)-\gamma\right]\right)},
\end{aligned}$$

or

$$F(p) = 2\delta + \rho - \sqrt{(2\delta+\rho)^2 + \left(4(\rho+\delta)\frac{\delta}{\gamma}\left[(1-\alpha)\left(\frac{\rho/\delta+(1-\alpha)}{\alpha}\right)-\gamma\right]\right)}.$$

Thus, $F(p) \geq 0$ (and hence $\lambda(p)+\delta$) iff $\gamma \geq (1-\alpha)\left[\frac{\rho/\delta+(1-\alpha)}{\alpha}\right]$.

a) γ is the inverse of the elasticity of substitution. Everything else equal, a higher γ implies that the agent is less willing to substitute intertemporally. She wants a flat consumption path, as close to the steady state consumption as possible. This implies that she will invest little (even starting with investment below its steady state value) and it will take longer to converge to the steady state.

b) In the limit, we know that if $\delta = 0$ then investment is zero in the steady state. Thus, along the transition path, investment is always coming from above its steady state value. For δ close enough to zero this is still true.

c) The marginal productivity of capital is $f'(k) = \alpha A k^{\alpha-1}$. Close to the steady state (i.e. linearizing it around k^*) it satisfies

$$f'(k) \simeq f'(k^*) + f''(k^*)(k-k^*).$$

Thus

$$(\alpha-1)f'(k)/k$$

$$\begin{aligned} f'(k) &\simeq f'(k^*) + (\alpha - 1) f'(k^*) \frac{k - k^*}{k^*} \\ &= f'(k^*) \left(1 + (1-\alpha) \frac{k^* - k}{k^*}\right) \\ &= (\rho + \delta) \left(1 + (1-\alpha) \frac{k^* - k}{k^*}\right), \end{aligned}$$

where the last line uses that in the steady state $f'(k^*) = \rho + \delta$. Thus, close to the steady state, for $k < k^*$ we have $\partial f'(k^*) / \partial \alpha < 0$. This means that as α declines, the marginal productivity of capital increases (close to k^*) and hence, the benefits from investing increases. Hence investment will come from above its steady state value.

Exercise 3. Consider the following set of parameters. The first set is

$$\alpha = 0.3,\ A = 1,\ \gamma = 2,\ \rho = 0.075,\ \delta = 0.075$$

and the second is

$$\alpha = 0.4,\ A = 1,\ \gamma = 2,\ \rho = 0.05,\ \delta = 0.10$$

Compute $\lambda_1 + \delta$ for these two set of parameters.

Ans: For the first set of parameters, $\lambda_1 + \delta = -0.04$. For the second set, $\lambda_1 + \delta = 0.01$.

Now we examine the behavior of the ratio x/y, where y denotes GDP. This ratio is a natural business cycles indicator, high in the expansions and low in the recessions. It has the added advantage of being independent of common "trends" in x and y.

We focus in the following quantity:

$$\mu(k) = \left(\frac{k}{x(k)/y(k)} d \frac{x(k)/y(k)}{dk}\right)$$

where $x(k)$, and $y(k)$ are investment and GDP as function of capita (i.e. $\mu(k)$ is the elasticity of the investment to GDP ratio with respect to capital). We are interested in this quantity evaluated at $k = k^*$ which we simply refer to as μ. This can be written as

$$\begin{aligned} \mu(k(t)) &= \left(\frac{k(t)}{x(k(t))/y(k(t))} d \frac{x(k(t))/y(k(t))}{dk}\right) \\ &= \left(\frac{1}{x(t)/y(t)} d \frac{x(t)/y(t)}{dt}\right) \Big/ \left(\frac{1}{k(t)} \frac{dk(t)}{dt}\right) \end{aligned}$$

Then we study

$$\upsilon(t) = \frac{1}{x(t)/y(t)} d \frac{x(t)/y(t)}{dt}$$

for the linear approximation to the solution of the model.

Exercise 4. Show that

$$\mu = \frac{\lambda_1}{\delta} + 1 - \alpha$$

[Hint: Start by finding the linearized approximation of $\mu(k(t))$. To that end, note that $\upsilon(t) = \frac{1}{x(t)}\frac{dx(t)}{dt} - \frac{1}{y(t)}\frac{dy(t)}{dt}$. Then use the linear approximations

$$\begin{aligned} x(t) - x^* &= (\lambda_1 + \delta)(k(t) - k^*), \\ y(t) - y^* &= (\rho + \delta)(k(t) - k^*), \\ k(t) - k^* &= (k(0) - k^*)\exp(\lambda_1 t), \\ dk(t)/dt &= (k(0) - k^*)\exp(\lambda_1 t)\lambda_1 \end{aligned}$$

to obtain the derivatives. Replace them into your expression for $\mu(k(t))$. Lastly, take the limit as t goes to infinity and use the equations relating x^*, k^* and y^* to obtain μ].

Ans: We have

$$\upsilon(t) = \frac{1}{x(t)}\frac{dx(t)}{dt} - \frac{1}{y(t)}\frac{dy(t)}{dt},$$

where

$$\begin{aligned} \frac{1}{x(t)}\frac{dx(t)}{dt} &= \frac{1}{x(t)}[k(0) - k^*]\exp(\lambda_1 t)[\lambda_1 + \delta]\lambda_1 \\ \frac{1}{y(t)}\frac{dy(t)}{dt} &= \frac{1}{y(t)}[k(0) - k^*]\exp(\lambda_1 t)[\rho + \delta]\lambda_1, \end{aligned}$$

where we used the linear approximations:

$$\begin{aligned} x(t) - x^* &= (\lambda_1 + \delta)(k(t) - k^*), \\ y(t) - y^* &= (\rho + \delta)(k(t) - k^*), \\ k(t) - k^* &= (k(0) - k^*)\exp(\lambda_1 t), \\ dk(t)/dt &= (k(0) - k^*)\exp(\lambda_1 t)\lambda_1. \end{aligned}$$

Thus

$$\upsilon(t) = [k(0) - k^*]\exp(\lambda_1 t)\lambda_1\left(\frac{\lambda_1 + \delta}{x(t)} - \frac{\rho + \delta}{y(t)}\right).$$

Consider

$$\mu(k(t)) \equiv k(t)\frac{\upsilon(t)}{dk(t)/dt} = k(t)\frac{\upsilon(t)}{[k(0) - k^*]\exp(\lambda_1 t)\lambda_1} = k(t)\left(\frac{\lambda_1 + \delta}{x(t)} - \frac{\rho + \delta}{y(t)}\right),$$

and let $t \to \infty$, so that

$$\mu = k^* \left(\frac{\lambda_1 + \delta}{x^*} - \frac{\rho + \delta}{y^*} \right).$$

Using

$$\begin{aligned} \alpha y^* &= (\rho + \delta) k^* \\ x^* &= \delta k^*, \end{aligned}$$

we arrive at

$$\mu = \left(\frac{\lambda_1 + \delta}{\delta} - \frac{(\rho + \delta)}{(\rho + \delta)/\alpha} \right) = \frac{\lambda_1}{\delta} + 1 - \alpha.$$

Exercise 5. Show that

$$\lim_{\rho/\delta \to 0} \mu = (1 - \alpha) \left[1 - \frac{1}{\sqrt{\gamma \alpha}} \right].$$

Is the path of $x(t)/y(t)$ increasing or decreasing through time if $k(0) < k^*$, ρ/δ is small, and $\gamma\alpha > 1$? Does this setting of parameters made investment/GDP in the model procyclical?

