Theory of Income, Fall 2008
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Problem Set 4

1 Neoclassical Growth Model: Exogenous Growth

Let the exogenous time augmenting productivity growth gross rate be $\lambda > 1$. Feasibility is given by

$$c_t + k_{t+1} = F\left(k_t, \lambda^t n_t\right) - (1-\delta) k_t,$$

where F is a neoclassical constant returns to scale production function, k_t is capital, c_t is consumption, n_t is labor, and δ is the depreciation rate of capital. The endowment of time per period is normalized to 1, so that leisure is $l_t = 1 - n_t$. Preferences are given by

$$\sum_{t=0}^{\infty} \beta^t v\left(c_t, 1 - n_t\right).$$

The consumer has a budget constraint given by

$$\sum_{t=0}^{\infty} p_t \left[c_t + x_t\right] = \sum_{t=0}^{\infty} p_t \left[w_t n_t + k_t v_t\right],$$

and the law of motion of capital is

$$k_{t+1} = x_t + (1-\delta) k_t,$$

where p_t is the Arrow-Debreu price of consumption goods at time t in terms of time zero consumption good, and w_t and v_t are the real wage and rental rate of capital in terms of consumption goods at period t.

The firm's problem is

$$\max_{k_t, n_t} \; F\left(k_t, \lambda^t n_t\right) - w_t n_t - v_t k_t.$$

Exercise 1. Let r_t be the time t interest rate, i.e. $p_t/p_{t+1} = 1 + r_t$. Use the budget constraint of the household, and the law of motion of capital to show that, as long as $x_t > 0$,

$$v_{t+1} = r_t + \delta.$$

[Hint: Consider an investment of 1 at t, renting it on $t+1$ and consuming the undepreciated capital at $t+1$].

Ans: In fact, there is no need to use the budget constraint of the household to obtain this result. Consider instead the following arbitrage argument: the household borrows one unit of good at period t and invests it to construct capital and rent it to the firm the following period. The next period the household has to repay $(1 + r_t)$ and receives $(1 - \delta) + v_{t+1}$. That is, since a fraction δ of the capital depreciates, the household receives $(1 - \delta)$ units of capital plus the rental rate v_{t+1}. Since net cash-flows at t are zero, they have to be zero at $t + 1$ as well. Thus,

$$1 - \delta + v_{t+1} - (1 + r_t) = 0$$

or

$$v_{t+1} = r_t + \delta,$$

as desired.

Exercise 2. Write down the first order condition w.r.t. c_t, n_t and k_{t+1}. Use μ for the multiplier on the budget constraint [Hint: Replace x_t in the household budget constraint using the law of motion of capital]. Combine the FOC for c_t for two consecutive periods to obtain a relationship between the marginal rate of substitution between c_t and c_{t+1} and r_t. Combine the FOC with respect to c_t and n_t to obtain a relationship between the marginal rate of substitution between n_t and c_t and w_t.

Ans: Replacing x_t into the household budget constraint, we note that the latter becomes

$$\sum_{t=0}^{\infty} p_t \left[w_t n_t - c_t + (v_t + 1 - \delta) k_t - k_{t+1} \right] = 0.$$

Thus, the Lagrangian of the household's problem is given by

$$\max_{\{c_t, n_t, k_{t+1}\}} \sum_{t=0}^{\infty} \beta^t U (c_t, 1 - n_t) + \mu \left[\sum_{t=0}^{\infty} p_t \left[w_t n_t - c_t + (v_t + 1 - \delta) k_t - k_{t+1} \right] \right].$$

The FOCs w.r.t. c_t, n_t and k_{t+1} are, respectively

$$\beta^t U_c (c_t, 1 - n_t) = \mu p_t, \quad t = 0, 1, 2, .. \tag{1}$$

$$\beta^t U_{1-n} (c_t, 1 - n_t) = \mu p_t w_t, \quad t = 0, 1, 2, .. \tag{2}$$

and

$$-p_t + p_{t+1} (v_{t+1} + 1 - \delta) = 0, \quad t = 0, 1, 2, .. \tag{3}$$

So, we see that the last equation implies the no-arbitrage condition $(1 + r_t) \equiv \frac{p_t}{p_{t+1}} = v_{t+1} + 1 - \delta$.

Now divide (1) at period $t+1$ with the same equation at period t to get

$$\frac{\beta^{t+1} U_c\left(c_{t+1}, 1-n_{t+1}\right)}{\beta^t U_c\left(c_t, 1-n_t\right)}=\frac{p_{t+1}}{p_t}.$$

Canceling terms and using $(1+r_t) \equiv \frac{p_t}{p_{t+1}}$ we get

$$\frac{\beta U_c\left(c_{t+1}, 1-n_{t+1}\right)}{U_c\left(c_t, 1-n_t\right)}=\frac{1}{1+r_t}, \tag{4}$$

that is, the marginal rate of substitution between consumption at $t+1$ and t must equal its relative price. This is the standard Euler equation.

Dividing (1) with (2) we obtain

$$\frac{U_c\left(c_t, 1-n_t\right)}{U_{1-n}\left(c_t, 1-n_t\right)}=\frac{1}{w_t}, \tag{5}$$

that is, the marginal rate of substitution between consumption and leisure must equal the relative price of consumption versus leisure.

Finally, the firm's problem is

$$\max_{k_t, n_t} F\left(k_t, \lambda^t n_t\right)-w_t n_t-\upsilon_t k_t,$$

which gives the following optimality conditions:

$$F_k\left(k_t, \lambda^t n_t\right)=\upsilon_t, \tag{6}$$

and

$$F_n\left(k_t, \lambda^t n_t\right) \lambda^t=w_t. \tag{7}$$

Exercise 3. Use the expression for the rental rate of capital υ_t for $t \geq 1$ and the law of motion of capital for $t \geq 0$ (solving for x_t) to show that the household's budget constraint can be written as

$$\sum_{t=0}^{\infty} p_t c_t=p_0 k_0\left(\upsilon_0+1-\delta\right)+\sum_{t=0}^{\infty} p_t w_t n_t.$$

Ans: Start with the budget constraint

$$\sum_{t=0}^{\infty} p_t\left[c_t+k_{t+1}-(1-\delta) k_t\right]=\sum_{t=0}^{\infty} p_t\left[w_t n_t+\upsilon_t k_t\right].$$

Consider the terms involving only capital:

$$\begin{aligned}
\sum_{t=0}^{\infty} p_t \left[k_{t+1} - (1-\delta+\upsilon_t) k_t\right] &= \sum_{t=0}^{\infty} p_t k_{t+1} - \sum_{t=0}^{\infty} p_t (1-\delta+\upsilon_t) k_t \\
&= \sum_{t=0}^{\infty} p_t k_{t+1} - \sum_{t=1}^{\infty} p_t (1-\delta+\upsilon_t) k_t - p_0 (1-\delta+\upsilon_0) k_0 \\
&= \sum_{t=0}^{\infty} p_t k_{t+1} - \sum_{t=0}^{\infty} p_{t+1} (1-\delta+\upsilon_{t+1}) k_{t+1} - p_0 (1-\delta+\upsilon_0) k_0 \\
&= \sum_{t=0}^{\infty} \left[p_t - p_{t+1}(1-\delta+\upsilon_{t+1})\right] k_{t+1} - p_0 (1-\delta+\upsilon_0) k_0.
\end{aligned}$$

Thus, the budget constraint becomes

$$\sum_{t=0}^{\infty} p_t c_t = \sum_{t=0}^{\infty} p_t w_t n_t + \sum_{t=0}^{\infty} \left[p_{t+1}(1-\delta+\upsilon_{t+1}) - p_t\right] k_{t+1} + p_0 (1-\delta+\upsilon_0) k_0.$$

Using the equilibrium condition $p_{t+1}(1-\delta+\upsilon_{t+1}) = p_t$, this equation reduces to

$$\sum_{t=0}^{\infty} p_t c_t = p_0 (1-\delta+\upsilon_0) k_0 + \sum_{t=0}^{\infty} p_t w_t n_t,$$

as was to be shown.

