Theory of Income, Fall 2008
Fernando Alvarez, U of C

Problem Set 3

Introduction In this problem set we are are going to extend the results of the notes on the Security Market Model in a variety of ways. First, we will present a partial characterization of security prices with complete and incomplete markets. This characterization, extends the result of prices being equivalent in the notes ($q = Dp$).

Second, we will specialize the utility function, dividends, and non-trade incomes to a case that it is tractable enough to solve the equilibrium completely (the normal-exponential case). In the context of this example, we will analyze P.O. and the determinants of the equilibrium prices (risk-premium) for complete and incomplete market economies.

Third, we will consider 3 related economies: one atemporal but with uncertainty, one without uncertainty but with two periods, and one with two periods, where in the second period there is uncertainty. The point of this example is to analyze the determinants of interest rates and risk premium separately and simultaneously.

Finally, we will add uncertainty to an Overlapping Generation Model (OLG). This set-up will have many features in common with the previous example, since the problem for the young has two periods, and uncertainty in the second. It introduces securities that live for more than two periods. Most importantly, conceptually it illustrates an application of the 1st Welfare theorem highlighting that efficiency of the C.E. is attained when the expected returns are higher than the growth rate of total consumption.

1 Arbitrage Opportunities

Consider a one period pure exchange economy with uncertainty. Let there be m states of the world and one physical commodity. Assume that there are security markets as described in class note 4 with K securities, each with price q_k, and dividends d_{ks} in state s. Agent i is endowed with θ_k^i units of security k and with a vector $\tilde{e}^i \in R_+^m$ of goods.

Definition. Arbitrage Opportunities.

We say that D, q admits arbitrage opportunities if there is a portfolio h such that:

$$\sum_{k=1}^{K} q_k h_k \leq 0 \text{ and } \sum_{k=1}^{K} d_{ks} h_k \geq 0 \text{ for all } s = 1, ..., m,$$

with at least one of the $m + 1$ inequalities being strict.

Notice that we can write the conditions for an arbitrage opportunity in vector form as

$$q^T h \leq 0 \text{ and } D^T h \geq \bar{0},$$

with at least one strict inequality, where $\bar{0}$ is a $m-$dimensional vector of zeroes.

Exercise 1. Assume that $u^i : R_+^m \to R$ is strictly increasing and that there exists an asset that pays positive in all states of nature and strictly positive in at least one of them (i.e. one of the rows of D is positive with at least one entry being strictly positive, for example a risk free bond). Show that if (q, D) admits arbitrage opportunities then the agent problem in the security market economy has no solution.

Exercise 2. Show that if (D, q) can be written as

$$q = D\ \eta,$$

for a strictly positive vector $\eta \in R_{++}^m$, i.e.,

$$q_k = \sum_{s=1}^{m} d_{ks} \eta_s; \text{ for all } k = 1, 2, .., K, \tag{1}$$

then there are no arbitrage opportunities.

Exercise 3. Consider the following corollary to the Separating Hyperplane Theorem:

Theorem: Linear Separation of closed convex cones*: Suppose that A and B are closed convex cones in R^n that intersect precisely at zero. If A does not contain a linear subspace other than $\{\bar{0}\}$, then there is a non-zero vector $\lambda \in R^n$ such that $\lambda y < \lambda z$ for all y in B and non-zero z in A. (i.e. the two sets can be strictly separated by a linear function.)*

Comment 1: a set X is a cone if $x \in X$ implies that $\alpha x \in X$ for all $\alpha \geq 0$ (i.e. if a point x belongs to the set, then any ray passing from zero through x for α positive belongs to the set).

Comment 2: Suppose $L \subseteq A$ is a linear subspace of A other than $\{\bar{0}\}$. By definition, if $x, y \in L$ then $\alpha x + \beta y \in L$ for any pair of real numbers α, β. If A does not contain a linear subspace other than $\{\bar{0}\}$ then there is no such a set as L (informally, A contains no lines).

Assume that (q, D) admits no arbitrage opportunities. Show that there must exist a strictly positive vector $\eta \in R^m_{++}$ such that $q = D\eta$, i.e., (1) must hold.

Hints:

1. Consider the sets $A = R^{m+1}_+$ (i.e. the non-negative orthant) and

$$\begin{aligned} B &= \Big\{(y_0, y_1, ..., y_m) : \text{ there is a } h \in R^K \text{ such that} \\ &\quad y_0 = -q^T h \text{ and } (y_1, y_2, ..., y_m)^T = D^T h\Big\}. \end{aligned}$$

Show that both are closed, convex cones.

2. Argue (show) that if there are no arbitrage opportunities, then $A \cap B = \{\bar{0}\}$ (the sets A and B meet precisely at the vector zero).

3. Use the above corollary to the Separating Hyperplane Theorem to show that there is a vector $\lambda \in R^{m+1}$ such that

$$\lambda\, y < \lambda\, z,$$

for all $y \in B$ and all $z \in \bar{A}$, where $\bar{A} = A - \{\bar{0}\}$ (i.e. all non-zero elements in A).

4. Use the definitions of $\bar{A}$ and B to argue that $\lambda y \leq 0$ for all $y \in B$.

5. Use the definition of B to show that if $y \in B$ then $(-y) \in B$.

6. Use 4 and 5 to show that $\lambda y = 0$ for all $y \in B$.

7. Use 3 and 6 to argue that $\lambda z > 0$ for all $z \in \bar{A}$.

8. Use 7 to show that λ is strictly positive, i.e. $\lambda \in R^{m+1}_{++}$.

9. Use 6 and 8 to show that η defined as

$$\eta_s = \frac{\lambda_s}{\lambda_0} \text{ for } s = 1, 2, ..., m,$$

satisfies

$$q_k = \sum_{k=1}^{K} d_{ks}\ \eta_s,$$

for all $k = 1, 2, ..., K$.