Ans: Since

$$\mu = \frac{\lambda_1}{\delta} + 1 - \alpha = \frac{\frac{\rho}{\delta} - \sqrt{\left(\frac{\rho}{\delta}\right)^2 + \frac{4(1-\alpha)}{\gamma\alpha} \left(\frac{\rho}{\delta} + 1\right) \left(\frac{\rho}{\delta} + (1 - \alpha)\right)}}{2} + 1 - \alpha,$$

we have

$$\lim_{\rho/\delta \to 0} \mu = \frac{-\sqrt{\frac{4(1-\alpha)}{\gamma\alpha} (1 - \alpha)}}{2} + 1 - \alpha = (1 - \alpha) \left[1 - \frac{1}{\sqrt{\alpha\gamma}} \right].$$

If $k(0) < k^*$, ρ/δ is small, and $\gamma\alpha > 1$, then $\mu > 0$ and hence, close to the steady state

$$\frac{d\left(x(k)/y(k)\right)}{dk} > 0,$$

which implies that x/y is increasing in the transition. Since the transition is interpreted as an expansion, this parameter setting is not consistent with the procyclicality of investment.

Exercise 6. Show that

$$\lim_{\rho/\delta \to \infty} \mu = -\infty.$$

Is the path of $x(t)/y(t)$ increasing or decreasing through time if $k(0) < k^*$ and ρ/δ is very large? Is this consistent with the interpretation of the adjustment to steady state as an expansion?

Ans: Letting $\zeta = \rho/\delta$ and $L(\zeta) = \lambda_1/\delta$:

$$\frac{\lambda_1}{\delta} + 1 - \alpha = L(\zeta) + 1 - \alpha = \frac{\zeta - \sqrt{\zeta^2 + \frac{4(1-\alpha)}{\gamma\alpha}(\zeta+1)(\zeta+(1-\alpha))}}{2} + 1 - \alpha.$$

Since $L(\zeta) < 0$, and

$$\begin{aligned}\lim_{\zeta\to\infty} \frac{L(\zeta)}{\zeta} &= \lim_{\zeta\to\infty} \frac{1 - \sqrt{1 + \frac{4(1-\alpha)}{\gamma\alpha}(1+1/\zeta)(1+(1-\alpha)/\zeta)}}{2} \\ &= \frac{1 - \sqrt{1 + \frac{4(1-\alpha)}{\gamma\alpha}}}{2} < 0,\end{aligned}$$

then $L(\zeta) \to -\infty$ as $\zeta \to \infty$, and thus $\mu \to -\infty$ as $\rho/\delta \to \infty$.

If $k(0) < k^*$, and ρ/δ is large then $\mu < 0$ and hence, close to the steady state

$$\frac{dx(k)/y(k)}{dk} < 0$$

which implies that x/y is decreasing in the transition. Since the transition is interpreted as an expansion, this parameter setting is consistent with the procyclicality of investment.

Exercise 7. Consider the following set of parameters. The first set is

$$\alpha = 0.3,\ A = 1,\ \gamma = 2,\ \rho = 0.075,\ \delta = 0.075$$

and the second is

$$\alpha = 0.4,\ A = 1,\ \gamma = 2,\ \rho = 0.05,\ \delta = 0.10$$

For each setting of parameters compute $\lambda_1/\delta + (1-\alpha)$ and comment if they are consistent with the procyclicality of x/y.

Ans: For the first set of parameters $\mu = \lambda_1/\delta + (1-\alpha) = -0.85$. For the second set, $\mu = \lambda_1/\delta + (1-\alpha) = -0.29$. Both are consistent with the procyclicality of x/y.

Exercise 8. Suppose that the economy is in steady state with $A = 1$. Use the first set of parameters ($\alpha = 0.3,\ \gamma = 2,\ \rho = 0.075,\ \delta = 0.075$) for all the calculations.

i. What is the steady state investment to GDP ratio x^*/y^*?

Assume that A unexpectedly changes to $A' = (1+\varepsilon)^{1-\alpha}$, or approximately $(1-\alpha)\varepsilon/100$ %, for small ε. Assume that A' will stay at that level forever.

ii. What is the % change in the steady state capital? Denote this capital by k^{**}. Does the steady state value x/y depend on A?

iii. Use the definition of μ above to compute the change in the investment/GDP ratio x/y on impact. Your answer should be a function of μ, (x^*/y^*) and ε. [Hint. Let $z(k) = x(k)/y(k)$, use a first order approximation for z around k^{**} and evaluate it at $k = k^*$].

iv. Using the reference numerical values for all parameters, compute the new value of x/y just after the change in productivity if $\varepsilon = 0.1$ (10%).

Ans:

i.

$$\frac{x^*}{y^*} = \frac{\delta k^*}{(\rho+\delta)\, k^*/\alpha} = \alpha\frac{\delta}{\delta+\rho} = 0.3\ \frac{.075}{.075+.075} = .15.$$

ii.

$$\begin{aligned} \rho+\delta &= \alpha A\,(k^*)^{\alpha-1} \\ \rho+\delta &= \alpha A\,(1+\varepsilon)^{1-\alpha}\,(k^{**})^{\alpha-1}. \end{aligned}$$

Thus

$$1 = \left(\frac{1}{1+\varepsilon}\right)^{1-\alpha}\left(\frac{k^*}{k^{**}}\right)^{a-1},$$

or

$$\left(\frac{k^*}{k^{**}}\right)^{1-\alpha} = \left(\frac{1}{1+\varepsilon}\right)^{1-\alpha},$$

or

$$k^{**} = (1+\varepsilon)\,k^*,$$

so the steady state capital increases by $\varepsilon/100\%$. x^*/y^* is independent of A.

iii. Let $z(k) = x(k)/y(k)$.

$$\mu = \frac{k^{**}}{z(k^{**})}\frac{dz(k^{**})}{dk},$$

so

$$\begin{aligned} z(k^*) &\cong z(k^{**}) + \frac{dz(k^{**})}{dk}(k^* - k^{**}) \\ &= z(k^{**}) + z(k^{**})\,\frac{k^{**}}{z(k^{**})}\frac{dz(k^{**})}{dk}\frac{k^*-k^{**}}{k^{**}} \\ &= z(k^{**})\left(1+\mu\frac{k^*-(1+\varepsilon)\,k^*}{(1+\varepsilon)\,k^*}\right) \\ &= z(k^{**})\left(1-\frac{\mu\varepsilon}{1+\varepsilon}\right). \end{aligned}$$

Hence

$$z(k^*) = \left(\frac{x^*}{y^*}\right)\left(1-\frac{\mu\varepsilon}{1+\varepsilon}\right).$$

iv.

$$0.15 \ \times \left(1 - \frac{(-.85) \ \times 0.1}{1 + 0.1}\right) = 0.162,$$

so on impact investment/GDP is about 16% (an increase of about 8%).

3 Investment Specific Technological Progress

Consider the following version of the neoclassical growth model

$$\max \int_0^\infty e^{-\rho t} u\left(c\left(t\right)\right) dt$$

$$\begin{aligned} \text{s.t.} \quad p_k x\left(t\right) + c\left(t\right) &= F\left[k(t), e^{\gamma t}\right], \\ \dot{k}\left(t\right) &= e^{\eta t} x\left(t\right) - \delta k\left(t\right), \quad \text{all } t \geq 0, \\ \text{given } k\left(0\right) &= k_0 > 0, \end{aligned}$$

where F has constant returns to scale, and (inelastically supplied) labor is normalized at unity. Labor-augmenting technical change occurs at the rate $\gamma \geq 0$, and investment-specific technical change at the rate η. The constant p_k corresponds to the initial (time $t = 0$) price of investment in terms of consumption.

Exercise 1. Formulate the Hamiltonian using c as the only control variable and write the first order conditions for an optimum.

