Theory of Income, Fall 2008
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Problem Set 4

1 Neoclassical Growth Model: Exogenous Growth

Let the exogenous time augmenting productivity growth gross rate be $\lambda > 1$. Feasibility is given by

$$c_t + k_{t+1} = F\left(k_t, \lambda^t n_t\right) - (1-\delta)\, k_t,$$

where F is a neoclassical constant returns to scale production function, k_t is capital, c_t is consumption, n_t is labor, and δ is the depreciation rate of capital. The endowment of time per period is normalized to 1, so that leisure is $l_t = 1 - n_t$. Preferences are given by

$$\sum_{t=0}^{\infty} \beta^t v\left(c_t, 1-n_t\right).$$

The consumer has a budget constraint given by

$$\sum_{t=0}^{\infty} p_t \left[c_t + x_t\right] = \sum_{t=0}^{\infty} p_t \left[w_t n_t + k_t v_t\right],$$

and the law of motion of capital is

$$k_{t+1} = x_t + (1-\delta)\, k_t,$$

where p_t is the Arrow-Debreu price of consumption goods at time t in terms of time zero consumption good, and w_t and v_t are the real wage and rental rate of capital in terms of consumption goods at period t.

The firm's problem is

$$\max_{k_t, n_t} \; F\left(k_t, \lambda^t n_t\right) - w_t n_t - v_t k_t.$$

Exercise 1. Let r_t be the time t interest rate, i.e. $p_t/p_{t+1} = 1 + r_t$. Use the budget constraint of the household, and the law of motion of capital to show that, as long as $x_t > 0$,

$$v_{t+1} = r_t + \delta.$$

[Hint: Consider an investment of 1 at t, renting it on $t+1$ and consuming the undepreciated capital at $t+1$].

Exercise 2. Write down the first order condition w.r.t. c_t, n_t and k_{t+1}. Use μ for the multiplier on the budget constraint [Hint: Replace x_t in the household budget constraint using the law of motion of capital]. Combine the FOC for c_t for two consecutive periods to obtain a relationship between the marginal rate of substitution between c_t and c_{t+1} and r_t. Combine the FOC with respect to c_t and n_t to obtain a relationship between the marginal rate of substitution between n_t and c_t and w_t.

Exercise 3. Use the expression for the rental rate of capital v_t for $t \geq 1$ and the law of motion of capital for $t \geq 0$ (solving for x_t) to show that the household's budget constraint can be written as

$$\sum_{t=0}^{\infty} p_t c_t = p_0 k_0 \left(v_0 + 1 - \delta\right) + \sum_{t=0}^{\infty} p_t w_t n_t.$$

Definition: A *balanced growth path* is given by an initial capital k_0 and λ such that it is optimal to set

$$\begin{aligned} c_t &= c_0 \lambda^t, \\ k_t &= k_0 \lambda^t, \\ n_t &= n_0, \\ w_t &= \lambda^t w_0, \\ r_t &= r_0, \end{aligned}$$

for all $t \geq 0$.

Exercise 4. Write down the FOC for the household imposing a balanced growth path. Use the FOC for the household and firm's problem as well as feasibility to write down the system of equations in c_0, n_0, k_0, w_0, r_0 that a balanced growth path must satisfy.

Exercise 5. Show that if preferences are of the form

$$v\left(c, 1-n\right) = \frac{c^{1-\gamma}}{1-\gamma} h\left(1-n\right), \tag{1}$$

for $\gamma \neq 1$, or

$$v\left(c, 1-n\right) = \log c + h\left(1-n\right),$$

then there is a balanced growth path.

Exercise 6. Assume that $v\left(c, l\right)$ is strictly concave and increasing in $\left(c, l\right)$ and have the form described in (1). Consider first the case where $\gamma \in (0, 1)$. What are the properties of $h\left(l\right)$? i.e. is it positive or negative, increasing or decreasing, concave or convex? Next, consider the case where $\gamma = 1$. What are the properties of $h\left(l\right)$? i.e. is it increasing or decreasing, concave or convex? Finally, consider the case where $\gamma > 1$. What are the properties of $h\left(l\right)$? i.e. is it positive or negative, increasing or decreasing, concave or convex?