Definition: A *balanced growth path* is given by an initial capital k_0 and λ such that it is optimal to set

$$\begin{aligned}
c_t &= c_0 \lambda^t, \\
k_t &= k_0 \lambda^t, \\
n_t &= n_0, \\
w_t &= \lambda^t w_0, \\
r_t &= r_0,
\end{aligned}$$

for all $t \geq 0$.

Exercise 4. Write down the FOC for the household imposing a balanced growth path. Use the FOC for the household and firm's problem as well as feasibility to write down the system of equations in c_0, n_0, k_0, w_0, r_0 that a balanced growth path must satisfy.

Ans: Evaluating the firm's FOCs at the balanced growth path we get

$$F_k\left(k_0\lambda^t, n_0\lambda^t\right) = r_0 + \delta,$$

and

$$F_n\left(k_0\lambda^t, n_0\lambda^t\right)\lambda^t = w_0\lambda^t.$$

Since $F(k,n)$ is homogeneous of degree one, all its partial derivatives are homogeneous of degree zero. Thus, we can rewrite the above equations as

$$F_k(k_0, n_0) = r_0 + \delta, \tag{8}$$

and

$$F_n(k_0, n_0) = w_0. \tag{9}$$

The household's FOCs evaluated at the balanced growth path are

$$U_c\left(c_0\lambda^t, 1-n_0\right) = \beta(1+r_0)\,U_c\left(c_0\lambda^{t+1}, 1-n_0\right), \tag{10}$$

and

$$\frac{U_{1-n}\left(c_0\lambda^t, 1-n_0\right)}{U_c\left(c_0\lambda^t, 1-n_0\right)} = w_0\lambda^t. \tag{11}$$

Finally, feasibility in the balanced growth path is given by

$$c_0\lambda^t + k_0\lambda^{t+1} = F\left(k_0\lambda^t, \lambda^t n_0\right) + (1-\delta)\,k_0\lambda^t,$$

or, using that $F\left(k_0\lambda^t, \lambda^t n_0\right) = \lambda^t F(k_0, n_0)$,

$$c_0 + k_0\lambda = F(k_0, n_0) + (1-\delta)\,k_0. \tag{12}$$

Note that we have 5 equations, (8), (9), (10), (11) and (12), in 4 unknowns: (c_0, n_0, w_0, r_0) (recall that k_0 is given). Does that mean that the model is overidentified? No, in fact the equations that come from the firm's problem determine only one equilibrium quantity: the capital to labor ratio. To see this, dividing (8) and (9) and using the homogeneity of degree zero of the partial derivatives we obtain

$$\frac{F_k(k_0/n_0, 1)}{F_n(k_0/n_0, 1)} = \frac{r_0+\delta}{w_0}, \tag{13}$$

so we can only obtain the capital to labor ratio. The intuition for this is that since technology displays constant returns to scale, the scale of the firm remains undetermined in equilibrium. In other words, the absolute demand for capital and labor is not determined by looking at the firm's problem.

Therefore, the quantities (c_0, n_0, w_0, r_0) are pinned down by the system of equations given by (10), (11), (12) and (13).

Exercise 5. Show that if preferences are of the form

$$v(c, 1-n) = \frac{c^{1-\gamma}}{1-\gamma} h(1-n), \tag{14}$$

for $\gamma \neq 1$, or

$$v(c, 1-n) = \log c + h(1-n),$$

then there is a balanced growth path.

Ans: We must show that the above system of equations has a solution. Let's focus on the case $\gamma \neq 1$. From the household's conditions (10) and (11) we obtain

$$\left(c_0\lambda^t\right)^{-\gamma} \upsilon(1-n_0) = \beta(1+r_0)\left(c_0\lambda^{t+1}\right)^{-\gamma} \upsilon(1-n_0),$$

and

$$\frac{(1-\gamma)^{-1}\left(c_0\lambda^t\right)^{1-\gamma} \upsilon'(1-n_0)}{\left(c_0\lambda^t\right)^{-\gamma} \upsilon(1-n_0)} = w_0\lambda^t.$$

The first equation gives

$$1 = \beta(1+r_0)\lambda^{-\gamma}, \tag{15}$$

so we solved for the equilibrium interest rate $r_0 = \lambda^{\gamma}/\beta - 1$. The second equation implies

$$\frac{c_0}{1-\gamma}\frac{\upsilon'(1-n_0)}{\upsilon(1-n_0)} = w_0,$$

which can be used to solve for consumption as a function of labor and the wage rate:

$$c_0 = f(n_0, w_0) \equiv (1-\gamma) w_0 \frac{\upsilon(1-n_0)}{\upsilon'(1-n_0)}.$$

Using equations (15) and (13) we can solve for the capital to labor ratio as a function of w_0 and parameters:

$$\frac{F_k(k_0/n_0, 1)}{F_n(k_0/n_0, 1)} = \frac{\lambda^{\gamma}/\beta + \delta - 1}{w_0},$$

or

$$k_0/n_0 \equiv g(w_0),$$

where $g(w_0)$ solves

$$\frac{F_k(g(w_0), 1)}{F_n(g(w_0), 1)} = \frac{\lambda^{\gamma}/\beta + \delta - 1}{w_0}.$$

Using feasibility and the definition of c_0 and k_0/n_0 we notice that

$$c_0 + k_0(\lambda - (1-\delta)) = F(k_0, n_0),$$

or

$$f(n_0, w_0) + \frac{k_0}{n_0} n_0 (\lambda - (1-\delta)) = n_0 F\left(\frac{k_0}{n_0}, 1\right),$$

or

$$f(n_0, w_0) + g(w_0) n_0 (\lambda - (1-\delta)) = n_0 F(g(w_0), 1).$$

Consider solving this equation for n_0 as a function of w_0:

$$n_0 \equiv h(w_0).$$

Finally, since k_0 is given, we can use the equation $k_0/n_0 \equiv g(w_0)$ or $k_0 = h(w_0) g(w_0)$ to solve for w_0 as a function of k_0 (one equation in one unknown). Once we know w_0 we can solve for the rest of the equilibrium variables.

The case of log utility can be done in exactly the same way and is left as an exercise.

Exercise 6. Assume that $v(c,l)$ is strictly concave and increasing in (c,l) and have the form described in (14). Consider first the case where $\gamma \in (0,1)$. What are the properties of $h(l)$? i.e. is it positive or negative, increasing or decreasing, concave or convex? Next, consider the case where $\gamma = 1$. What are the properties of $h(l)$? i.e. is it increasing or decreasing, concave or convex? Finally, consider the case where $\gamma > 1$. What are the properties of $h(l)$? i.e. is it positive or negative, increasing or decreasing, concave or convex?