Ans: Using the feasibility constraint in the law of motion for capital we obtain

$$\dot{k}_t = e^{\eta t} \frac{F\left(k_t, e^{\gamma t}\right) - c_t}{p_k} - \delta k_t.$$

Thus, the Hamiltonian is

$$H = u\left(c_t\right) + \lambda_t \left[e^{\eta t} \frac{F\left(k_t, e^{\gamma t}\right) - c_t}{p_k} - \delta k_t\right],$$

where λ_t is the co-state variable. The FOCs for an optimum are

$$0 = \frac{\partial H}{\partial c} = u'\left(c_t\right) - \lambda_t \frac{e^{\eta t}}{p_k}, \tag{6}$$

and

$$\dot{\lambda} = \rho \lambda - \frac{\partial H}{\partial k} = \lambda_t \rho + \lambda_t \delta - \lambda_t \frac{e^{\eta t}}{p_k} F_k\left(k_t, e^{\gamma t}\right),$$

or

$$\frac{\dot{\lambda}_t}{\lambda_t} = \rho + \delta - \frac{e^{\eta t}}{p_k} F_k\left(k_t, e^{\gamma t}\right). \tag{7}$$

Exercise 2. Reduce the system obtained in 1 to a pair of differential equations in (c, k).

Ans: Take logs to (6)

$$\log u'(c_t) = \log \lambda_t + \eta t - \log p_k.$$

Differentiate w.r.t. time,

$$\frac{c_t u''(c_t)}{u'(c_t)} \frac{\dot{c}_t}{c_t} = \frac{\dot{\lambda}_t}{\lambda_t} + \eta.$$

Solving for $\dot{\lambda}/\lambda$ and introducing the result into (7) we obtain

$$\frac{c_t u''(c_t)}{u'(c_t)} \frac{\dot{c}_t}{c_t} = \eta + \rho + \delta - \frac{e^{\eta t}}{p_k} F_k\left(k_t, e^{\gamma t}\right).$$

Thus, the pair of differential equations in (c, k) are

$$\frac{\dot{c}_t}{c_t} = -\frac{1}{c_t u''(c_t)/u'(c_t)} \left[\frac{e^{\eta t}}{p_k} F_k\left(k_t, e^{\gamma t}\right) - (\rho + \delta + \eta)\right],$$

$$\dot{k}_t = e^{\eta t} \frac{F\left(k_t, e^{\gamma t}\right) - c_t}{p_k} - \delta k_t.$$

Use the following functional form for the remaining of this question:

$$\begin{aligned} u(c) &= \frac{c^{1-\sigma}}{1-\sigma}, \quad \sigma > 0, \\ F\left(k, e^{\gamma t}\right) &= A\, k^{\alpha} e^{(1-\alpha)\gamma t}, \quad 0 < \alpha < 1. \end{aligned}$$

Balanced Growth Path. Suppose $\gamma, \eta > 0$. Consider the growth rates of capital and consumption in the long run. Conjecture that in the long run the ratio of consumption to capital falls at the rate η. That is, conjecture that along the balanced growth path $c(t)/k(t) = x^* e^{-\eta t}$, where x^* is a constant that must be determined.

Exercise 3. Calculate the long run growth rates for capital (g_k), consumption (g_c) and output as function of the parameters $\alpha, \sigma, \delta, \gamma$ and η.

Ans: Using the above functional forms the system of differential equations becomes

$$\frac{\dot{c}_t}{c_t} = \frac{1}{\sigma} \left[\frac{e^{\eta t} \alpha A k_t^{\alpha-1} \left(e^{\gamma t}\right)^{1-\alpha}}{p_k} - (\rho + \delta + \eta)\right], \tag{8}$$

$$\dot{k}_t = \frac{e^{\eta t}}{p_k}\left[Ak_t^{\alpha}\left(e^{\gamma t}\right)^{1-\alpha} - c_t\right] - \delta k_t,$$

or, dividing by k_t, the last equation becomes

$$\frac{\dot{k}_t}{k_t} = \frac{e^{\eta t}}{p_k}\left[Ak_t^{\alpha-1}\left(e^{\gamma t}\right)^{1-\alpha} - \frac{c_t}{k_t}\right] - \delta. \tag{9}$$

Rewrite (8) as

$$g_c = \frac{1}{\sigma}\left[\frac{\alpha A}{p_k}e^{[\eta+\gamma(1-\alpha)]t}k_t^{\alpha-1} - (\rho+\delta+\eta)\right]. \tag{10}$$

Since in a balanced growth path the left hand side is constant (i.e. independent of t), so has to be the right hand side. But the RHS is constant if and only if

$$e^{[\eta+\gamma(1-\alpha)]t}k_t^{\alpha-1},$$

is constant. Taking logs of that expression we find

$$[\eta+\gamma(1-\alpha)]\,t - (1-\alpha)\,k_t = \text{constant},$$

and differentiating w.r.t. time we have

$$\eta+\gamma(1-\alpha) - (1-\alpha)\frac{\dot{k}_t}{k_t} = 0,$$

or

$$g_k = \gamma + \frac{\eta}{1-\alpha}.$$

That is, we pinned down the growth rate of capital in the balanced growth path.

Now using (9) we obtain

$$g_k + \delta = \left[\frac{1}{\alpha}\frac{\alpha A}{p_k}e^{[\eta+\gamma(1-\alpha)]t}k_t^{\alpha-1} - \frac{e^{\eta t}}{p_k}\frac{c_t}{k_t}\right]. \tag{11}$$

Moreover, as we showed above the term $e^{[\eta+\gamma(1-\alpha)]t}k_t^{\alpha-1}$ is constant in a balanced growth path. Thus it immediately follows that

$$e^{\eta t}\frac{c_t}{k_t} = x^* = \text{constant}$$

in a steady state (noticed that we didn't use the conjecture, we proved it!). Hence (taking logs and differentiating w.r.t. t)

$$\eta + \frac{\dot{c}_t}{c_t} = \frac{\dot{k}_t}{k_t},$$

or

$$g_c = \gamma + \frac{\eta}{1-\alpha} - \eta,$$

or

$$g_c = \gamma + \frac{\alpha}{1-\alpha}\eta.$$

Furthermore (even though not asked), we can obtain x^*. Rewrite (11) as

$$\gamma + \frac{\eta}{1-\alpha} + \delta = \frac{1}{\alpha}\frac{\alpha A}{p_k}e^{[\eta+\gamma(1-\alpha)]t}k_t^{\alpha-1} - \frac{x^*}{p_k}.$$

Use (10) to get

$$\left(\gamma + \frac{\eta\alpha}{1-\alpha}\right)\sigma + \rho + \delta + \eta = \frac{\alpha A}{p_k}e^{[\eta+\gamma(1-\alpha)]t}k_t^{\alpha-1},$$

and introducing the last equation into the previous one

$$\frac{x^*}{p_k} = \frac{1}{\alpha}\left[\left(\gamma + \frac{\eta}{1-\alpha} - \eta\right)\sigma + \rho + \delta + \eta\right] - \left(\gamma + \frac{\eta}{1-\alpha} + \delta\right),$$

which gives x^*.

Exercise 4. How would you measure the contributions to long term growth of labor-augmenting technical change and investment-specific technical change for the US economy? [Sketch only, 10 lines maximum. Hint: A good answer should mention measuring TFP (total factor productivity) being careful on the units on which GDP and factors are measured, and the rate at which the investment deflator and the consumption deflator grow. Notice also that labor-augmenting technical change is closely related to TFP when the production function is Cobb-Douglas].