Exercise 4. Is it important that portfolio positions h are unrestricted, i.e. $h \in R^K$, to obtain the previous proposition? What are the interpretations of $h \in R^K$ and of $h \in R_+^K$ in terms of portfolio constraints? What would have happen with the previous result if $h \in R_+^K$, i.e. which step of the proof would have been violated?

Exercise 5. Show that markets are complete if and only if there is a unique η such that (1) holds. (Hint: write (1) as a system of linear equations

$$q = D\ \eta,$$

what do you know about the uniqueness of the solution η?)

Exercise 6. Consider the case where there are three states $m = 3$ and two securities $K = 2$ with

$$\begin{bmatrix} d_{11} & d_{12} & d_{13} \\ d_{21} & d_{22} & d_{23} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix},$$

and

$$\begin{bmatrix} q_1 \\ q_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \end{bmatrix}.$$

Show that in this case there is a one-dimensional space indexed by $a \in (0, 2)$ for the state prices $\eta > 0$ satisfying (1)

$$\begin{aligned} \eta_3 &= a, \\ \eta_1 &= 2 - a, \\ \eta_2 &= 2 - a. \end{aligned}$$

Exercise 7. Consider the case where there are three states $m = 3$ and two securities

$K = 2$ with

$$\begin{bmatrix} d_{11} & d_{12} & d_{13} \\ d_{21} & d_{22} & d_{23} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix},$$

and

$$\begin{bmatrix} q_1 \\ q_2 \end{bmatrix} = \begin{bmatrix} 2 \\ -2 \end{bmatrix}.$$

Show that (q, D) admits arbitrage opportunities, in particular display a portfolio h for which $q^T h = 0$ and $D^T h > \bar{0}$.

Exercise 8. Assume that (q, D) admits no arbitrage opportunities. Denote by η any of its corresponding state prices. Assume that there is a portfolio h such that it gives a risk-free payoffs, i.e. $D^T h = (1, 1, ..., 1)$. Denote the price of this portfolio by q_0 so that

$$q_0 = \sum_{s=1} \eta_s,$$

and define the interest rate r as satisfying $q_0 = 1/(1 + r)$. Define the risk-adjusted probabilities φ_s as

$$\varphi_s = \frac{\eta_s}{\sum_{s'=1}^{m} \eta_{s'}},$$

for all $s = 1, 2, ..., m$. Show that the price of any asset k is given by the expected discounted payoffs, where the expectation is taken using the risk-adjusted probability φ,i.e.

$$q_k = E_\varphi \left[\frac{d_k}{1 + r} \right] = \sum_{s=1}^{m} \frac{d_{ks}}{1 + r} \varphi_s,$$

for all $k = 1, 2, ..., K$.

Exercise 9. Consider the agent's i problem in a security market economy. Replace her consumption x by

$$x_s = \sum_{k=1}^{K} d_{ks} h_k + \tilde{e}_s^i,$$

for each $s = 1, 2..., m$ in her utility function u^i, so that we have $u^i \left(D^T h + \tilde{e}^i \right)$ or

$$u^i \left(\sum_{k=1}^{K} d_{k1} h_k + \tilde{e}_1^i, ..., \sum_{k=1}^{K} d_{ks} h_k + \tilde{e}_s^i, ..., \sum_{k=1}^{K} d_{km} h_k + \tilde{e}_m^i \right).$$

The agent's problem is

$$\max_h u^i \left(D^T h + \tilde{e}^i \right),$$

subject to

$$q^T h = q^T \theta^i.$$

1) Write down the first order conditions of this problem for each h_k, $k = 1, 2, ...K$ using μ_i for the multiplier of the budget constraint.

2) Use these FOCs to show that

$$\begin{aligned} q_k &= \sum_{s=1}^{K} d_{ks}\ \eta_s^i \text{ for all } k = 1, ..., K, \\ \eta_s^i &= \frac{1}{\mu_i} \frac{\partial u^i}{\partial x_s} \left(x^i \right). \end{aligned}$$

3) Suppose that the allocation $\left\{ x^i \right\}$ is PO. Show that η_s^i are the same for all agents [Hint: use the FOC for the λ weighted planning problem with $\lambda_i = 1/\mu_i$ and $\gamma = \eta$].

4) Suppose that markets are not complete and that η^i differ across agents. Is the allocation PO?

Exercise 10. Specialize u^i to expected utility so that

$$u^i(x) = \sum_{s=1}^{m} v^i(x_s)\, \pi_s,$$

and $u^i \left(D^T h + \tilde{e}^i \right)$ is given by

$$u^i \left(D^T h + \tilde{e}^i \right) = \sum_{s=1}^{m} v^i \left(\sum_{k=1}^{K} d_{ks} h_k + \tilde{e}_s^i \right) \pi_s.$$

Repeat the analysis of the previous exercise to show that in this case the η^i solving

$$q_k = \sum_{s=1}^{K} d_{ks}\ \eta_s^i \text{ for all } k = 1, ..., K,$$

are given by

$$\eta_s^i = \frac{1}{\mu_i} \frac{\partial v^i \left(x_s^i\right)}{\partial x} \pi_s,$$

for all $s = 1, 2..., m$. Show that the corresponding risk-adjusted probabilities φ_s^i are given by

$$\varphi_s^i = \frac{\partial v^i \left(x_s^i\right) / \partial x}{\sum_{s'=1}^m \partial v^i \left(x_{s'}^i\right) / \partial x \ \pi_{s'}} \ \pi_s.$$