Ans: Consider the following first and second order derivatives (here $l = 1 - n$ is leisure). For $\gamma \neq 1$,

$$\begin{aligned}
v_c(c,l) &= c^{-\gamma} h(l) > 0 \\
v_l(c,l) &= \frac{c^{1-\gamma}}{1-\gamma} h'(l) > 0 \\
v_{cc}(c,l) &= -\gamma c^{-\gamma-1} h(l) < 0 \\
v_{ll}(c,l) &= \frac{c^{1-\gamma}}{1-\gamma} h''(l) < 0,
\end{aligned}$$

and for $\gamma = 1$,

$$\begin{aligned}
v_c(c,l) &= 1/c > 0, \\
v_l(c,l) &= h'(l) > 0, \\
v_{cc}(c,l) &= -1/c^2, \\
v_{ll}(c,l) &= h''(l).
\end{aligned}$$

i) If $\gamma \in (0,1)$, from $v_c > 0$ we have $h(l) > 0$, from $v_l > 0$ we require $h'(l) > 0$ and from $v_{ll} < 0$ we obtain $h''(l) < 0$ ($h(l)$ is positive, increasing and concave).

ii) If $\gamma = 1$, $h(l)$ can be positive or negative. From $v_l > 0$ we obtain $h'(l) > 0$ and from $v_{ll} < 0$ we require $h''(l) < 0$ ($h(l)$ is increasing and concave).

iii) If $\gamma > 1$ we have $(1-\gamma) < 0$. From $v_c > 0$ we require $h(l) > 0$, from $v_l > 0$ we obtain $h'(l) < 0$ and from $v_{ll} < 0$ we require $h''(l) > 0$ ($h(l)$ is positive, decreasing and convex).

Exercise 7. Let $v(c, 1-n)$ be

$$v(c, 1-n) = g(c - n^{\sigma}/\sigma),$$

for $\sigma > 1$ and g strictly increasing and concave. What is the income elasticity of the labor supply for this utility function? Show that this preferences are inconsistent with a balanced growth path. [Hint: In a BGP we must have

$$\frac{v_l(c_0\lambda^t, 1-n_0)}{v_c(c_0\lambda^t, 1-n_0)} = w_0\lambda^t,$$

but

$$\frac{v_l(c_0\lambda^t, 1-n_0)}{v_c(c_0\lambda^t, 1-n_0)} = \frac{g'(c_0\lambda^t - n_0^{\sigma}/\sigma)}{g'(c_0\lambda^t - n_0^{\sigma}/\sigma)} n_0^{\sigma-1} = n_0^{\sigma-1} = w_0\lambda^t,$$

so compare the LHS and RHS of the last equality].

Ans: Following the hint, if there exists a BGP, it must satisfy

$$n_0^{\sigma-1} = w_0\lambda^t.$$

But this equation implies that labor is not constant over time, so these preferences are inconsistent with a BGP. In general, the condition equating the marginal rate of substitution with the wage rate is

$$\frac{g'(c_t - n_t^{\sigma}/\sigma)}{g'(c_t - n_t^{\sigma}/\sigma)} n_t^{\sigma-1} = w_t,$$

so that the elasticity of labor supply with respect to the wage rate is $1/(\sigma - 1)$. For a balanced growth path to exist, this elasticity must be zero.

Exercise 8. Show that if the economy admits a balanced growth path for an open set of parameters β, λ and δ, preferences must be of the form in (14). [Hint: Write down an Euler equation-like expression relating the marginal rate of substitution of consumption between t and $t+1$ with r. Impose the balanced growth condition on this, noticing that this expression must be satisfied for all t. Differentiate this expression with respect to t to obtain a differential equation, whose solution implies that v is of the form $B(1-n) + c^{1-\gamma(1-n)}h(1-n)$ or $B(1-n) + \log c + h(1-n)$ where γ is a function of $(1-n)$. Use this, and the condition that marginal rates of substitution equal relative prices on a balanced growth path to establish the required result].

Ans: First, we know that in any balanced growth path the following equation must hold for all t

$$U_c\left(c_0\lambda^t, 1-n_0\right) = \beta\left(1+r_0\right)U_c\left(c_0\lambda^{t+1}, 1-n_0\right). \tag{16}$$

We want to differentiate the last expression with respect to t. To do that, recall that

$$\lambda^t = e^{\log \lambda^t} = e^{t\log\lambda},$$

so that $d\lambda^t/dt = \lambda^t \log\lambda$. Taking the derivative w.r.t. t we obtain

$$U_c\left(c_0\lambda^t, 1-n_0\right)c_0\lambda^t\log\lambda = \beta\left(1+r_0\right)U_{cc}\left(c_0\lambda^{t+1}, 1-n_0\right)c_0\lambda^{t+1}\log\lambda.$$

Dividing the last expression by equation (16) we find

$$\frac{U_{cc}\left(c_0\lambda^t, 1-n_0\right)}{U_c\left(c_0\lambda^t, 1-n_0\right)}c_0\lambda^t = \frac{U_{cc}\left(c_0\lambda^{t+1}, 1-n_0\right)}{U_c\left(c_0\lambda^{t+1}, 1-n_0\right)}c_0\lambda^{t+1}.$$

Moreover, noting that $c_t = c_0\lambda^t$, this expression implies that any BGP must satisfy

$$\frac{U_{cc}\left(c_t, 1-n_0\right)}{U_c\left(c_t, 1-n_0\right)}c_t = \frac{U_{cc}\left(c_{t+1}, 1-n_0\right)}{U_c\left(c_{t+1}, 1-n_0\right)}c_{t+1},$$

for all c_t. Equivalently,

$$\frac{U_{cc}\left(c, 1-n\right)}{U_c\left(c, 1-n\right)}c = \text{constant (independent of } c\text{)}.$$

Note that it could be the case that the constant may depend on n. So, let us rewrite the last condition, without loss of generality, as

$$\frac{U_{cc}\left(c, 1-n\right)}{U_c\left(c, 1-n\right)}c = -\gamma\left(1-n\right)$$

where $\gamma\left(1-n\right)$ means that the constant is function of $\left(1-n\right)$. To avoid cumbersome notation, keep n implicit and rewrite the last equation as

$$\frac{u''\left(c\right)}{u'\left(c\right)}c = -\gamma$$

where $u\left(c\right) \equiv U\left(c, 1-n\right)$. That's a second order differential equation, so we will find its solution. Rewrite it as

$$u''\left(c\right)c = -\gamma u'\left(c\right)$$

Now we will integrate both sides of the equation with respect to c. For the left side expression, use integration by parts, i.e.,

$$\int u''(c)\,cdc = u'(c)\,c - \int u'(c)\,dc.$$

Then

$$u'(c)\,c - \int u'(c)\,dc = -\gamma \int u'(c)\,dc,$$

or

$$u'(c)\,c = (1-\gamma)\int u'(c)\,dc$$

or

$$u'(c)\,c - (1-\gamma)\,u(c) = A,$$

where A is a constant of integration. Multiply both sides of the equation by $c^{-(2-\gamma)}$ to obtain

$$u'(c)\,c^{-(1-\gamma)} - (1-\gamma)\,u(c)\,c^{-(2-\gamma)} = Ac^{-(2-\gamma)}.$$