Ans: Taking logs of the production function we obtain

$$\log y_t = (1-\alpha)\gamma t + \alpha \ \log k_t.$$

Differentiating this expression w.r.t. time yields

$$g_y = (1-\alpha)\gamma + \alpha g_k,$$

or, rearranging,

$$g_{\text{TFP}} \equiv (1-\alpha)\gamma = g_y - \alpha g_k,$$

which gives an expression for TFP growth in terms of measurable quantities: the rate of growth of real GDP, g_y, the rate of growth of the real stock of capital, g_k and capital income's share in total production, α. Dividing this expression by $(1-\alpha)$ we obtain the rate of labor-augmenting technical change, γ. Moreover, recall that

$$g_k = \gamma + \frac{\eta}{1-\alpha}.$$

Hence

$$\eta = (1-\alpha)(g_k - \gamma),$$

gives the rate of investment-specific technical change. Finally, since

$$g_c = g_y = \gamma + \frac{\alpha}{1-\alpha}\eta,$$

then γ/g_y and $\frac{\alpha}{1-\alpha}\eta/g_y$ are the contributions to long term growth of labor-augmenting technical change and investment-specific technical change.

For the rest of the question consider the case where $\eta = \gamma = 0$.

Exercise 5. Draw the phase diagram corresponding to the system in b. Have c in the vertical axis and k in the horizontal axis. Indicate the $\dot{k} = 0$ and $\dot{c} = 0$ locus, the steady state values k^*, c^*. Include arrows indicating the direction of movement in each relevant quadrant, display the saddle path clearly, and indicate typical paths of trajectories that start close but not on the saddle path.

Ans: The system of differential equations for $\eta = \gamma = 0$ is

$$\frac{\dot{c}_t}{c_t} = \frac{1}{\sigma}\left[\alpha A k_t^{\alpha-1}/p_k - (\rho+\delta)\right], \tag{12}$$

$$\dot{k}_t = \left(A k_t^{\alpha} - c_t\right)/p_k - \delta k_t. \tag{13}$$

The dynamics of equations (13) and (12) are shown in the phase diagram of Figure 3 below. As usual, the $\dot{k}_t = 0$ locus is strictly concave and strictly increasing (decreasing) for all k that satisfy $f'(k) > (<) p_k \delta$. In turn, the $\dot{c}_t = 0$ locus is a vertical line in the (k, c) plane. Starting from a situation in which $\dot{c}_t = 0$, if we increase (decrease) the stock of capital then consumption will start falling (growing), as can be seen directly from equation (12). We summarize this with the arrows pointing south (north) to the right (left) of the $\dot{c}_t = 0$ locus. Similarly, starting from a situation in which $\dot{k}_t = 0$, if we increase (decrease) consumption then the stock of capital will start falling (growing), as can be seen directly from equation (13). This is summarized with the arrows pointing west (east) above (below) the $\dot{k}_t = 0$ locus.

Starting from any initial level of capital there exists a unique trajectory (the saddle path) that converges to the steady state, (k^*, c^*). The trajectory is unique because any other (k, c) pair off the saddle path would lead to a trajectory that eventually violates the necessary conditions for optimality. For instance, if the economy starts with a capital stock of k_0

then the optimal level of consumption is c_0. Any level of consumption above c_0 will lead to zero consumption and zero capital stock in finite time. This would clearly violate the Euler equation at that time. In turn, any level of consumption above c_0 will lead to a very large amount of capital, which would violate the transversality condition.

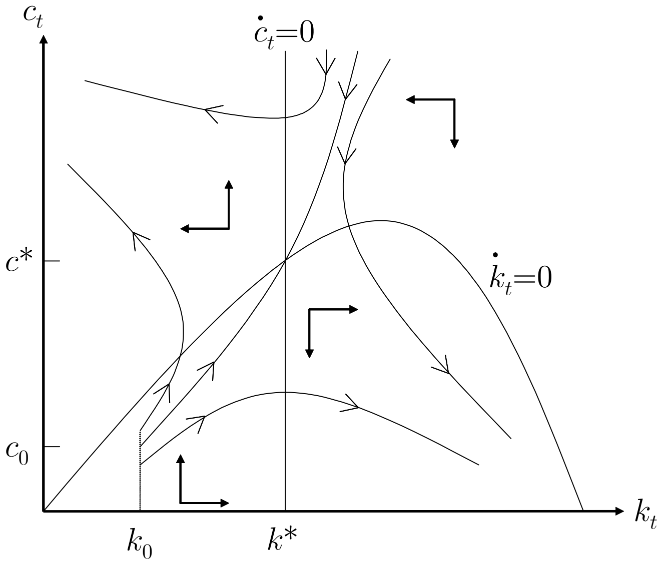

Figure 3. Phase diagram of the neoclassical growth model.

Exercise 6. Find an expression for the steady state capital stock k^* as a function of p_k, α, ρ and δ. Find an expression for the value of capital relative to GDP, i.e. $p_k k^*/y^*$ where y^* is the steady state output.

Ans: (12) at the steady state becomes

$$0 = \frac{1}{\sigma}\left[\alpha A\left(k^*\right)^{\alpha-1}/p_k - (\rho+\delta)\right].$$

Thus

$$k^* = \left[\frac{\alpha A}{p_k(\rho+\delta)}\right]^{1/(1-\alpha)}.$$

To obtain $p_k k^*/y^*$, recall that $f\left(k^*\right) = k^* f'\left(k^*\right)/\alpha$. Thus,

$$\begin{aligned} \frac{p_k k^*}{y^*} &= \frac{p_k k^*}{k^* f'\left(k^*\right)/\alpha} \\ &= \frac{\alpha}{f'\left(k^*\right)/p_k} \\ &= \frac{\alpha}{\rho+\delta}, \end{aligned}$$

where the last line uses that in the steady state the marginal product of capital equals $\rho + \delta$.

Exercise 7. Find an expression for the steady state consumption c^* as a function of p_k, α, ρ and δ. Find an expression for the value of consumption relative to GDP, i.e. c^*/y^* where y^* is the steady state output.

Ans: Feasibility at the steady state is

$$p_k \delta k^* + c^* = f(k^*).$$

Thus

$$\begin{aligned} c^* &= f(k^*) - p_k \delta k^* \\ &= \frac{k^* f'(k^*)}{\alpha} - p_k \delta k^* \\ &= \left(\frac{f'(k^*)/p_k}{\alpha} - \delta \right) p_k k^*, \end{aligned}$$

or

$$c^* = \left(\frac{\rho + \delta}{\alpha} - \delta \right) p_k^{-\alpha/(1-\alpha)} \left(\frac{\alpha A}{\rho + \delta} \right)^{1/(1-\alpha)}.$$

On the other hand,

$$\frac{c^*}{y^*} = 1 - \frac{p_k \delta k^*}{y^*},$$

or, using the result in exercise 6,

$$\frac{c^*}{y^*} = 1 - \frac{\alpha \delta}{\rho + \delta}.$$

For the remaining of this question, consider the case of no depreciation, i.e. $\delta = 0$.

Exercise 8. What is the steady state value of investment x^*? What is the steady state value of investment $p_k x^*/y^*$ relative to GDP.

Ans: $x^* = \delta k^* = 0$ and $p_k x^*/y^* = 0$.

Exercise 9. Assume that $k(0)$ is smaller than the its steady state value k^*. Draw a figure with time t in the horizontal axis, and output $y(t)$, investment $x(t)$, and consumption $c(t)$ in the vertical axis. Label the steady state values for output, investment and consumption, y^*, x^* and c^* in the vertical axis. Make sure $y(t) = c(t) + p_k x(t)$ and that x^* is as in exercise 8.

Ans: The dynamics of capital and consumption can be read off the phase diagram in Figure 3. Figure 4 below plots the associated dynamics. We see that 1) investment converges from above to $x^* = 0$, and 2) consumption and output converge from below to $c^* = y^*$, with consumption being less than output along the transition path (since investment is strictly positive).