Exercise 7. Let $v(c, 1-n)$ be

$$v(c, 1-n) = g(c - n^{\sigma}/\sigma),$$

for $\sigma > 1$ and g strictly increasing and concave. What is the income elasticity of the labor supply for this utility function? Show that this preferences are inconsistent with a balanced growth path. [Hint: In a BGP we must have

$$\frac{v_l(c_0\lambda^t, 1-n_0)}{v_c(c_0\lambda^t, 1-n_0)} = w_0\lambda^t,$$

but

$$\frac{v_l(c_0\lambda^t, 1-n_0)}{v_c(c_0\lambda^t, 1-n_0)} = \frac{g'(c_0\lambda^t - n_0^{\sigma}/\sigma)}{g'(c_0\lambda^t - n_0^{\sigma}/\sigma)} n_0^{\sigma-1} = n_0^{\sigma-1} = w_0\lambda^t,$$

so compare the LHS and RHS of the last equality].

Exercise 8. Show that if the economy admits a balanced growth path for an open set of parameters β, λ and δ, preferences must be of the form in (1). [Hint: Write down an Euler equation-like expression relating the marginal rate of substitution of consumption between t and $t+1$ with r. Impose the balanced growth condition on this, noticing that this expression must be satisfied for all t. Differentiate this expression with respect to t to obtain a differential equation, whose solution implies that v is of the form $B(1-n) + c^{1-\gamma(1-n)}h(1-n)$ or $B(1-n) + \log c + h(1-n)$ where γ is a function of $(1-n)$. Use this, and the condition that marginal rates of substitution equal relative prices on a balanced growth path to establish the required result].

2 Deriving the Euler Equation in Continuous Time

In this question we obtain the continuous time Euler Equation by taking limits of the discrete time Euler equation. The point of this is to realize that although the expression for the continuous time counterpart is less intuitive that the one for the discrete time -which has the natural interpretation of equating marginal cost to marginal benefit- they are really the same.

The idea is to consider a sequence of discrete time dynamic programing problems. In each problem the length of time between periods where the state is decided is denoted by Δ. Decisions are taken at times $0,\ \Delta,\ 2\Delta,\ 3\Delta, ...$ The sequence of states to be chosen is

$$\{x_{\Delta(t+1)}\}_{t=0}^{\infty} = \{x_{\Delta}, x_{2\Delta}, x_{3\Delta}, ...\},$$

where x_0 is given. Setting $\Delta = 1$ we obtain the standard problem analyzed in the class notes.

We adjust the discount factor for each problem accordingly letting

$$\beta = \frac{1}{1+\Delta\rho},$$

so that ρ has the interpretation of a discount rate.

For each Δ we write the period return function during the interval of time of length Δ as F, and the corresponding return function per unit of time as $\hat{F}$. They satisfy:

$$F\left(x_t, x_{t+\Delta}\right) \equiv \Delta\ \hat{F}\left(x_t, \frac{1}{\Delta}\left(x_{t+\Delta} - x_t\right)\right),$$

where $t = i\Delta$ for some integer i. It helps to write these return functions as

$$F\left(x, y\right) \equiv \Delta\ \hat{F}\left(x, \frac{1}{\Delta}\left(y - x\right)\right),$$

or

$$F\left(x, \dot{x}\Delta + x\right) = \Delta\ \hat{F}\left(x, \dot{x}\right),$$

where we define $\dot{x}$ as the change per unit of time on the state:

$$\dot{x} \equiv \frac{y-x}{\Delta},$$

or using time subscripts:

$$\dot{x}_t = \frac{x_{t+\Delta} - x_t}{\Delta},$$

for $t = i\Delta$ and any integer i.

Likewise we can define the feasible correspondence $\hat{\Gamma}$ for the change per unit of time $\dot{x}$ in terms of the feasible correspondence for levels Γ as

$$\hat{\Gamma}\left(x\right) = \left\{\dot{x} : y \in \Gamma\left(x\right),\ y = \dot{x}\Delta + x\right\}.$$

Thus for each Δ we consider the problem

$$\max_{\left\{x_{(t+1)\Delta}\right\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \left(\frac{1}{1+\Delta\rho}\right)^t\ F\left(x_{t\Delta},\ x_{(t+1)\Delta}\right),$$

subject to

$$x_{(t+1)\Delta} \in \Gamma\left(x_{t\Delta}\right)\ ,$$

for all $t \geq 0$, where x_0 given.