Exercise 11. Assume that

$$\theta_k^i = 0,$$

for all $k = 1, ..., K$ and $i \in I$. Furthermore assume that

$$\operatorname{cov}\left[\frac{\partial v^i \left(\tilde{e}^i\right)}{\partial x}, d_k\right] = 0 \text{ for all } k \text{ and } i,$$

i.e.

$$\sum_{s=1}^m \left(\frac{\partial v^i \left(\tilde{e}_s^i\right)}{\partial x}\right) \left(d_{ks} - \sum_{s'=1}^m d_{ks'} \pi_{s'}\right) \ \pi_s = 0,$$

for all $k = 1, 2, ..., K$ and $i \in I$.

1) Show that the equilibrium in this case has

$$\begin{aligned} h^i &= 0, \\ x^i &= \tilde{e}^i, \\ q_k &= E\left[d_k\right] = \sum_{s=1}^m d_{ks} \pi_s \text{ all } k = 1, 2, ..., K. \end{aligned}$$

2) Moreover assume that

$$\sum_{i \in I} \tilde{e}_s^i = \sum_{i \in I} \tilde{e}_{s'}^i > 0,$$

for all $s, s' = 1, 2, ..., m$ but that $\tilde{e}^i$ has strictly positive variance. Is the equilibrium allocation PO?

3) Are the η^i all the same?

4) Consider the particular case where

$$\begin{aligned} m &= 4, \pi_s = 1/4 \text{ all } s, \\ v^i(c) &= -\frac{1}{2}(5-c)^2, \quad I = 2, \end{aligned}$$

endowments are

$$\begin{bmatrix} \tilde{e}_1^1 \\ \tilde{e}_2^1 \\ \tilde{e}_3^1 \\ \tilde{e}_4^1 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix},$$

$$\begin{bmatrix} \tilde{e}_1^2 \\ \tilde{e}_2^2 \\ \tilde{e}_3^2 \\ \tilde{e}_4^2 \end{bmatrix} = \begin{bmatrix} 4 \\ 3 \\ 2 \\ 1 \end{bmatrix},$$

payoffs of securities

$$\begin{bmatrix} d_{11} \\ d_{12} \\ d_{13} \\ d_{14} \end{bmatrix} = \begin{bmatrix} 1/4+\alpha \\ 1/4+\beta \\ 1/4-\alpha \\ 1/4-\beta \end{bmatrix},$$

$$\begin{bmatrix} d_{21} \\ d_{22} \\ d_{23} \\ d_{24} \end{bmatrix} = \begin{bmatrix} 1/4+\gamma \\ 1/4+\delta \\ 1/4 \\ 1/4-(\gamma+\delta) \end{bmatrix},$$

with

$$\begin{aligned} \alpha + 2\beta - 3\alpha - 4\beta &= 0, \\ 4\alpha + 3\beta - 2\alpha - \beta &= 0, \end{aligned}$$

and

$$\begin{aligned} \gamma + 2\delta - 4(\gamma+\delta) &= 0, \\ 4\gamma + 3\delta - (\gamma+\delta) &= 0, \end{aligned}$$

and endowment of securities $\theta^i = 0$ for $i = 1, 2$. Compute q, h^i, and η^1 and η^2 for this example.

2 Equilibrium in the Security Market

We start with the case of no labor market income at all, i.e. $\hat{e}_s^i = 0$ for all i and s. In this case, the budget constraint is given by:

$$\begin{aligned}\sum_{k=1}^{K} q_k h_k^i &= \sum_{k=1}^{K} q_k \theta_k^i \\ x_s^i &= \sum_{k=1}^{K} h_k^i d_{ks} \text{ for } s = 1, 2, ..., m.\end{aligned}$$

a) Write down and expression for the equilibrium aggregate consumption $\bar{e}_s$ in state s. Your answer should imply:

$$\bar{e} = D^T \bar{\theta},$$

where $\bar{\theta} \in R^K$. Give an interpretation of $\bar{\theta}$ in terms of the elements that define the security market economy (one line maximum).

b) Consider an example with two agents: $i = 1, 2$. Agent 1 has subutility function

$$v^1(x) = \log(x),$$

and agent 2,

$$\begin{aligned}v^2(x) &= x - 1 \text{ for } x \leq 1 \text{ and} \\ &= \log x \text{ for } x > 1.\end{aligned}$$

The aggregate endowment takes 4 values:

$$(\bar{e}_1, \bar{e}_2, \bar{e}_3, \bar{e}_4) = (0.5,\ 1.5,\ \ 2.5,\ 3.5),$$

Solve the planning problem for $\lambda_1 = \lambda_2 > 0$. [Hint: Consider the following two problems:

$$\max_{x_s^1} \log\left(x_s^1\right) + \left[\left(\bar{e}_s - x_s^1\right) - 1\right],$$

and

$$\max_{x_s^1} \log\left(x_s^1\right) + \log\left(\bar{e}_s - x_s^1\right).$$

Write the solution of each of these problems, obtaining x_s^2 as a function of $\bar{e}_s$. Analyze which of the problems applies as a function of $\bar{e}_s$, that is, for which solution we are using the relevant expression for the utility function of agent 2. Note that we allow the consumption of agent 2 to be negative].