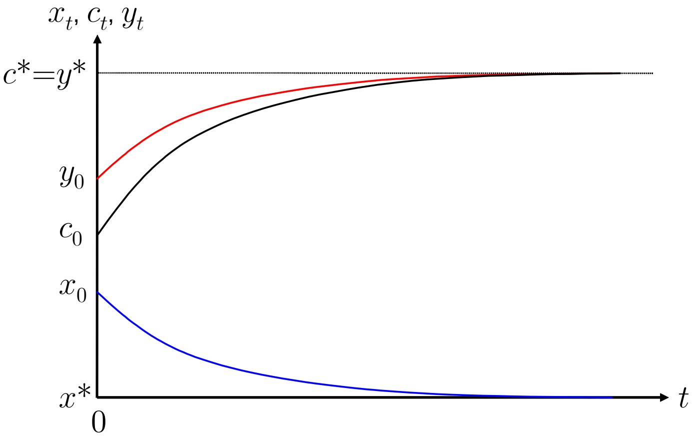

Figure 4. Dynamics of investment, consumption and output for $k\left(0\right) < k^*$.

Exercise 10. Assume that $k\left(0\right) = k^*$ is the steady state value of capital for the price p_k. Assume that at time $t = 0$, the price of capital decreases to p_k' with $p_k' < p_k$ and will remain there forever. Denote the new steady state values as y^{**}, c^{**} and x^{**}. Draw a figure with time t in the horizontal axis, and output $y\left(t\right)$, investment $x\left(t\right)$ and consumption $c\left(t\right)$ in the vertical axis. Make sure $y\left(t\right) = c\left(t\right) + p_k' x\left(t\right)$ and that x^* and x^{**} are as in exercise 8. Label the steady states values for output y^* and y^{**}, investment x^* and x^{**} and consumption c^{**} and c^* in the vertical axis. Show whether $c^* < c\left(0\right)$ or $c^* \geq c\left(0\right)$ and whether $y^* < y\left(0\right)$ or $y^* \geq y\left(0\right)$.

What do you learn about the "cyclicality" of consumption? That is, would the transition caused by a permanent decrease in p_k starting from a steady state capital look like an economic boom? (recall that in a boom consumption and GDP both increase together). How do you think your answer will change for a higher (positive) value of the depreciation δ?

Ans: The dynamics of consumption and capital can be read off the phase diagram in Figure 5. We see that a fall in the price of capital shifts the $\dot{c}_t = 0$ locus to the right (to $\dot{c}_t' = 0$) while it leaves the $\dot{k}_t = 0$ locus unmodified. Figure 6 plots the associated dynamics. In particular, 1) investment converges from above to $x^* = x^{**} = 0$, 2) consumption and output converge from below to $c^{**} = y^{**} > c^* = y^*$, with consumption being less than output along the transition path (since investment is strictly positive), and 3) $y(0) = y^* = c^*$ (since the capital stock is given at time zero) and $c(0) < y(0) = c^*$ (consumption must fall on impact to free up resources for investment).

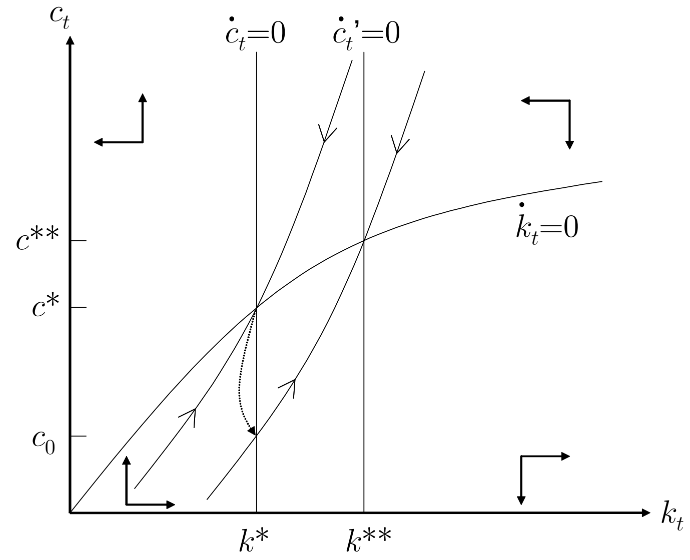

Figure 5. The impact of a fall in the price of capital, p_k, when $\delta = 0$.

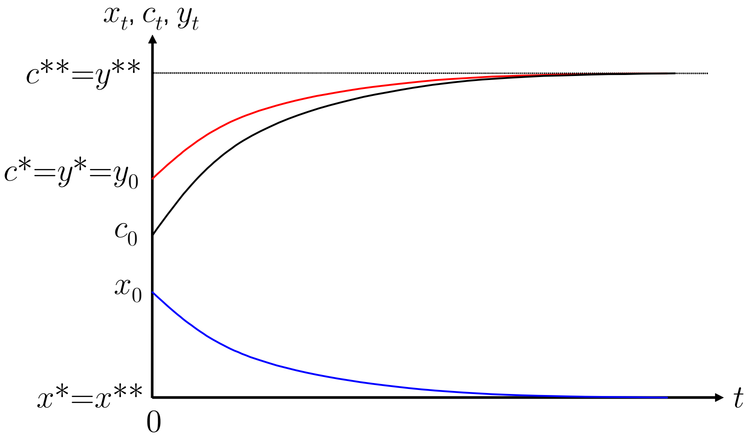

Figure 6. Dynamics of investment, consumption and output due to a fall in the price of capital, p_k, when $\delta = 0$.

Since consumption and output move together along the transition path (i.e., consumption is procyclical), a fall in the price of capital will look like an economic boom. The case of a strictly positive depreciation rate, $\delta > 0$, is left as an exercise.

4 Habit Formation model

Assume that agents select $c(t)$ and $l(t)$ for $t \geq 0$ to maximize

$$\int_0^\infty e^{-\rho t} \left[u\left(c(t) - \eta s(t)\right) - Al(t)\right] dt,$$

subject to

$$\begin{aligned} c(t) &= wl(t), \\ \dot{s}(t) &= c(t) - \delta s(t), \end{aligned}$$

for all $t \geq 0$, for given $s(0)$.

In this problem c is consumption, l labor, w the real wage, ρ the discount factor, s the undepreciated stock of past consumption or the habit level, δ the depreciation rate of consumption in the habit level, and $u(c - \delta s) - Al$ the period utility function.

As it is clear from the expression above the agent has no access to a savings account or a storage technology. At each point in time the agent must decide how much to work (or consume). The problem is not static because we assume that preferences depend on labor l, and consumption c *relative to the depreciated stock of consumption* s.

We can write this problem as

$$\max_{\{c(t)\}_{t=0}^\infty} \int_0^\infty e^{-\rho t} \left[u\left(c(t) - \eta s(t)\right) - \frac{A}{w} c(t)\right] dt,$$

subject to

$$\dot{s}(t) = c(t) - \delta s(t),$$

for all $t \geq 0$ given $s(0)$. We will ignore the non-negativity conditions for $l(t)$.

We will assume that $0 < \eta < \delta$, $\delta > 0, \rho > 0$ and $w > 0$. At some points we will also assume that u has constant relative risk aversion γ, i.e.

$$u(x) = \frac{x^{1-\gamma} - 1}{1 - \gamma},$$

for $\gamma > 0$.

Exercise 1. Write the Hamiltonian $H(c, s, \lambda)$. Which is the control and which is the state?

Ans:

$$H(c, s, \lambda) = u(c - \eta s) - \frac{A}{w} c + \lambda (c - \delta s).$$

The control is c and the state is s.