Equivalently, we can write this problem as a choice of the sequence of discrete time changes

$\{\dot{x}_{t\Delta}\}_{t=0}^{\infty}$:

$$\max_{\{\dot{x}_{t\Delta}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \left(\frac{1}{1+\Delta\rho}\right)^t \ \Delta \ \hat{F}\left(x_{t\Delta}, \ \dot{x}_{t\Delta}\right),$$

subject to

$$\dot{x}_{t\Delta} \in \hat{\Gamma}\left(x_{t\Delta}\right),$$

$$x_{t\Delta+\Delta} = x_{t\Delta} + \dot{x}_{t\Delta} \ \Delta,$$

for all $t \geq 0$, and for given x_0.

We emphasize that the optimal sequence $\left\{x_{(t+1)\Delta}\right\}_{t=0}^{\infty}$ that solves the problem with interval of length Δ is a function of Δ.

For future reference, we also introduce the notation for the changes per unit of time on the change per unit of time of the state, denoting it by $\ddot{x}_t$:

$$\ddot{x}_t \equiv \frac{1}{\Delta}\left(\dot{x}_{t+\Delta} - \dot{x}_t\right),$$

for all $t = i\Delta$ and an integer i.

Exercise 1. Derive a formula for F_y and F_x in terms of $\partial\hat{F}/\partial x$ and $\partial\hat{F}/\partial\dot{x}$. In particular use the relationship between F and $\hat{F}$ to show that

$$F_y\left(x, \dot{x}\Delta + x\right) = \frac{\partial}{\partial\dot{x}}\hat{F}\left(x, \dot{x}\right),$$

$$F_x\left(x, \dot{x}\Delta + x\right) = \Delta\frac{\partial}{\partial x}\hat{F}\left(x, \dot{x}\right) - \frac{\partial}{\partial\dot{x}}\hat{F}\left(x, \dot{x}\right).$$

Exercise 2. Write the Euler Equations for the problem where we chose the sequence of levels of the state: $\left\{x_{(t+1)\Delta}\right\}_{t=0}^{\infty}$. Your Euler equation should involve $F_y, \ F_x, \ \Delta, \ \rho$ and be evaluated at $x_t, \ x_{t+\Delta}$ and $x_{t+2\Delta}$. [Hint: This is the standard problem].

Exercise 3. Rewrite the Euler equation obtained in 2 replacing the $x_{t+\Delta}$ in F_y in terms of Δ , x_t and $\dot{x}_t$, and replacing the $x_{t+2\Delta}$ in F_x in terms of Δ , $x_{t+\Delta}$ and $\dot{x}_{t+\Delta}$.

Exercise 4. Use the relationship between the derivatives of F and $\hat{F}$ found in 1 into your expression for the Euler equation found in exercise 3.

Exercise 5. Show that by rearranging the terms in the expression found in 4, the Euler equation can be written as:

$$\frac{1}{\Delta}\left[\frac{\partial}{\partial\dot{x}}\hat{F}\left(x_{t+\Delta}, \dot{x}_{t+\Delta}\right) - \frac{\partial}{\partial\dot{x}}\hat{F}\left(x_t, \dot{x}_t\right)\right] = \rho\frac{\partial}{\partial\dot{x}}\hat{F}\left(x_t, \dot{x}_t\right) + \frac{\partial}{\partial x}\hat{F}\left(x_{t+\Delta}, \dot{x}_{t+\Delta}\right).$$

Assumptions. The next steps consists on taking the limit of the above expression as $\Delta \to 0$. For this we will assume that as we take the limit as $\Delta \to 0$, the solutions are such that the resulting path $x\left(t\right)$ is twice differentiable with respect to time, so that the following limits are well defined and given by the corresponding expressions:

$$\begin{aligned}
\lim_{\Delta\to 0} x_{t+\Delta} &= x_t, \\
\lim_{\Delta\to 0} \frac{x_{t+\Delta}-x_t}{\Delta} &= \dot{x}_t, \\
\lim_{\Delta\to 0} \dot{x}_{t+\Delta} &= \dot{x}_t, \\
\lim_{\Delta\to 0} \frac{\dot{x}_{t+\Delta}-\dot{x}_t}{\Delta} &= \ddot{x},
\end{aligned}$$

for all t.