c) Continue with the example developed in part *b*. Suppose there is a security market with two securities, $K = 2$, with payoffs given by

$$d_{1s} = 0.5 \text{ for } s = 1, 2, 3, 4,$$

for the first security and

$$(d_{21},\ d_{22},\ d_{23},\ d_{24}) = (0,\ 1,\ 2,\ 3),$$

for the second security. The aggregate supply of these securities satisfies:

$$\begin{aligned} \theta_1^1 + \theta_1^2 &= 1, \\ \theta_2^1 + \theta_2^2 &= 1. \end{aligned}$$

Show that there is an endowment of securities θ_k^i for $i = 1, 2$ and $k = 1, 2$ satisfying the two restrictions on their aggregate supply such that the security market equilibrium reproduces the P.O. allocations obtained in the A-D economy with $\lambda_1 = \lambda_2$. [Hint: use your answer to parts *a*) and *b*)].

d) Consider now the case where $L = R^m$ and $X = L$ (so we are allowing negative consumption) and the subutility functions $v^i : R \to R$ are exponential with parameter ρ_i :

$$v^i(x) = (-1/\rho_i)\exp(-\rho_i x).$$

The parameter ρ_i measures the curvature of v^i, defined as $-v''(x)/v'(x)$, and is referred to as the coefficient of "absolute risk aversion". Its reciprocal, $1/\rho_i$, is called "risk tolerance".

d.0) To help interpreting the results below, show the following: if x is normally distributed with $x \sim N\left(\mu, \sigma^2\right)$, then the insurance premium $\hat{p}$ defined implicitly by

$$v^i\left(Ex - \hat{p}\right) = E\left[v^i\left(x\right)\right],$$

satisfies:

$$\hat{p} = \frac{1}{2}\rho^i\sigma^2.$$

Give a (one line) intuitive explanation of this formula. [Hint: use the formula for the expected value of the exponential of a normal variable in the notes, as well as the definition of the insurance premium above].

Now you should solve the planner's problem for the A-D economy with weights λ_i. In particular:

$d.1$) Obtain an expression for x_s^i as a function of λ_i and γ_s using the FOC w.r.t. x_s^i and letting γ_s be the multiplier of the resource constraint at state s.

$d.2$) Use $d.1$ and market clearing $\left(\sum_{i\in I} x_s^i = \bar{e}_s\right)$ to obtain an expression for $\log \gamma_s$ as a function of $\bar{e}_s$, π_s, $\{\rho_i\}_{i\in I}$ and $\{\lambda_i\}_{i\in I}$. Use the following normalization for λ_i:

$$\sum_{i'\in I} \frac{1}{\rho_{i'}} \log \lambda_{i'} = 0.$$

Give an economic interpretation of the effects of $\bar{e}_s$, π_s and $\{\rho_i\}$ in your formula for γ_s (three lines maximum).

$d.3$) Use your answer to $d.2$ and $d.1$ to write the value of x_s^i as a function of $\bar{e}_s$, $\{\rho_i\}_{i\in I}$ and $\{\lambda_i\}_{i\in I}$. Use the following normalization for λ_i:

$$\sum_{i'\in I} \frac{1}{\rho_{i'}} \log \lambda_{i'} = 0.$$

In particular, you have to show that

$$x_s^i = \frac{1/\rho_i}{\left(\sum_{i'\in I} \frac{1}{\rho_{i'}}\right)} \bar{e}_s + \frac{1}{\rho_i} \log \lambda_i.$$

Give an economic interpretation of the terms and form of this equation (two lines maximum).

$d.4$) Consider now a security market equilibrium. Assume that security $k = 1$ is a riskless bond, so it pays $d_{1s} = 1$ for all $s = 1, 2, ..., m$. Describe a portfolio h_s^i for each agent $i \in I$ and each security $k = 1, 2, ..., K$, so that the consumption of each agent in the security market economy will coincide with the consumption that solves the planner's problem found in $d.3$). [Hint: you can write the holdings for agent i of each security $k = 2, 3, ..., K$ as a fraction of the aggregate portfolio of each security in proportion to the relative risk tolerance of agent i, and include an additional adjustment to the amount of security $k = 1$ held according to the weight of agent i].

$d.5$) Use the value of γ_s solved out in $d.2$) to find q_k, the value of a security k in the CE that gives the same allocation than the planning problem in the A-D economy with weights $\{\lambda_i\}$. The expression for q_k should be a function of the probabilities $\{\pi_s\}_{s\in S}$, the risk aversion parameters $\{\rho_i\}_{i\in I}$, the aggregate endowment $\{\bar{e}_s\}_{s\in S}$, and the payoffs $\{d_{ks}\}_{s\in S}$.

$d.6$) Use your answers to $d.4$ and $d.5$ to find the weights $\{\lambda_i\}$ that correspond to an equilibrium for an arbitrary endowment of securities $\left\{\theta^i\right\}_{i\in I}$. [Hint: replace the h^i_k obtained in $d.4$ and the q_k obtained in $d.5$ into the budget constraint $q^T\ h^i = q^T\ \theta^i$ and solve for λ_i].

Now we turn to the case of positive labor risk. Assume that $\hat{e} = \left(\hat{e}^1, \hat{e}^2, .., \hat{e}^I\right)$ and $d = (d_1, d_2, ..., d_K)$. We let asset $k = 1$ be the riskless asset, so it has zero variance and $d_1 = 1$. We let the remaining $K - 1$ asset have expected values of dividends given by μ_d. We use μ_e for the vector of expected values of labor earnings for all agents with $\mu_e(i)$ denoting the one for agent i. We use $\hat{d} = (d_2, d_1, ..., d_K)$ to denote the risky assets payoffs.