Exercise 2. Write the first order conditions (these are two equations $H_c = 0$ and $\dot{\lambda} = ...$). Your answer should be in terms of the parameters ρ, δ, η, w and the function u.

Ans:

$$\begin{aligned} 0 &= u' - \frac{A}{w} + \lambda \\ \dot{\lambda} &= \lambda\rho - H_s = \lambda\rho + u'\left(c - \eta s\right)\eta + \delta\lambda, \end{aligned}$$

or

$$\begin{aligned} 0 &= u' - \frac{A}{w} + \lambda \\ \dot{\lambda} &= \lambda\left(\rho + \delta\right) + u'\left(c - \eta s\right)\eta. \end{aligned}$$

Exercise 3. Use the previous equations, including the one for $\dot{s}$, to solve for steady state c as a function of the parameters of the model.

Ans:

$$\begin{aligned} \dot{s} &= 0 \text{ gives } c^* = \delta s^*. \\ \dot{\lambda} &= 0 \text{ gives } -\lambda^*\left(\rho + \delta\right) = u'\left(c^* - \eta s^*\right)\eta, \end{aligned}$$

or

$$\left[u'\left(c^* - \eta s^*\right) - \frac{A}{w}\right]\left(\rho + \delta\right) = u'\left(c^* - \eta s^*\right)\eta,$$

or

$$u'\left(c^* - \eta s^*\right) = \frac{A}{w}\frac{\left(\rho + \delta\right)}{\rho + \delta - \eta},$$

or

$$u'\left(c^*\left[1 - \frac{\eta}{\delta}\right]\right) = \frac{A}{w}\frac{\left(\rho + \delta\right)}{\rho + \delta - \eta},$$

which gives the steady state level of consumption c^*.

Exercise 4. Show that the long run elasticity of consumption and labor supply is

$$\frac{w}{c^*}\frac{dc^*}{dw} = \left[\frac{u'}{-cu''}\left(c^*\left(1 - \frac{\eta}{\delta}\right)\right)\right] \; / \; \left[1 - \frac{\eta}{\delta}\right],$$

i.e. this is the percentage change in steady state consumption for a 1% change in wages.

Ans:

$$\begin{aligned} u''\left(c^*\left[1-\frac{\eta}{\delta}\right]\right)\left[1-\frac{\eta}{\delta}\right]\frac{dc^*}{dw} &= -\frac{A}{w^2}\frac{(\rho+\delta)}{\rho+\delta-\eta} \\ u''\left(c^*\left[1-\frac{\eta}{\delta}\right]\right)\left[1-\frac{\eta}{\delta}\right]\frac{w}{c^*}\frac{dc^*}{dw} &= -\frac{A}{w}\frac{(\rho+\delta)}{\rho+\delta-\eta}\frac{1}{c^*} = -u'\left(c^*\left[1-\frac{\eta}{\delta}\right]\right)\frac{1}{c^*}, \end{aligned}$$

or

$$\frac{w}{c^*}\frac{dc^*}{dw} = \left[\frac{u'}{-cu''}\left(c^*\left(1-\frac{\eta}{\delta}\right)\right)\right] \;/\; \left[1-\frac{\eta}{\delta}\right].$$

Exercise 5. To be able to draw the phase diagram for this model we want to eliminate λ from the system of three equations (one for $\dot{\lambda}$, one for $\dot{s}$ and $H_c = 0$). This is analogous to what we did in the neoclassical growth model. You must show that you can solve for λ and express the system of two differential equations in $\dot{s}$ and $\dot{c}$ as

$$\begin{aligned} \dot{s} &= c-\delta s, \\ \dot{c} &= \frac{u'}{-u''}\left[\eta-(\rho+\delta)\right]+\left[\frac{1}{-u''}\right]\frac{A}{w}(\rho+\delta)+\eta(c-\delta s). \end{aligned}$$

Ans: Differentiating $H_c = 0$ w.r.t. time:

$$\dot{\lambda}+u''(c-\eta s)(\dot{c}-\eta\dot{s}) = 0.$$

Using $\dot{s}$:

$$\dot{\lambda} = -u''(c-\eta s)(\dot{c}-\eta(c-\delta s)).$$

Inserting it in $\dot{\lambda}$:

$$-u''(c-\eta s)(\dot{c}-\eta(c-\delta s)) = \left[-u'(c-\eta s)+\frac{A}{w}\right](\rho+\delta)+u'(c-\eta s)\eta,$$

or

$$-u''(c-\eta s)\dot{c} = u'(c-\eta s)\left[\eta-(\rho+\delta)\right]+\frac{A}{w}(\rho+\delta)-u''(c-\eta s)(c-\delta s)\eta,$$

or

$$\dot{c} = \frac{u'(c-\eta s)}{-u''(c-\eta s)}\left[\eta-(\rho+\delta)\right]+\left[\frac{1}{-u''(c-\eta s)}\right]\frac{A}{w}(\rho+\delta)+\eta(c-\delta s). \tag{14}$$

Exercise 6. Define the function $\theta(s)$ as giving the combinations of $(c,s) = (\theta(s), s)$ such that $\dot{c} = 0$, where $\dot{c}$ is given in question 5. Show that, at the steady state level s^*

$$\theta'(s^*) = \frac{\eta\left[(\rho+2\delta)-\eta\right]}{(\rho+\delta)}.$$

Ans: Evaluating (14) at $\dot{c} = 0$ and multiplying by $-u''$ we obtain (note that at $\dot{c} = 0$, we have $c = \theta(s)$)

$$0 = u'(\theta(s) - \eta s)[\eta - (\rho + \delta)] + \frac{A}{w}(\rho + \delta) - u''(\theta(s) - \eta s)(\theta(s) - \delta s)\eta.$$

Differentiating the last expression w.r.t. s we find

$$\begin{aligned} 0 &= u''(\theta(s) - \eta s)[\eta - (\rho + \delta)]\left(\theta'(s) - \eta\right) - u'''(\theta(s) - \eta s)(\theta(s) - \delta s)\eta\left(\theta'(s) - \eta\right) \\ &\quad -u''(\theta(s) - \eta s)\left(\theta'(s) - \delta\right)\eta. \end{aligned}$$

Evaluating at the steady state $c^* = \theta(s^*) = \delta s^*$, we find

$$0 = u''\left(c^*\left[1 - \frac{\eta}{\delta}\right]\right)[\eta - (\rho + \delta)]\left(\theta'(s^*) - \eta\right) - u''\left(c^*\left[1 - \frac{\eta}{\delta}\right]\right)\left(\theta'(s^*) - \delta\right)\eta,$$

or, dividing by u'',

$$[\eta - (\rho + \delta)]\left(\theta'(s^*) - \eta\right) - \left(\theta'(s^*) - \delta\right)\eta = 0.$$

Thus

$$\theta'(s^*) = \frac{\eta[(\rho + 2\delta) - \eta]}{(\rho + \delta)}.$$

Exercise 7. Show that, since $\eta < \delta$, then

$$0 < \theta'(s^*) < \delta.$$

Ans: Since $\eta < \delta$, then $\theta'(s^*) > 0$. For the other inequality, we will show that $\theta'(s^*) - \delta < 0$:

$$\begin{aligned} \theta'(s^*) - \delta &= \frac{\eta[\rho + 2\delta - \eta]}{(\rho + \delta)} - \delta \\ &= \frac{\eta[\rho + \delta + (\delta - \eta)]}{(\rho + \delta)} - \delta \\ &= \eta\left[1 + \frac{(\delta - \eta)}{\rho + \delta}\right] - \delta \\ &= \eta\frac{(\delta - \eta)}{\rho + \delta} - (\delta - \eta) \\ &= (\delta - \eta)\left[\frac{\eta}{\rho + \delta} - 1\right] \\ &= (\delta - \eta)\left[\frac{\eta - \delta - \rho}{\rho + \delta}\right] < 0, \end{aligned}$$

since $\delta > \eta$.