Exercise 6. Use the Assumptions to show that the limit of the RHS of the expression in 5 is

$$\lim_{\Delta\to 0}\left[\rho\frac{\partial}{\partial \dot{x}}\hat{F}(x_t,\dot{x}_t)+\frac{\partial}{\partial x}\hat{F}(x_{t+\Delta},\dot{x}_{t+\Delta})\right]=\rho\frac{\partial}{\partial \dot{x}}\hat{F}(x_t,\dot{x}_t)+\frac{\partial}{\partial x}\hat{F}(x_t,\dot{x}_t).$$

Exercise 7. Taking the limit of the LHS of the expression in 5 is more subtle. Use the expressions for the limits as $\Delta \to 0$ in the Assumptions to show that the limit as $\Delta \to 0$ of the LHS of the EE derived in 5

$$\lim_{\Delta\to 0}\frac{1}{\Delta}\left[\frac{\partial}{\partial \dot{x}}\hat{F}(x_{t+\Delta},\dot{x}_{t+\Delta})-\frac{\partial}{\partial \dot{x}}\hat{F}(x_t,\dot{x}_t)\right],$$

requires the use of L'Hôpital's rule for its evaluation.

Exercise 8. We now apply L'Hôpital's rule to evaluate the limit as $\Delta \to 0$ of the LHS of the EE derived in 5. To do so use the definitions

$$\begin{aligned}
x_{t+\Delta} &= x_t + \dot{x}_t\Delta, \\
\dot{x}_{t+\Delta} &= \dot{x}_t + \ddot{x}_t\Delta,
\end{aligned}$$

so that

$$\frac{\partial}{\partial \dot{x}}\hat{F}(x_{t+\Delta},\dot{x}_{t+\Delta})=\frac{\partial}{\partial \dot{x}}\hat{F}\left(x_t+\dot{x}_t\Delta,\ \dot{x}_t+\ddot{x}_t\Delta\right),$$

in computing the derivative

$$\frac{\partial}{\partial \Delta}\frac{\partial}{\partial \dot{x}}\hat{F}(x_{t+\Delta},\dot{x}_{t+\Delta}).$$

Show that

$$\lim_{\Delta\to 0}\frac{1}{\Delta}\left[\frac{\partial}{\partial \dot{x}}\hat{F}(x_{t+\Delta},\dot{x}_{t+\Delta})-\frac{\partial}{\partial \dot{x}}\hat{F}(x_t,\dot{x}_t)\right]=\frac{\partial^2}{\partial x\partial \dot{x}}\hat{F}(x_t,\ \dot{x}_t)\ \dot{x}_t+\frac{\partial^2}{\partial \dot{x}\partial \dot{x}}\hat{F}(x_t,\ \dot{x}_t)\,\ddot{x}_t.$$

[Hint: Use the assumptions to take the limit].

Exercise 9. Use your answers to question 5, 6 and 8 to obtain the continuous time Euler equation:

$$\frac{\partial^2}{\partial \dot{x} \partial x} \hat{F}\left(x_t, \dot{x}_t\right) \ \dot{x}_t + \frac{\partial^2}{\partial \dot{x} \partial \dot{x}} \hat{F}\left(x_t, \dot{x}_t\right) \ddot{x}_t = \rho \frac{\partial}{\partial \dot{x}} \hat{F}\left(x_t, \dot{x}_t\right) + \frac{\partial}{\partial x} \hat{F}\left(x_t, \dot{x}_t\right).$$

3 Continuous time agent's problem

Exercise 1. Let the discrete time budget constraint for a problem with length of time period Δ be

$$a_{t+\Delta} + \Delta c_t + \Delta \tau_t = \Delta w_t \left(1 - \bar{\tau}_t\right) + \left(1 + \Delta r_t \left(1 - \bar{\tau}_t\right)\right) a_t,$$

where a_t are assets, w_t wages, τ_t lump sum taxes, $\bar{\tau}_t$ income tax rate, and r_t the interest rate. Show that as Δ goes to zero this gives the following asset accumulation equation:

$$\dot{a}\left(t\right) + c\left(t\right) + \tau\left(t\right) = \left(1 - \bar{\tau}\left(t\right)\right)\left[w\left(t\right) + r\left(t\right) a\left(t\right)\right].$$

[Hint: Rearrange the discrete time expression, divide by Δ, and take limits].