Assume that all agents use expected utility and have exponential utility function with absolute risk aversion coefficient ρ, so that $u^i\left(x^i\right) = E\left[v\left(x^i\right)\right]$, where

$$v(x) = -(1/\rho_i)\exp(-\rho_i\ x).$$

$e.1$) In an equilibrium, agents maximize

$$E\left[v^i\left(\hat{e}^i + \sum_{k=1}^{K} h^i_k d_k\right)\right],$$

subject to

$$q^T h^i = q^T \theta^i.$$

Assume that $\hat{e}^i$ and $\hat{d}$ are independent. Show that with this assumption and with the exponential utility function v^i this problem is equivalent to

$$\max_{h^i} E\left[v^i\left(\sum_{k=1}^{K} h_k^i d_k\right)\right],$$

subject to $q^T h^i = q^T \theta^i$.

$e.2)$ Given your result in question $d)$, are equilibrium prices of securities q affected by the the presence of random labor income that is independent of the asset payoffs and returns?

$e.3)$ Describe the consumption allocation x_s^i in the equilibrium with independent labor income [Hint: use your answer to $d)$]. Is this equilibrium allocation Pareto Optimal in the A-D (contingent claims) economy?

$e.4)$ From now on assume that asset and labor income are normally distributed. Let the variance $\operatorname{var}\left(\hat{d}\right) = \Omega_{dd}$, $\operatorname{var}(\hat{e}) = \Omega_{ee}$, and $\operatorname{cov}\left(\hat{d}, \hat{e}\right) = \Omega_{ed}$, with typical elements $\mu_e(i)$, $\mu_d(k)$, $\Omega_{ee}(i,j)$, $\Omega_{dd}(k,s)$, etc.

$$\begin{bmatrix} \hat{e} \\ \hat{d} \end{bmatrix} \sim N\left(\begin{bmatrix} \mu_e \\ \mu_d \end{bmatrix}, \begin{bmatrix} \Omega_{ee} & \Omega_{ed} \\ \Omega_{ed} & \Omega_{dd} \end{bmatrix}\right).$$

Let $x^i(h) = \sum_{k=1}^{K} h_k^i \ d_k + \hat{e}^i$, so that

$$\begin{aligned} E\left(x^i(h)\right) &= \sum_{k=1}^{K} h_k \ E(d_k) + E\left(\hat{e}^i\right) \\ &= h_1 + \left(\hat{h}\right)^T \mu_d \ + \mu_e(i), \end{aligned}$$

where we let $\hat{h} = (h_2, h_3, ..., h_K)$ denote the $K-1$ vector of portfolio holdings for risky assets and

$$\operatorname{var}\left(x^i(h)\right) = \hat{h}^T \ \Omega_{dd} \ \hat{h} + \Omega_e(i,i) + 2\sum_{k=1}^{K-1} \hat{h}_k \ \Omega_{ed}(i,k).$$

Show that

$$E\left[v\left(x^i(h)\right)\right] = -(1/\rho_i)\exp\left(-\rho_i E\left[x^i(h)\right] + \frac{1}{2}\rho_i^2 \operatorname{var}\left(x^i(h)\right)\right),$$

or

$$-\frac{1}{\rho_i}\log\left(\ (-\rho_i) \ \ E\left[v\left(x^i(h)\right)\right] \ \right) = E\left[x^i(h)\right] - \frac{1}{2}\rho_i \operatorname{var}\left(x^i(h)\right).$$

$e.5$) Consider the increasing transformation $f^i : R_- \to R$,

$$f^i(u) = -\frac{1}{\rho_i}\log(-\rho_i\ u),$$

where $u^i(h) = E\left[v\left(x^i(h)\right)\right]$. Show that

$$w^i(h) = f\left(u^i(h)\right) = E\left[x^i(h)\right] - \frac{1}{2}\rho_i \operatorname{var}\left(x^i(h)\right).$$

$e.6$) Consider the problem

$$\max_h\ E\left[x^i(h)\right] - \frac{1}{2}\rho_i \operatorname{var}\left(x^i(h)\right),$$

subject to

$$q^T h = q^T \theta,$$

where the expressions for $E\left[x^i(h)\right]$ and $\operatorname{var}\left(x^i(h)\right)$ are given above.

Normalizations. From now on we normalize $q_1 = 1$ (which, since $d_1 = 1$ means that we normalize the rate of interest to zero). We also normalize $\mu_d(k) = 1$ for each $k = 2, ..., K$. We denote the vector μ_d by ι, a $K-1$ vector of ones. This means that $1/q_k$ is the gross expected return of asset k, and since the interest rate is zero, it is also the gross risk premium. We can think of $1 - q_k$ as a measure of the risk premium of asset k.