Now note that

$$\frac{d\left(u(c)\,c^{-(1-\gamma)}\right)}{dc} = u'(c)\,c^{-(1-\gamma)} - (1-\gamma)\,u(c)\,c^{-(2-\gamma)}.$$

Thus, we have

$$\frac{d\left(u(c)\,c^{-(1-\gamma)}\right)}{dc} = Ac^{-(2-\gamma)}.$$

Finally, integrate the last equation with respect to c to obtain

$$u(c)\,c^{-(1-\gamma)} = -\frac{Ac^{-(1-\gamma)}}{(1-\gamma)} + \frac{\upsilon}{1-\gamma}$$

where $\upsilon/(1-\gamma)$ is another constant of integration. Now solving for $u(c)$ we get

$$u(c) = B + \frac{c^{(1-\gamma)}}{(1-\gamma)}\upsilon,$$

or, making n explicit,

$$U(c, 1-n) = B(1-n) + \frac{c^{(1-\gamma(1-n))}}{(1-\gamma(1-n))}\upsilon(1-n)$$

This is the solution of the differential equation given by the intertemporal condition, where we make explicit the fact that in general the constants will depend on n. Now we must also make sure that the intratemporal condition is satisfied. Recall the intratemporal condition:

$$\frac{U_{1-n}\left(c_0\lambda^t, 1-n_0\right)}{U_c\left(c_0\lambda^t, 1-n_0\right)} = w_0\lambda^t.$$

Using the U obtained above and letting $\tilde{v}\left(1-n\right) \equiv \frac{v(1-n)}{1-\gamma(1-n)}$ the last equation reads

$$\frac{B'\left(1-n_0\right) + \left(c_0\lambda^t\right)^{1-\gamma(1-n_0)}\left[\tilde{v}'\left(1-n_0\right) - \gamma'\left(1-n_0\right)\tilde{v}\left(1-n_0\right)\log(c_0\lambda^t)\right]}{\left(1-\gamma\left(1-n_0\right)\right)\tilde{v}\left(1-n_0\right)\left(c_0\lambda^t\right)^{-\gamma(1-n)}} = w_0\lambda^t.$$

Now, notice that the above equation must hold for all t. It follows that it is necessary and sufficient that

$$B'\left(1-n_0\right) = \gamma'\left(1-n_0\right) = 0,$$

otherwise the equation will depend on t, concluding that the only preferences consistent with a BGP are

$$U\left(c, 1-n\right) = a + \frac{c^{1-\gamma}}{1-\gamma}v\left(1-n\right),$$

where a and γ are constants and $\gamma \neq 1$.

The case of log utility can be done in exactly the same way and is left as an exercise.

2 Deriving the Euler Equation in Continuous Time

In this question we obtain the continuous time Euler Equation by taking limits of the discrete time Euler equation. The point of this is to realize that although the expression for the continuous time counterpart is less intuitive that the one for the discrete time -which has the natural interpretation of equating marginal cost to marginal benefit- they are really the same.

The idea is to consider a sequence of discrete time dynamic programing problems. In each problem the length of time between periods where the state is decided is denoted by Δ. Decisions are taken at times $0,\ \Delta,\ 2\Delta,\ 3\Delta, ...$ The sequence of states to be chosen is

$$\left\{x_{\Delta(t+1)}\right\}_{t=0}^{\infty} = \left\{x_{\Delta}, x_{2\Delta}, x_{3\Delta}, ...\right\},$$

where x_0 is given. Setting $\Delta = 1$ we obtain the standard problem analyzed in the class notes. We adjust the discount factor for each problem accordingly letting

$$\beta = \frac{1}{1+\Delta\rho},$$

so that ρ has the interpretation of a discount rate.

For each Δ we write the period return function during the interval of time of length Δ as

F, and the corresponding return function per unit of time as $\hat{F}$. They satisfy:

$$F\left(x_t, x_{t+\Delta}\right) \equiv \Delta\ \hat{F}\left(x_t, \frac{1}{\Delta}\left(x_{t+\Delta} - x_t\right)\right),$$

where $t = i\Delta$ for some integer i. It helps to write these return functions as

$$F\left(x, y\right) \equiv \Delta\ \hat{F}\left(x, \frac{1}{\Delta}\left(y - x\right)\right),$$

or

$$F\left(x, \dot{x}\Delta + x\right) = \Delta\ \hat{F}\left(x, \dot{x}\right),$$

where we define $\dot{x}$ as the change per unit of time on the state:

$$\dot{x} \equiv \frac{y - x}{\Delta},$$

or using time subscripts:

$$\dot{x}_t = \frac{x_{t+\Delta} - x_t}{\Delta},$$

for $t = i\Delta$ and any integer i.

Likewise we can define the feasible correspondence $\hat{\Gamma}$ for the change per unit of time $\dot{x}$ in terms of the feasible correspondence for levels Γ as

$$\hat{\Gamma}\left(x\right) = \left\{\dot{x} : y \in \Gamma\left(x\right),\ y = \dot{x}\Delta + x\right\}.$$

Thus for each Δ we consider the problem

$$\max_{\left\{x_{(t+1)\Delta}\right\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \left(\frac{1}{1+\Delta\rho}\right)^t\ F\left(x_{t\Delta},\ x_{(t+1)\Delta}\right),$$

subject to

$$x_{(t+1)\Delta} \in \Gamma\left(x_{t\Delta}\right)\ ,$$

for all $t \geq 0$, where x_0 given.

Equivalently, we can write this problem as a choice of the sequence of discrete time changes $\left\{\dot{x}_{t\Delta}\right\}_{t=0}^{\infty}$:

$$\max_{\left\{\dot{x}_{t\Delta}\right\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \left(\frac{1}{1+\Delta\rho}\right)^t\ \Delta\ \hat{F}\left(x_{t\Delta},\ \dot{x}_{t\Delta}\right),$$

subject to

$$\dot{x}_{t\Delta} \in \hat{\Gamma}\left(x_{t\Delta}\right),$$

$$x_{t\Delta+\Delta} = x_{t\Delta} + \dot{x}_{t\Delta}\ \Delta,$$

for all $t \geq 0$, and for given x_0.

We emphasize that the optimal sequence $\left\{x_{(t+1)\Delta}\right\}_{t=0}^{\infty}$ that solves the problem with interval of length Δ is a function of Δ.

For future reference, we also introduce the notation for the changes per unit of time on the change per unit of time of the state, denoting it by $\ddot{x}_t$:

$$\ddot{x}_t \equiv \frac{1}{\Delta}\left(\dot{x}_{t+\Delta} - \dot{x}_t\right),$$

for all $t = i\Delta$ and an integer i.