From now on assume that the utility function u has CRRA γ.

Exercise 8. Show that $d\dot{c}/dc > 0$, where $\dot{c}$ is the equation found in exercise 5 evaluated at steady state values c^* and s^*. [Hint. Using that u has CRRA it is easy to show that $d\left(\dot{c}/c\right)/dc > 0$. This, of course, implies that $d\dot{c}/dc > 0$].

Ans: We have

$$\begin{aligned}
\dot{c} &= \frac{u'\left(c-\eta s\right)}{-u''\left(c-\eta s\right)}\left[\eta-\left(\rho+\delta\right)\right]+\left[\frac{1}{-u''\left(c-\eta s\right)}\right]\frac{A}{w}\left(\rho+\delta\right)+\eta\left(c-\delta s\right)\\
&= \left(c-\eta s\right)\frac{u'\left(c-\eta s\right)}{-u''\left(c-\eta s\right)\left(c-\eta s\right)}\left[\eta-\left(\rho+\delta\right)+\frac{A}{w}\frac{\left(\rho+\delta\right)}{u'\left(c-\eta s\right)}\right]+\eta\left(c-\delta s\right).
\end{aligned}$$

But with the CRRA preferences we have $-u''\left(c-\eta s\right)\left(c-\eta s\right)/u'\left(c-\eta s\right)=\gamma$. Hence

$$\dot{c}=\left(c-\eta s\right)\frac{1}{\gamma}\left[\eta-\left(\rho+\delta\right)+\frac{A}{w}\frac{\left(\rho+\delta\right)}{u'\left(c-\eta s\right)}\right]+\eta\left(c-\delta s\right).$$

Differentiating this expression w.r.t c we obtain

$$\begin{aligned}
\frac{d\dot{c}}{dc} &= \frac{1}{\gamma}\left[\eta-\left(\rho+\delta\right)+\frac{A}{w}\frac{\left(\rho+\delta\right)}{u'\left(c-\eta s\right)}\right]-\left(c-\eta s\right)\frac{1}{\gamma}\frac{A}{w}\left(\rho+\delta\right)\left[\frac{u''\left(c-\eta s\right)}{\left(u'\left(c-\eta s\right)\right)^2}\right]+\eta\\
&= \frac{1}{\gamma}\left[\eta-\left(\rho+\delta\right)+\frac{A}{w}\frac{\left(\rho+\delta\right)}{u'\left(c-\eta s\right)}\right]+\left[-\frac{u''\left(c-\eta s\right)\left(c-\eta s\right)}{u'\left(c-\eta s\right)}\right]\frac{1}{\gamma}\frac{A}{w}\frac{\left(\rho+\delta\right)}{u'\left(c-\eta s\right)}+\eta\\
&= \frac{1}{\gamma}\left[\eta-\left(\rho+\delta\right)+\frac{A}{w}\frac{\left(\rho+\delta\right)}{u'\left(c-\eta s\right)}\right]+\frac{A}{w}\frac{\left(\rho+\delta\right)}{u'\left(c-\eta s\right)}+\eta.
\end{aligned}$$

But, from question 3, we know that in the steady state

$$\eta-\left(\rho+\delta\right)+\frac{A}{w}\frac{\left(\rho+\delta\right)}{u'\left(c^*-\eta s^*\right)}=0.$$

Thus,

$$\begin{aligned}
\left.\frac{d\dot{c}}{dc}\right|_{c^*,s^*} &= \frac{A}{w}\frac{\left(\rho+\delta\right)}{u'\left(c^*-\eta s^*\right)}+\eta\\
&= \left(\rho+\delta\right) > 0.
\end{aligned}$$

Exercise 9. Draw the phase diagram. To simplify draw the phase diagram as if both the $\dot{s}=0$ and the $\dot{c}=0$ loci were linear. Put s in the x-axis and c in the y-axis. Make sure to draw arrows with the directions in which s and c will move in all relevant quadrants. Make sure that you include some arrows that intersect the $\dot{s}=0$ and $\dot{c}=0$ loci at points different from the steady state, and that the slope of these arrows are consistent with the $\dot{c}$ and $\dot{s}$ equations (either they cross vertically or horizontally). Make sure your graph is readable, the points you'll obtain depend on it! Plot the saddle path, make sure that it crosses the right quadrants, and that it has the right slope (i.e. that it is stepper or flatter than the relevant $\dot{c} = 0$ or $\dot{s} = 0$ loci).

Ans: The system of differential equations is

$$\dot{c} = (c - \eta s)\frac{1}{\gamma}\left[\eta - (\rho + \delta) + \frac{A}{w}\frac{(\rho + \delta)}{u'(c - \eta s)}\right] + \eta (c - \delta s), \tag{15}$$

and

$$\dot{s} = c - \delta s. \tag{16}$$

The dynamics of these equations are shown in the phase diagram of Figure 7 below.

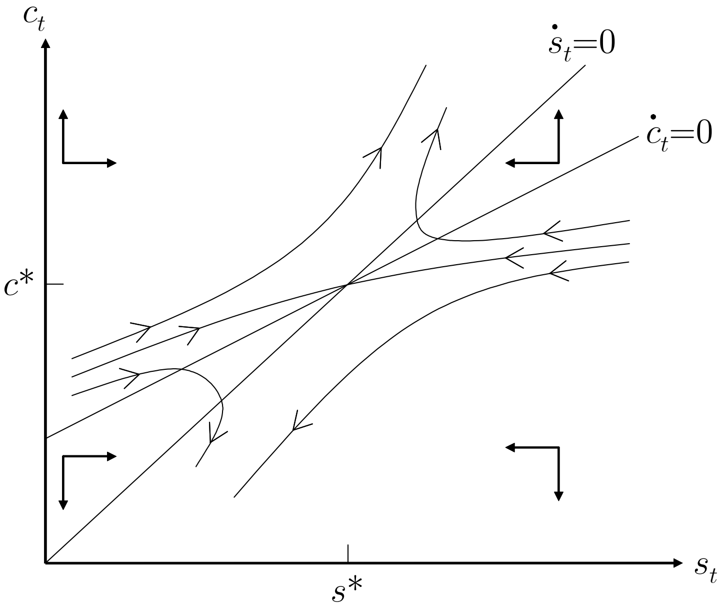

Figure 7. Phase diagram of the habit formation model.

The $\dot{s}_t = 0$ locus is a line emanating from the origin with slope $\delta > 0$. In turn, we draw the $\dot{c}_t = 0$ locus as a line with slope $0 < \theta'(s) < \delta$. Starting from a situation in which $\dot{c}_t = 0$, if we increase (decrease) the habit level then consumption will start falling (growing), as can be checked from equation (15). We summarize this with the arrows pointing south (north) to the right (left) of the $\dot{c}_t = 0$ locus. Similarly, starting from a situation in which $\dot{s}_t = 0$, if we increase (decrease) consumption then the habit level will start growing (falling), as can be seen directly from equation (16). This is summarized with the arrows pointing east (west) above (below) the $\dot{s}_t = 0$ locus.