Use the budget constraint to write

$$h_1 = q^T\theta - \hat{q}^T\hat{h},$$

where $\hat{q}$ denotes the $K-1$ vector of prices of risky assets $\hat{q} = (q_2, q_3, ..., q_K)$. Thus, the problem is equivalent to

$$\max_{\hat{h}}\ \left[q^T\theta + \hat{h}^T(\iota - \hat{q})\ + \mu_e(i)\ \right] - \frac{\rho_i}{2}\left[\hat{h}^{\ T}\ \Omega_{dd}\ \hat{h} + \Omega_e(i,i) + 2\sum_{k=1}^{K}\hat{h}_k\ \Omega_{ed}(i,k)\right].$$

Write down the FOC for this problem w.r.t. $\hat{h}_k$ for $k = 2, ..., K$. Collect these FOCs in vector form. Use $\Omega(i,\cdot)$ to denote the vector of the $K-1$ covariances of $\hat{e}^i$. Derive an expression for $\hat{h}^i$ as a function of Ω_{dd}^{-1}, ρ_i, $\hat{q}$, ι and $\Omega_{ed}(i,\cdot)$. Derive an expression for

$h_1^i(\hat{q})$ [Hint: your expression should be a function of Ω_{dd}^{-1}. ρ_i, $\hat{q}$, ι and $\Omega_{ed}(i,\cdot)$]. Assume (only for this last part of $e.6$) that Ω_{dd} is diagonal, solve for $\hat{h}_k^i$ explicitly, and give an economic explanation of each of the terms on it.

$e.7$) What is the effect on the demands for risky assets $\hat{h}^i$ and for the riskless asset h_1^i if an agent has higher wealth (say a higher value of θ^i)?

$e.8$) Use that in equilibrium the following $K-1$ equations have to be solved

$$\sum_{i=1}^{I} \hat{\theta}^i = \sum_{i=1}^{I} \hat{h}^i(\hat{q}),$$

where $\hat{h}^i(\hat{q})$ denotes the optimal demand of risky asset, and $\hat{\theta}^i$ denotes the endowment of the $K-1$ risky assets for agent i. Write an explicit solution for $\hat{q}$. Your solution should be a function of the reciprocal of the average risk tolerance, $1/\left(\sum_i 1/\rho_i\right)$, the variance Ω_{dd} of assets, the vector of the endowment of risky assets, $\sum_i \hat{\theta}^i$, and the sum of the covariances, $\sum_i \Omega_{ed}(i,\cdot)$. Give an economic explanation of the effect of each of these terms (four lines maximum).

In the next exercise we will study the risk sharing implications for equilibrium allocations and Pareto Optimal allocations in the context of a pure exchange economy, with one good, two periods, S states of nature, and preferences given by expected utility. We will first specialize to the case of a one period economy and then solve the two period version. This was a midterm question in 2007.

3 Uncertainty

Consider a one period pure exchange economy with uncertainty. Let there be S states of the world and one physical commodity. We will index a commodity by the state, so there are $m = S$ goods, or $L = R^m$. Thus, we can interpret the vector x as the consumption of the good in each of the different states. We will write x_s for the good in state s. Thus the utility function u^i is a function of a vector on R^m. The endowment of agent $i \in I$, denoted by e^i are also indexed by state s. We let $\bar{e}_s$ be the aggregate endowment in state s.

Assume u^i is given by

$$u^i (x_1, x_2, ..., x_s) = \left(v^i\right)^{-1} \left(\sum\nolimits_{s=1}^{m} v^i (x_s) \pi_s\right) \qquad (2)$$

where $\left(v^i\right)^{-1}(\cdot)$ is the inverse function of v^i, $v^i : R \to R$ is the sub-utility function of agent i and $\pi_s \in R^m_+$ is the common probability of state s, with $\sum_{s=1}^m \pi_s = 1$. We assume that v^i are differentiable, strictly increasing and strictly concave.

Question 1. State, without proof, a result that characterizes the Pareto Optimal allocation, i.e., how x^i_s depends on $\bar{e}_s$. State carefully all the characteristics of the functions that are mentioned (for instance, what is the domain of the functions).

Hint: This comes straight from the notes.

Question 2. Take two arbitrary states, s and s', write the Marginal Rate of Substitution (MRS) between them using (2)

Question 3. Assume that $v^i (x) = x^{1-\gamma} / (1 - \gamma)$. What is the form of the function $g^i (\bar{e}_s)$ that describes the PO allocation described in your answer to Question 1?

Consider the security market economy as the one described in your lecture notes. Recall that the budget constraint is written in two (set of) equations. The first equation is given by

$$\sum_{k=1}^{K} h_k^i q_k = \sum_{k=1}^{K} \theta_k^i q_k$$

The second equation is one for each state $s = 1, .., m$:

$$x_s^i = \sum_{k=1}^{K} h_k^i d_{ks} + \hat{e}_s^i$$

Let security $k = 1$ be a risk-less bond, i.e. it pays $d_{ks} = 1$ in all states of nature. Security $k = 2$ is similar to a stock, it pays $d_{ks} = \bar{e}_s$. Assume complete markets, that $v^i(x) = x^{1-\gamma}/(1-\gamma)$ and that $\log \bar{e} \sim N(\mu, \sigma^2)$.

Question 4. Write down and expression for the risk premium.

Hint: recall that the risk premium is given by $(1 + r_2)/(1 + r_1)$. With this particular utility form, this is given by

$$\frac{1 + r_2}{1 + r_1} = \frac{E[\bar{e}^{-\gamma}] E[\bar{e}]}{E[\bar{e}^{-\gamma}\bar{e}]}$$

Also, recall that if X is lognormally distributed, then

$$\begin{aligned} E[X] &= \exp\left(\mu + \frac{1}{2}\sigma^2\right) \\ E[X^{-\gamma}] &= \exp\left(-\gamma\mu + \gamma^2 \frac{1}{2}\sigma^2\right) \end{aligned}$$

4 A Two Period Economy

Consider a version of an endowment economy in which there is no uncertainty and there are only two periods, then $L = R^2$. Let $u^i(x_1, x_2) = F^i(x_0, x_1)$.