Exercise 1. Derive a formula for F_y and F_x in terms of $\partial\hat{F}/\partial x$ and $\partial\hat{F}/\partial\dot{x}$. In particular use the relationship between F and $\hat{F}$ to show that

$$F_y\left(x, \dot{x}\Delta + x\right) = \frac{\partial}{\partial\dot{x}}\hat{F}\left(x, \dot{x}\right),$$

$$F_x\left(x, \dot{x}\Delta + x\right) = \Delta\frac{\partial}{\partial x}\hat{F}\left(x, \dot{x}\right) - \frac{\partial}{\partial\dot{x}}\hat{F}\left(x, \dot{x}\right).$$

Ans: Differentiating

$$F\left(x, \dot{x}\Delta + x\right) = \Delta\ \hat{F}\left(x, \dot{x}\right),$$

w.r.t. x and $\dot{x}$ we obtain:

$$F_x\left(x, \dot{x}\Delta + x\right) + F_y\left(x, \dot{x}\Delta + x\right) = \Delta\frac{\partial}{\partial x}\hat{F}\left(x, \dot{x}\right),$$

and

$$F_y\left(x, \dot{x}\Delta + x\right) = \frac{\partial}{\partial\dot{x}}\hat{F}\left(x, \dot{x}\right).$$

Inserting this in the previous expression we have

$$F_x\left(x, \dot{x}\Delta + x\right) = \Delta\frac{\partial}{\partial x}\hat{F}\left(x, \dot{x}\right) - \frac{\partial}{\partial\dot{x}}\hat{F}\left(x, \dot{x}\right).$$

Exercise 2. Write the Euler Equations for the problem where we chose the sequence of levels of the state: $\left\{x_{(t+1)\Delta}\right\}_{t=0}^{\infty}$. Your Euler equation should involve $F_y,\ F_x,\ \Delta,\ \rho$ and be evaluated at $x_t,\ x_{t+\Delta}$ and $x_{t+2\Delta}$. [Hint: This is the standard problem].

Ans: The Euler equation is

$$F_y\left(x_t, x_{t+\Delta}\right) + \left(\frac{1}{1+\Delta\rho}\right)F_x\left(x_{t+\Delta}, x_{t+2\Delta}\right) = 0.$$

Exercise 3. Rewrite the Euler equation obtained in 2 replacing the $x_{t+\Delta}$ in F_y in terms of Δ , x_t and $\dot{x}_t$, and replacing the $x_{t+2\Delta}$ in F_x in terms of Δ , $x_{t+\Delta}$ and $\dot{x}_{t+\Delta}$.

Ans: Using the definition of $\dot{x}_t$ we can write the EE as

$$F_y\left(x_t, \dot{x}_t\Delta + x_t\right) + \left(\frac{1}{1+\Delta\rho}\right) F_x\left(x_{t+\Delta}, \dot{x}_{t+\Delta}\Delta + x_{t+\Delta}\right) = 0.$$

Exercise 4. Use the relationship between the derivatives of F and $\hat{F}$ found in 1 into your expression for the Euler equation found in exercise 3.

Ans: Using the relationship between the derivatives of F and $\hat{F}$ we can write this EE as

$$\frac{\partial}{\partial \dot{x}}\hat{F}\left(x_t, \dot{x}_t\right) + \left(\frac{1}{1+\Delta\rho}\right)\left[\Delta\frac{\partial}{\partial x}\hat{F}\left(x_{t+\Delta}, \dot{x}_{t+\Delta}\right) - \frac{\partial}{\partial \dot{x}}\hat{F}\left(x_{t+\Delta}, \dot{x}_{t+\Delta}\right)\right] = 0.$$

Exercise 5. Show that by rearranging the terms in the expression found in 4, the Euler equation can be written as:

$$\frac{1}{\Delta}\left[\frac{\partial}{\partial \dot{x}}\hat{F}\left(x_{t+\Delta}, \dot{x}_{t+\Delta}\right) - \frac{\partial}{\partial \dot{x}}\hat{F}\left(x_t, \dot{x}_t\right)\right] = \rho\frac{\partial}{\partial \dot{x}}\hat{F}\left(x_t, \dot{x}_t\right) + \frac{\partial}{\partial x}\hat{F}\left(x_{t+\Delta}, \dot{x}_{t+\Delta}\right).$$

Ans: Multiplying both sides of the answer in 4 by $(1+\Delta\rho)$ yields

$$(1+\Delta\rho)\frac{\partial}{\partial \dot{x}}\hat{F}\left(x_t, \dot{x}_t\right) + \left[\Delta\frac{\partial}{\partial x}\hat{F}\left(x_{t+\Delta}, \dot{x}_{t+\Delta}\right) - \frac{\partial}{\partial \dot{x}}\hat{F}\left(x_{t+\Delta}, \dot{x}_{t+\Delta}\right)\right] = 0,$$

or rearranging,

$$\frac{1}{\Delta}\left[\frac{\partial}{\partial \dot{x}}\hat{F}\left(x_{t+\Delta}, \dot{x}_{t+\Delta}\right) - \frac{\partial}{\partial \dot{x}}\hat{F}\left(x_t, \dot{x}_t\right)\right] = \rho\frac{\partial}{\partial \dot{x}}\hat{F}\left(x_t, \dot{x}_t\right) + \frac{\partial}{\partial x}\hat{F}\left(x_{t+\Delta}, \dot{x}_{t+\Delta}\right).$$

Assumptions. The next steps consists on taking the limit of the above expression as $\Delta \to 0$. For this we will assume that as we take the limit as $\Delta \to 0$, the solutions are such that the resulting path $x(t)$ is twice differentiable with respect to time, so that the following limits are well defined and given by the corresponding expressions:

$$\begin{aligned}
\lim_{\Delta\to 0} x_{t+\Delta} &= x_t, \\
\lim_{\Delta\to 0} \frac{x_{t+\Delta} - x_t}{\Delta} &= \dot{x}_t, \\
\lim_{\Delta\to 0} \dot{x}_{t+\Delta} &= \dot{x}_t, \\
\lim_{\Delta\to 0} \frac{\dot{x}_{t+\Delta} - \dot{x}_t}{\Delta} &= \ddot{x},
\end{aligned}$$

for all t.

Exercise 6. Use the Assumptions to show that the limit of the RHS of the expression in 5 is

$$\lim_{\Delta\to 0}\left[\rho\frac{\partial}{\partial\dot{x}}\hat{F}\left(x_t,\dot{x}_t\right)+\frac{\partial}{\partial x}\hat{F}\left(x_{t+\Delta},\dot{x}_{t+\Delta}\right)\right]=\rho\frac{\partial}{\partial\dot{x}}\hat{F}\left(x_t,\dot{x}_t\right)+\frac{\partial}{\partial x}\hat{F}\left(x_t,\dot{x}_t\right).$$

Ans: Use the assumption that $x(t)$ is twice differentiable and hence the expressions for the limits written above.

Exercise 7. Taking the limit of the LHS of the expression in 5 is more subtle. Use the expressions for the limits as $\Delta\to 0$ in the Assumptions to show that the limit as $\Delta\to 0$ of the LHS of the EE derived in 5

$$\lim_{\Delta\to 0}\frac{1}{\Delta}\left[\frac{\partial}{\partial\dot{x}}\hat{F}\left(x_{t+\Delta},\dot{x}_{t+\Delta}\right)-\frac{\partial}{\partial\dot{x}}\hat{F}\left(x_t,\dot{x}_t\right)\right],$$

requires the use of L'Hôpital's rule for its evaluation.