Exercise 2. Show that the following present value budget constraint

$$\int_t^\infty \left[c\left(s\right) + \tau\left(s\right) - w\left(s\right)\left(1 - \bar{\tau}\left(s\right)\right)\right] e^{-\int_t^s r(u)(1-\bar{\tau}(u))du} ds = a\left(t\right),$$

is a solution of the previous asset accumulation equation. [Hint: Differentiate this expression with respect to time].

Exercise 3. Formulate the problem of an agent with utility

$$\int_0^\infty e^{-\rho t} U\left(c\left(t\right)\right) dt,$$

of choosing consumption subject to the present value budget constraint (at time $t = 0$) obtained in the previous exercise. Write the Lagrangian using λ for the multiplier of the present value budget constraint. Show that the FOC with respect to $c\left(t\right)$ is:

$$e^{-\rho t} U'\left(c\left(t\right)\right) = \lambda e^{-\int_0^t r(s)(1-\bar{\tau}(s))ds}.$$

Show that this equation implies

$$\frac{\dot{c}\left(t\right)}{c\left(t\right)} = \left[\left(1 - \bar{\tau}\left(t\right)\right) r\left(t\right) - \rho\right] \ / \ \left[-c\left(t\right) \frac{U''\left(c\left(t\right)\right)}{U'\left(c\left(t\right)\right)}\right].$$

[Hint: Differentiate both sides of the FOC with respect to time].

Exercise 4. Consider the budget constraint of the government with purchases g_t, lump sum taxes τ_t and income taxes at rate $\bar{\tau}\left(t\right)$, and government assets (i.e. minus government debt) b_t :

$$b_{t+\Delta} + \Delta g_t = \bar{\tau}_t \left(\Delta w_t + \Delta r_t a_t \right) + \Delta \tau_t + b_t \left(1 + \Delta r_t \right).$$

Show that, as Δ goes to zero it implies:

$$\dot{b}(t) + g(t) = \bar{\tau}(t) \left(w(t) + r(t) a(t) \right) + \tau(t) + b(t) r(t),$$

and that it corresponds to the following present value budget constraint:

$$b(t) + \int_t^{\infty} \left[\tau(s) + \bar{\tau}(s) \left(w(s) + r(s) a(s) \right) - g(s) \right] e^{-\int_t^s r(u) du} ds = 0.$$

Exercise 5. Walras' law. Show that if i) $a, c, \tau, \bar{\tau}, w, r$ satisfy the asset accumulation equation for the households, ii) $b, g, \tau, \bar{\tau}, w, a, r$ satisfy the asset accumulation equation for the government, iii) there is equilibrium in the asset market, i.e.

$$a(t) + b(t) = k(t),$$

for all $t \geq 0$, and iv) firms maximize profits, so that:

$$\begin{aligned} r(t) &= f'(k(t)), \\ w(t) &= f(k(t)) - f'(k(t)) k(t), \end{aligned}$$

for all $t \geq 0$. Then the allocation is feasible, i.e.

$$\dot{k}(t) + c(t) + g(t) = f(k(t)),$$

holds for all $t \geq 0$.

Exercise 6. Ricardian Equivalence. Let a, b, τ, g, r, w, k be an equilibrium with lump sum taxes, so $\bar{\tau}(t) = 0$ all t. Consider the following fiscal policies with lump sum taxes τ' and debt b' satisfying:

$$\int_0^{\infty} \tau'(t) e^{-\int_0^t r(s) ds} dt = \int_0^{\infty} \tau(t) e^{-\int_0^t r(s) ds} dt,$$

and $b'(0) = b(0)$. Show that $a', b', \tau', g, r, w, k$ is also an equilibrium with lump sum taxes for some path of assets a' such that $a'(0) = a(0)$. [Hint: You must show that agents still maximize with the same choices c given τ', r, w, for some path of assets a' with $a'(0) = a(0)$ given, that firms maximize their profits, and that the government budget constraint also holds].