Question 5. Write down the Arrow-Debreu budget set.

Hint: One equation

Now we consider an alternative representation of the economy. Let $d = 1$ be the payoff of the security. Denote by q be the price of the security (measured in units of the good x_0), h^i the purchases of this security by agent i, θ^i the endowment of this security by agent i, and by $\hat{e}_0^i$ and $\hat{e}_1^i$ the endowment of goods in period 0 and 1.

Question 6. Write down the budget set of the corresponding security market economy. This consists of two equations, one that applies to period 0,when the sale and purchase of the security takes place, and the other for period 1.

Question 7. How does the net interest rate relate to q?

Question 8. How does the Arrow Debreu prices p relate to the price of the security q?(One line)

Question 9. .In this context, what will be the endowments in the Arrow Debreu economy e^i that correspond to the endowments in the security market θ^i, and $\hat{e}^i$?

Question 10. What is the MRS in this case? Interpret your results and relate them to the interest rate.

Specialize to the following Constant Elasticity of Substitution form:

$$F^i(x_0, x_1) = \left((1-\beta)\, x_0^{1-\rho} + \beta x_1^{1-\rho}\right)^{\frac{1}{1-\rho}}, \quad \text{for } \rho > 0$$

Question 11. Calculate the equilibrium interest rate only as a function of β, ρ and $\bar{e}_o/\bar{e}_1$ (the ratio of the aggregate endowments in period 0 over period 1).

Question 12. Why the equilibrium interest rate does not depend on the distribution of initial endowments?

4.1 A Two Period Economy with Uncertainty

Consider a two period pure exchange economy with uncertainty. In the current period, commodity "0" is consumed, while in the next period, commodities "1", ..."S" are consumed in state s. This means that there are $m = S+1$ commodities and $L = R^{S+1}$,. u^i is given by $u^i(x_0, x_1, ..., x_S)$, hence you have preferences over your consumption today, commodity "0", and consumption tomorrow according to what state you are in. The endowment of agent i, denoted by e^i also belong to R^{S+1}. We let $\bar{e}$ be the aggregate endowment.

Question 13. Consider the following securities. Denote by d_{ks} be the payoff of security $k \in K$ in sate s. Denote by q_k be the price of security k (measured in units of commodity

"0"), h_k^i the purchases of this security by agent i, θ_k^i the endowment of this security by agent i, and by $\hat{e}^i$ the endowment of goods.

Write the budget constraint for the security market as a set of two equations, one that applies to period 0 where consumption of x_0 and trade take place, and a set of equations for each state in period s.

Question 14. Assume that prices and payoffs in the security market are consistent with state prices of the Arrow-Debreu economy, and that the endowments of both economy are equivalent. Which Budget Set is larger?

Question 15. When are the Budget Sets of the security market and the Arrow Debreu economy the same?

From now on specialize the utility to be separable, so that we can highlight both how agents substitute consumption across the different states of the world as well as intertemporally. Let preferences by given by

$$u^i(x_0, x_1, ..., x_S) = F^i(x_0, \psi(x_1, ..., x_S)) \tag{3}$$

where

$$\psi(x_1, ..., x_S) = (v^i)^{-1}\left(\sum_{s=1}^{S} v^i(x_s)\,\pi_s\right) \tag{4}$$

π_s is the common probability of state s. We assume that v^i is strictly increasing and concave and that F^i is strictly increasing and concave in both arguments.

To understand these preferences, consider a bundle in which $x_1 = x_2 = ...x_S = x'$.

Question 16. What is $(v^i)^{-1}\left(\sum_{s=1}^{S} v^i(x)\,\pi_s\right)$ evaluated at $x = x'$?

Question 17. Compute F_1^i/F_2^i, where F_j^i is the partial derivative of the function F^i with respect to argument j. Give an intuitive interpretation of your result.

Question 18. Take two arbitrary states, s and s', where $s > 0$. What is the MRS between these two states? Be very explicit in all the terms you write. Does it depend on x_0? Does it depend on F^i? Why? (1 line).

It is easy to show that the following theorem holds for this economy:

Theorem 1 *Fix and arbitrary vector of λ weights. If x_s^i is a Pareto optimal allocation, then there are strictly increasing functions g^i, $g^i : R_+ \to R_+$, of the aggregate endowment, so that $x_s^i = g^i(\bar{e}_s)$ for all $i \in I$, $s = 1, .., S$.*

Question 19.You need to answer True or False to the following questions about the function g^i on the previous theorem:

1) Does $g^i(\cdot)$ depend on the λ (weights)?
True / False
2) Does $g^i(\cdot)$ depend on F^i?
True / False
3) Does $g^i(\cdot)$ depend on $\bar{e}_0$?
True / False
4) Does $g^i(\cdot)$ depend on $v^i(\cdot)$?
True / False
5) Does $g^i(\cdot)$ depend on π_s?
True / False

Question 20. Write down an expression for first order conditions for h_1^i, h_2^i and x_0^i. Use μ_i for the Lagrange multiplier of the budget constraint.

Question 21. Define the expected gross return of the securities 1 and 2, i.e $(1 + r_1) = E[d_1]/q_1$ and $(1 + r_2) = E[d_2]/q_2$. What is the relationshipt between r_1 and the interest rate? Using your answer to Question 20, do $1 + r_1$ and $1 + r_2$ depend on the choice of numeraire?