Ans: The LHS is

$$\begin{aligned}\lim_{\Delta\to 0}\frac{1}{\Delta}\left[\frac{\partial}{\partial\dot{x}}\hat{F}\left(x_{t+\Delta},\dot{x}_{t+\Delta}\right)-\frac{\partial}{\partial\dot{x}}\hat{F}\left(x_t,\dot{x}_t\right)\right] &= \frac{\lim_{\Delta\to 0}\left[\frac{\partial}{\partial\dot{x}}\hat{F}\left(x_{t+\Delta},\dot{x}_{t+\Delta}\right)-\frac{\partial}{\partial\dot{x}}\hat{F}\left(x_t,\dot{x}_t\right)\right]}{\lim_{\Delta\to 0}\ \Delta}\\ &= \frac{\frac{\partial}{\partial\dot{x}}\hat{F}\left(x_t,\dot{x}_t\right)-\frac{\partial}{\partial\dot{x}}\hat{F}\left(x_t,\dot{x}_t\right)}{0}=\frac{0}{0}.\end{aligned}$$

Thus we use L'Hôpital's rule and compute

$$\begin{aligned}\lim_{\Delta\to 0}\frac{1}{\Delta}\left[\frac{\partial}{\partial\dot{x}}\hat{F}\left(x_{t+\Delta},\dot{x}_{t+\Delta}\right)-\frac{\partial}{\partial\dot{x}}\hat{F}\left(x_t,\dot{x}_t\right)\right] &= \lim_{\Delta\to 0}\frac{\partial}{\partial\Delta}\left[\frac{\partial}{\partial\dot{x}}\hat{F}\left(x_{t+\Delta},\dot{x}_{t+\Delta}\right)-\frac{\partial}{\partial\dot{x}}\hat{F}\left(x_t,\dot{x}_t\right)\right]\\ &= \lim_{\Delta\to 0}\frac{\partial}{\partial\Delta}\frac{\partial}{\partial\dot{x}}\hat{F}\left(x_{t+\Delta},\dot{x}_{t+\Delta}\right).\end{aligned}$$

Exercise 8. We now apply L'Hôpital's rule to evaluate the limit as $\Delta\to 0$ of the LHS of the EE derived in 5. To do so use the definitions

$$\begin{aligned}x_{t+\Delta} &= x_t+\dot{x}_t\Delta,\\ \dot{x}_{t+\Delta} &= \dot{x}_t+\ddot{x}_t\Delta,\end{aligned}$$

so that

$$\frac{\partial}{\partial\dot{x}}\hat{F}\left(x_{t+\Delta},\dot{x}_{t+\Delta}\right)=\frac{\partial}{\partial\dot{x}}\hat{F}\left(x_t+\dot{x}_t\Delta,\ \dot{x}_t+\ddot{x}_t\Delta\right),$$

in computing the derivative

$$\frac{\partial}{\partial\Delta}\frac{\partial}{\partial\dot{x}}\hat{F}\left(x_{t+\Delta},\dot{x}_{t+\Delta}\right).$$

Show that

$$\lim_{\Delta\to 0}\frac{1}{\Delta}\left[\frac{\partial}{\partial \dot{x}}\hat{F}\left(x_{t+\Delta},\dot{x}_{t+\Delta}\right)-\frac{\partial}{\partial \dot{x}}\hat{F}\left(x_{t},\dot{x}_{t}\right)\right]=\frac{\partial^2}{\partial x\partial \dot{x}}\hat{F}\left(x_t,\ \dot{x}_t\right)\ \dot{x}_t+\frac{\partial^2}{\partial \dot{x}\partial \dot{x}}\hat{F}\left(x_t,\ \dot{x}_t\right)\ddot{x}_t.$$

[Hint: Use the assumptions to take the limit].

Ans: Applying L'Hôpital's rule, and replacing $x_{t+\Delta}$ and $\dot{x}_{t+\Delta}$ by the expressions above we can compute the derivative

$$\begin{aligned}\frac{\partial}{\partial \Delta}\frac{\partial}{\partial \dot{x}}\hat{F}\left(x_{t+\Delta},\dot{x}_{t+\Delta}\right) &= \frac{\partial}{\partial \Delta}\frac{\partial}{\partial \dot{x}}\hat{F}\left(x_t+\dot{x}_t\Delta,\ \dot{x}_t+\ddot{x}_t\Delta\right)\\ &= \frac{\partial^2}{\partial x\partial \dot{x}}\hat{F}\left(x_t+\dot{x}_t\Delta,\ \dot{x}_t+\ddot{x}_t\Delta\right)\dot{x}_t+\frac{\partial^2}{\partial \dot{x}\partial \dot{x}}\hat{F}\left(x_t+\dot{x}_t\Delta,\ \dot{x}_t+\ddot{x}_t\Delta\right)\ddot{x}_t,\end{aligned}$$

and taking the limit

$$\lim_{\Delta\to 0}\frac{\partial}{\partial \Delta}\frac{\partial}{\partial \dot{x}}\hat{F}\left(x_{t+\Delta},\dot{x}_{t+\Delta}\right)=\frac{\partial^2}{\partial x\partial \dot{x}}\hat{F}\left(x_t,\ \dot{x}_t\right)\ \dot{x}_t+\frac{\partial^2}{\partial \dot{x}\partial \dot{x}}\hat{F}\left(x_t,\ \dot{x}_t\right)\ddot{x}_t.$$

Summarizing,

$$\lim_{\Delta\to 0}\frac{1}{\Delta}\left[\frac{\partial}{\partial \dot{x}}\hat{F}\left(x_{t+\Delta},\dot{x}_{t+\Delta}\right)-\frac{\partial}{\partial \dot{x}}\hat{F}\left(x_{t},\dot{x}_{t}\right)\right]=\frac{\partial^2}{\partial x\partial \dot{x}}\hat{F}\left(x_t,\ \dot{x}_t\right)\ \dot{x}_t+\frac{\partial^2}{\partial \dot{x}\partial \dot{x}}\hat{F}\left(x_t,\ \dot{x}_t\right)\ddot{x}_t.$$

Exercise 9. Use your answers to question 5, 6 and 8 to obtain the continuous time Euler equation:

$$\frac{\partial^2}{\partial \dot{x}\partial x}\hat{F}\left(x_t,\dot{x}_t\right)\ \dot{x}_t+\frac{\partial^2}{\partial \dot{x}\partial \dot{x}}\hat{F}\left(x_t,\dot{x}_t\right)\ddot{x}_t=\rho\frac{\partial}{\partial \dot{x}}\hat{F}\left(x_t,\dot{x}_t\right)+\frac{\partial}{\partial x}\hat{F}\left(x_t,\dot{x}_t\right).$$

Ans: It follows from taking the limits in 5 as $\Delta\to 0$, which is done in 6 and 8.

3 Continuous time agent's problem

Exercise 1. Let the discrete time budget constraint for a problem with length of time period Δ be

$$a_{t+\Delta}+\Delta c_t+\Delta\tau_t=\Delta w_t\left(1-\bar{\tau}_t\right)+\left(1+\Delta r_t\left(1-\bar{\tau}_t\right)\right)a_t,$$

where a_t are assets, w_t wages, τ_t lump sum taxes, $\bar{\tau}_t$ income tax rate, and r_t the interest rate. Show that as Δ goes to zero this gives the following asset accumulation equation:

$$\dot{a}\left(t\right)+c\left(t\right)+\tau\left(t\right)=\left(1-\bar{\tau}\left(t\right)\right)\left[w\left(t\right)+r\left(t\right)a\left(t\right)\right].$$

[Hint: Rearrange the discrete time expression, divide by Δ, and take limits].