Exercise 10. Assume that at time $t = 0$, the habit levels $s(0)$ is given by the steady state value s^*. Then, "unexpectedly", agents learn at time $t = 0$ that w will be higher, say $\bar{w} > w$ for T periods, and then it will return to the steady state value w. That is, $w(t) = \bar{w}$ for $t = (0, T)$, and $w(t) = w$ for $t \geq T$. You have to analyze the path for optimal consumption and habit for the case of this transitory increase in wages. In this page you must draw a phase diagram with both saddle path corresponding to w and $\bar{w}$, but the flow (arrows) should correspond to $\bar{w}$. Remember that s cannot "jump" at time $t = 0$ but consumption can. Remember also that at time $t = T$ the system must land continually (with respect to time) in the saddle path corresponding to w. In this phase diagram you must draw the trajectory in the c-s space that the agent will chose. Make sure that this trajectory is clearly labeled, that includes arrows showing the direction of movement, that it starts at the right height (i.e. the qualitatively correct level of c) and that if it crosses any of the $\dot{c} = 0$ or $\dot{s} = 0$ it does so with the right slope. In the next page you must also draw two figures with time t in the horizontal axis and with the optimal path of c (one figure) and s in the vertical axis. Start these figures at some time $t < 0$ and clearly label the time periods $t = 0$, $t = T$ and include horizontal lines for the steady state values for consumption $c^*$and $\bar{c}^*$ and habit $s^*, \bar{s}^*$ that correspond to the values of w and $\bar{w}$, respectively. You should obtain the qualitative features of these time trajectories from your previous figure. Make sure your graphs are readable, the points you'll obtain depend on it!

Ans: From exercise 4, we know that in the steady state

$$\frac{dc^*}{dw} > 0.$$

Since $c^* = \delta s^*$, it follows that

$$\frac{ds^*}{dw} > 0,$$

as well. That is, a (permanent) rise in the wage rate will unambiguously increase the steady state levels of consumption and the habit level. This can be seen graphically in Figure 8 below. Indeed, a rise in w will shift the $\dot{c}_t = 0$ locus upward, and will have no effect on the $\dot{s}_t = 0$ locus. Thus, the new steady state values of c and s, $(\bar{c}^*, \bar{s}^*)$ are such that $\bar{c}^* > c^*$ and $\bar{s}^* > s^*$. The intuition for this result is clear: a permanent rise in w makes the individual richer, so it is reasonable to expect that he will enjoy a higher level of consumption (and habit level) in the long-run.

Notice that according to Figure 8, if the wage rise were permanent, consumption would rise on impact to $\bar{c}$ and settle immediately in the saddle path that characterizes the new dynamic system. Thereafter, consumption and the habit level would increase monotonically towards their new steady-state values. However, in our case the individual knows that at time $t = T$ the wage rate will fall back to its previous value. Thus, the dynamics of the system will be dictated by the $\dot{c}' = 0$ and $\dot{s} = 0$ loci for $t < T$, and by the old $\dot{c} = 0$ and $\dot{s} = 0$ loci for $t \geq T$. Notice that consumption cannot jump at time $t = T$ since this would violate the Euler equation at that time. Hence, the dynamics of capital and consumption for $t \in [0, T)$ must be such that the system reaches the old saddle path at time $t = T$ in a continuous fashion.

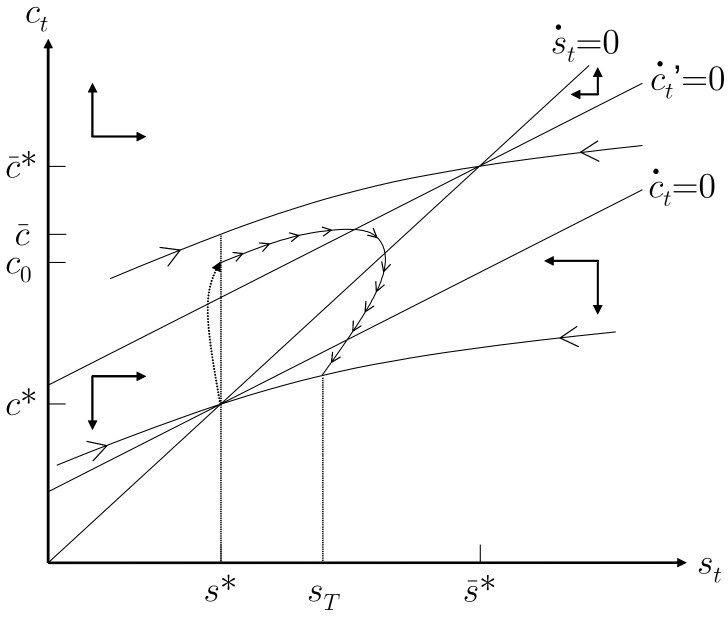

Figure 8. The impact of a temporary rise in the wage rate, w, if c_0 is above the $\dot{c}' = 0$ locus.

Moreover, consumption cannot stay constant or fall on impact. Otherwise, consumption would be decreasing afterwards and then will have to jump upwards at time $t = T$ so that the economy reaches the old saddle path. However, consumption cannot rise on impact all the way up to $\bar{c}$ either. If it did, consumption would follow the non-decreasing dynamics dictated by the new saddle path for $t \in [s, T)$ and then would have to jump downwards at time $t = T$ so that the economy lands in the old saddle path. These two observations imply that consumption must *rise on impact* to someplace strictly between c^* and $\bar{c}$, like c_0 as shown in Figure 8 above. This figure depicts the case where c_0 is above the $\dot{c}' = 0$ locus. After the initial jump in consumption, consumption and the habit level rise until the path crosses the $\dot{c}' = 0$ locus, when consumption starts to fall and the habit level keeps rising. Once the path crosses the $\dot{s} = 0$ locus, both consumption and the habit level fall until at time T the trajectory lands continuously in the old saddle path. Afterwards, consumption and the habit level monotonically decrease towards their old steady-state levels. Figure 9 shows the case where c_0 is below the $\dot{c}' = 0$ locus. The dynamics of consumption and the habit level are similar to that of the previous case once the trajectory crossed the $\dot{c}' = 0$ locus.

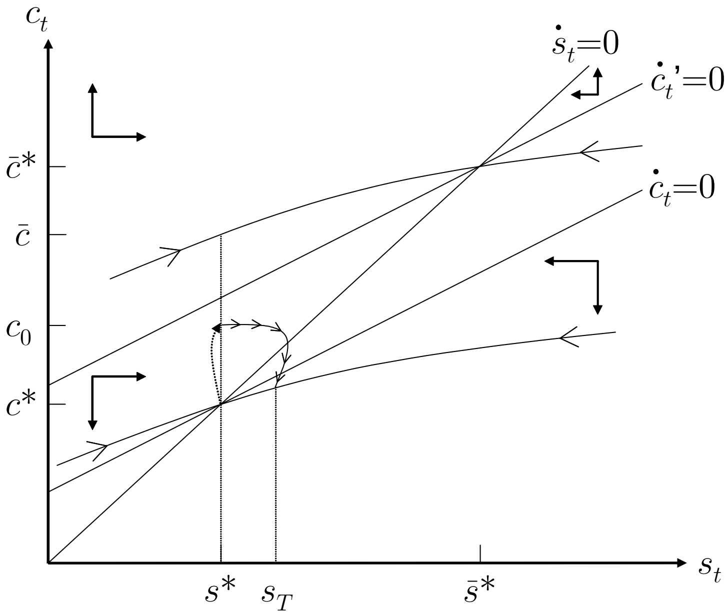

Figure 9. The impact of a temporary rise in the wage rate, w, if c_0 is below the $\dot{c}' = 0$ locus.

Notice that how much consumption falls on impact depends on how farther away in the future the wage rate will return to its previous level, T. In the limit as $T \to 0$ (i.e., the wage rate is constant), $c_0 \to c*$, whereas as $T \to +\infty$ (i.e., the wage increase is permanent), $c_0 \to \bar{c}$.

The dynamics of consumption and the habit level for each case can be easily read off the applicable saddle path diagram. This is left as an exercise.