Ans: Subtracting a_t from both sides of the budget constraint and dividing by Δ we obtain

$$\frac{a_{t+\Delta} - a_t}{\Delta} + c_t + \tau_t = w_t \left(1 - \bar{\tau}_t\right) + r_t \left(1 - \bar{\tau}_t\right) a_t.$$

Let $\Delta \to 0$ and use $\lim_{\Delta \to 0} \left(a_{t+\Delta} - a_t\right) / \Delta = \dot{a}(t)$ to obtain

$$\dot{a}(t) + c(t) + \tau(t) = \left(1 - \bar{\tau}(t)\right) \left[w(t) + r(t) a(t)\right].$$

Exercise 2. Show that the following present value budget constraint

$$\int_t^{\infty} \left[c(s) + \tau(s) - w(s)\left(1 - \bar{\tau}(s)\right)\right] e^{-\int_t^s r(u)(1-\bar{\tau}(u))du} ds = a(t),$$

is a solution of the previous asset accumulation equation. [Hint: Differentiate this expression with respect to time].

Ans: We will differentiate the above expression with respect to t. To that end, recall Leibniz's rule: Let $I(s) = \int_{a(s)}^{b(s)} f(x, s)\, dx$, then,

$$I'(s) = f(b, s) b'(s) - f(a, s) a'(s) + \int_{a(s)}^{b(s)} \frac{\partial f(x, s)}{\partial s} dx.$$

Using that formula, compute the derivative with respect to t of the above expression:

$$\begin{aligned} \dot{a}(t) &= -\left[c(t) + \tau(t) - w(t)\left(1 - \bar{\tau}(t)\right)\right] \underbrace{e^{-\int_t^t r(u)(1-\bar{\tau}(u))du}}_{=1} \\ &\quad + \underbrace{\left[\int_t^{\infty} \left[c(s) + \tau(s) - w(s)\left(1 - \bar{\tau}(s)\right)\right] e^{-\int_t^s r(u)(1-\bar{\tau}(u))du} ds\right]}_{=a(t)} \times r(t)\left(1 - \bar{\tau}(t)\right) \end{aligned}$$

or, rearranging,

$$\dot{a}(t) + c(t) + \tau(t) = \left(1 - \bar{\tau}(t)\right) \left[w(t) + r(t) a(t)\right].$$

Aside: Let us follow the backward procedure by solving the differential equation directly. To this end, let $R(s)$ be defined as

$$R(s) \equiv e^{-\int_t^s r(u)(1-\bar{\tau}(u))du}.$$

Then, integrating the budget constraint from period t to some period T gives

$$\int_t^T [c(s) + \tau(s) - (1 - \bar{\tau}(s)) w(s)] R(s) ds = \int_t^T [r(s)(1 - \bar{\tau}(s)) a(s) - \dot{a}(s)] R(s) ds. \tag{17}$$

Now, note that

$$\begin{aligned} \frac{d(a(s) R(s))}{ds} &= \dot{a}(s) R(s) + a(s) \dot{R}(s) = \dot{a}(s) R(s) - r(s)(1 - \bar{\tau}(s)) a(s) R(s) \\ &= -[r(s)(1 - \bar{\tau}(s)) a(s) - \dot{a}(s)] R(s), \end{aligned}$$

where we have used the fact that

$$\frac{\partial\left(-\int_t^s r(u)(1 - \bar{\tau}(u)) du\right)}{\partial s} = -r(s)(1 - \bar{\tau}(s)).$$

Thus, the RHS of equation (17) can be written as

$$-\int_t^T \frac{\partial(a(s) R(s))}{\partial s} ds = -\left. a(s) R(s) \right|_t^T = -a(T) R(T) + a(t),$$

where we have used the fact that

$$R(t) = e^{-\int_t^t r(u)(1 - \bar{\tau}(u)) du} = 1.$$

Using these results, and taking the limit as T goes to ∞ we find that expression (17) can be written as

$$\begin{aligned} \lim_{T \to \infty} \int_t^T [c(s) + \tau(s) - (1 - \bar{\tau}(s)) w(s)] R(s) ds &= -\lim_{T \to \infty} a(T) R(T) + a(t) \\ \int_t^\infty [c(s) + \tau(s) - (1 - \bar{\tau}(s)) w(s)] e^{-\int_t^s r(u)[1 - \bar{\tau}(u)] du} ds &= a(t), \end{aligned} \tag{18}$$

as long as the following no-Ponzi-game condition holds:

$$\lim_{T \to \infty} a(T) e^{-\int_t^T r(u)(1 - \bar{\tau}(u)) du} = 0.$$

Exercise 3. Formulate the problem of an agent with utility

$$\int_0^\infty e^{-\rho t} U(c(t)) dt,$$

of choosing consumption subject to the present value budget constraint (at time $t = 0$) obtained in the previous exercise. Write the Lagrangian using λ for the multiplier of the present value budget constraint. Show that the FOC with respect to $c(t)$ is:

$$e^{-\rho t}U'(c(t)) = \lambda e^{-\int_0^t r(s)(1-\bar{\tau}(s))ds}.$$

Show that this equation implies

$$\frac{\dot{c}(t)}{c(t)} = [(1-\bar{\tau}(t))\, r(t) - \rho] \ / \ \left[-c(t)\frac{U''(c(t))}{U'(c(t))}\right].$$

[Hint: Differentiate both sides of the FOC with respect to time].

Ans: The Lagrangian is

$$\int_0^\infty e^{-\rho t}U(c(t))\, dt + \lambda\left[a(0) - \int_0^\infty [c(t) + \tau(t) - w(t)(1-\bar{\tau}(t))]\, e^{-\int_0^t r(s)(1-\bar{\tau}(s))ds} dt\right].$$

The FOC w.r.t. $c(t)$ is then

$$e^{-\rho t}U'(c(t)) = \lambda e^{-\int_0^t r(s)(1-\bar{\tau}(s))ds}.$$

Differentiating the first order condition w.r.t. t we obtain

$$-\rho e^{-\rho t}U'(c(t)) + e^{-\rho t}U''(c(t))\,\dot{c}(t) = \lambda e^{-\int_0^t r(s)(1-\bar{\tau}(s))ds}\left[-r(t)(1-\bar{\tau}(t))\right].$$

Using the FOC again to eliminate λ we find

$$-\rho e^{-\rho t}U'(c(t)) + e^{-\rho t}U''(c(t))\,\dot{c}(t) = e^{-\rho t}U'(c(t))\left[-r(t)(1-\bar{\tau}(t))\right],$$

or

$$-\rho + \frac{U''(c(t))\, c(t)}{U'(c(t))}\frac{\dot{c}(t)}{c(t)} = \left[-r(t)(1-\bar{\tau}(t))\right],$$

or

$$\frac{\dot{c}(t)}{c(t)} = \frac{-U'(c(t))}{U''(c(t))\, c(t)}\left[r(t)(1-\bar{\tau}(t)) - \rho\right],$$

as was to be shown.

Exercise 4. Consider the budget constraint of the government with purchases g_t, lump sum taxes τ_t and income taxes at rate $\bar{\tau}(t)$, and government assets (i.e. minus government debt) b_t :

$$b_{t+\Delta} + \Delta g_t = \bar{\tau}_t(\Delta w_t + \Delta r_t a_t) + \Delta\tau_t + b_t(1 + \Delta r_t).$$

Show that, as Δ goes to zero it implies:

$$\dot{b}(t) + g(t) = \bar{\tau}(t)(w(t) + r(t)\, a(t)) + \tau(t) + b(t)\, r(t),$$

and that it corresponds to the following present value budget constraint:

$$b(t)+\int_t^{\infty}\left[\tau(s)+\bar{\tau}(s)(w(s)+r(s)a(s))-g(s)\right]e^{-\int_t^s r(u)du}ds=0.$$

Ans: Repeat the same steps of exercises 1 and 2.

Exercise 5. Walras' law. Show that if i) $a,c,\tau,\bar{\tau},w,r$ satisfy the asset accumulation equation for the households, ii) $b,g,\tau,\bar{\tau},w,a,r$ satisfy the asset accumulation equation for the government, iii) there is equilibrium in the asset market, i.e.

$$a(t)+b(t)=k(t),$$

for all $t\geq 0$, and iv) firms maximize profits, so that:

$$\begin{aligned} r(t) &= f'(k(t)), \\ w(t) &= f(k(t))-f'(k(t))k(t), \end{aligned}$$

for all $t\geq 0$. Then the allocation is feasible, i.e.

$$\dot{k}(t)+c(t)+g(t)=f(k(t)),$$

holds for all $t\geq 0$.

Ans: Adding the asset-accumulation equations for the household and the government we obtain the asset-accumulation equation for the economy as a whole:

$$\begin{aligned} \dot{a}(t)+\dot{b}(t)+c(t)+g(t)+\tau(t) &= [1-\bar{\tau}(t)][w(t)+r(t)a(t)]+\bar{\tau}(t)[w(t)+r(t)a(t)] \\ &\quad +\tau(t)+b(t)r(t) \\ \dot{a}(t)+\dot{b}(t)+c(t)+g(t) &= w(t)+r(t)[a(t)+b(t)]. \end{aligned}$$

If there is equilibrium in the asset markets, i.e.,

$$a(t)+b(t)=k(t) \qquad \forall t\geq 0,$$

and firms maximize profits, so that

$$\begin{aligned} r(t) &= f'(k(t)) \qquad \forall t\geq 0 \\ w(t) &= f(k(t))-f'(k(t))k(t) \qquad \forall t\geq 0, \end{aligned}$$

then it follows that

$$\begin{aligned}\dot{k}(t)+c(t)+g(t) &= f(k(t))-f'(k(t))\,k(t)+f'(k(t))\,k(t)\\ \dot{k}(t)+c(t)+g(t) &= f(k(t)) \qquad \forall t\geq 0, \end{aligned} \tag{19}$$

that is, the equilibrium allocation must be feasible.

Exercise 6. Ricardian Equivalence. Let a, b, τ, g, r, w, k be an equilibrium with lump sum taxes, so $\bar{\tau}(t)=0$ all t. Consider the following fiscal policies with lump sum taxes τ' and debt b' satisfying:

$$\int_0^\infty \tau'(t)\, e^{-\int_0^t r(s)ds}dt = \int_0^\infty \tau(t)\, e^{-\int_0^t r(s)ds}dt,$$

and $b'(0)=b(0)$. Show that $a', b', \tau', g, r, w, k$ is also an equilibrium with lump sum taxes for some path of assets a' such that $a'(0)=a(0)$. [Hint: You must show that agents still maximize with the same choices c given τ', r, w, for some path of assets a' with $a'(0)=a(0)$ given, that firms maximize their profits, and that the government budget constraint also holds].

Ans: First let's define a competitive equilibrium with lump sum taxes: It is a set of allocations $(a(t), c(t), g(t), b(t), \tau(t), r(t), w(t), k(t))$ for all t such that:

1. Given $\{\tau(t), r(t), w(t)\}$ and $a(0)$, $\{a(t), c(t)\}$ solve the consumer's problem,
2. Given $\{g(t)\}$ and $b(0)$, $\{b(t), \tau(t)\}$ are such that the government budget constraint is satisfied,
3. Firm's optimize given $\{r(t), w(t)\}$,
4. The asset market clears: $k(t)=a(t)+b(t)$, $\forall t\geq 0$, and
5. The goods market clears: $\dot{k}(t)+c(t)+g(t)=w(t)+r(t)\,k(t)$, $\forall t\geq 0$.

As we showed in the previous question, if the first four conditions are satisfied, the fifth is automatically satisfied.

Now suppose that government chooses another fiscal policy (τ', b') such that

$$\int_0^\infty \tau'(t)\, e^{-\int_0^t r(s)ds}dt = \int_0^\infty \tau(t)\, e^{-\int_0^t r(s)ds}dt \tag{20}$$

and $b'(0)=b(0)$.

We must show that $(a'(t), c(t), g(t), b'(t), \tau'(t), r(t), w(t), k(t))$ for all t is also an equilibrium with lump-sum taxes. To do this let's see that the above conditions 1 to 4 are still satisfied for those allocations and prices where a' will be defined below.

1. Given $\{\tau'(t), r(t), w(t)\}$ and $a(0)$, $\{c(t)\}$ still solves the consumer's problem for

some a' (to be found below). The consumer's problem is:

$$\max_{c(t)} \int_0^\infty e^{-\rho t} U\left(c\left(t\right)\right) dt,$$

$$\text{s.t.}: a\left(0\right) = \int_0^\infty \left[c\left(t\right) - w\left(t\right)\right] e^{-\int_0^t r(s)ds} dt + \int_0^\infty \tau'\left(t\right) e^{-\int_0^t r(s)ds} dt.$$

Using (20), the budget constraint is equivalent to

$$a\left(0\right) = \int_0^\infty \left[c\left(t\right) - w\left(t\right)\right] e^{-\int_0^t r(s)ds} dt + \int_0^\infty \tau\left(t\right) e^{-\int_0^t r(s)ds} dt.$$

Thus, if the new tax policy satisfies (20), the budget set of the consumer is exactly the same under the two fiscal regimes. Therefore, the optimal consumption choice will be the same (otherwise $\{c\left(t\right)\}$ wouldn't be an optimal choice for the initial fiscal policy). Notice, though, that the asset allocation need not be the same.

2. Given $\{g\left(t\right)\}$ (the same in both regimes) and using $b'\left(0\right) = b\left(0\right)$, by construction $\{b'\left(t\right), \tau'\left(t\right)\}$ satisfies the government budget constraint.

3. If $\{r\left(t\right), w\left(t\right)\}$ do not change, the firm's problem is exactly the same as before and the same policies are chosen.

4. Market clearing: Given the same $\{k\left(t\right)\}$, this condition reads

$$a'\left(t\right) + b'\left(t\right) = k\left(t\right)$$

that is, the level of assets held by the consumers adjust to the change in the government's assets in such a way that the total stock of capital remains unmodified.

In other words, we have shown that for the new fiscal policy (τ', b'), the allocation and prices $\left(a'\left(t\right), c\left(t\right), g\left(t\right), b'\left(t\right), \tau'\left(t\right), r\left(t\right), w\left(t\right), k\left(t\right)\right)$ for all t constitutes a competitive equilibrium, where $a'\left(t\right) = k\left(t\right) - b'\left(t\right)$.