Question 22. Define the multiplicative risk premium as $(1 + r_2)/(1 + r_1)$, the ratio of the expected gross return of the securities 2 to 1. Write down an expression for $(1 + r_2)/(1 + r_1)$. Does your answer depend on F^i or x_0^i ? Is your expression the same as the one for an economy with uncertainty but with only one period (like the one discussed in the class notes)?

Now, we further specialize the utility function. Let

$$F^i(x_0, \psi(x_1, ..., x_S)) = \left((1 - \beta)x_0^{1-\rho} + \beta(\psi(x_1, ..., x_S))^{1-\rho}\right)^{\frac{1}{1-\rho}},$$

for $\rho > 0$, $\psi(x_1, ..., x_S)$ given by (4) and $v^i(x) = x^{1-\gamma}/(1 - \gamma)$.

Question 23. Are the preferences described above identical and homothetic? (yes or no answer)

Assume complete markets, and that $\log \bar{e}_s/\bar{e}_0 \sim N(\mu, \sigma^2)$ and $\bar{e}_0 = 1$. Recall that i X is lognormally distributed, i.e. if $\log X$ is $N(\mu, \sigma^2)$ then

$$\begin{aligned} E[X] &= \exp\left(\mu + \frac{1}{2}\sigma^2\right) \\ E\left[X^{-\gamma}\right] &= \exp\left(-\gamma\mu + \gamma^2 \frac{1}{2}\sigma^2\right) \end{aligned}$$

Question 24. Use the answer to your previous question to find an expression for $1 + r_1$. Your answer should be a function of the preference parameters ρ, β, γ and the parameters describing the distribution of the aggregagte endowment σ and μ. How does this compare with your answer for the economy with two periods and no uncertainty?

Question 25. Use the answer to your previous question and the formulas for the expected value of a lognormal, to find expression for the mulitiplicative risk premium $(1 + r_2) / (1 + r_1)$ in this economy. Your expression should be a function of should be a function of the preference parameters ρ, β, γ and the parameters describing the distribution of the aggregagte endowment σ and μ. How does this compare with the expression for an economy with uncertainty and only one period, like the one in the class notes?

5 Uncertainty in an OLG Model

This was the core question for 2006-2007.

Time is discrete and starts at $t = 1$. Agents live for 2 periods, and have a labor endowment only when they are young. There is mass one of agents of each age, or if you prefer one agent per cohort. We index agents by the year in which they were born.

There is one unit a perishable "tree" in the economy. This tree gives a fruit, in consumption units, of d_t each period. The labor endowment of an agent born at time t is denoted by w_t. The young born at $t = 1, 2,$ have no endowment of trees. The old born at $t = 0$ have, at time $t = 1$, an endowment of 1 tree. Labor endowment and fruits are both proportional to the same random variable, Y_t :

$$w_t = (1 - \delta)\; Y_t \text{ and } d_t = \delta\; Y_t$$

where $\delta \in (0, 1)$ is a constant, and where Y_t satisfy

$$Y_{t+1} = Y_t \; z_{t+1}$$

with $Y_1 = 1$ and $\{z_{t+1}\}$ i.i.d. and $z_{t+1} > 0$.

There is one unit of the tree in the economy. Feasibility is then given by

$$c_t^t + c_t^{t-1} = w_t + d_t = Y_t \text{ for all } t \geq 1.$$

The preferences for the young born at $t \geq 1$ are:

$$(1 - \beta)\, u\left(c_t^t\right) + \beta E_t\left[u\left(c_{t+1}^t\right)\right]$$

where $E_t[\cdot]$ is the conditional expectation. The preferences for the old born at $t = 0$ are c_1^0 .

The budget constraint of agent born at t when young is:

$$c_t^t + s_t q_t^e + b_t q_t^b = w_t$$

where s_t are his purchases of the tree at price q_t^e, b_t are the purchases of a one period discount bond- a bond that pays one unit of consumption at $t + 1$, with current price q_t^b. At time $t + 1$, when this agent becomes old his consumption is financed by the fruits and sale of the trees, as well as the purchases of the discount bonds:

$$c_{t+1}^t = s_t \ \left[q_{t+1}^e + d_{t+1}\right] + b_t \ .$$

Question 1. [5 points] Using the budget constraint of the agent, replace c_{t+1}^t into the preferences and write down the objective function of the young agent born at time t as a function of s_t and b_t, taking current prices q_t^e, q_t^b and the distribution of future prices q_{t+1}^e and fruits d_{t+1} as given.

Question 2. [5 points] Write down the first order conditions (foc's) with respect to s_t and b_t of the problem stated above:

Question 3. [5 points] What has to be the equilibrium holding of trees of for generation $t \geq 1$, i.e. what has to be the equilibrium values of s_t for $t \geq 1$?

Question 4. [5 points] Use the answer for the previous question, and the assumption that initial old (born at $t = 0$) have zero initial endowment of the discount bonds ($b_0 = 0$) to find out what should be the equilibrium value of b_t be for all $t \geq 1$?

Question 5 . [10 points] Specialize the foc's obtained in Question 2 replacing the equilibrium values of s_t and b_t obtained in the Questions 3 and 4.

Question 6. [10 points] Assume from now on that preferences are given by $u(c) = \log(c)$. Guess that $q_t^e = Y_t \ Q^e$ and that $q_t^b = Q^b$ where Q^e and Q^b are two constants. Define $R_{t+1}^e = \left(q_{t+1}^e + d_{t+1}\right)/q_t^e$ and $R_t^b = 1/q_t^b$ as the gross returns on the trees and discount bonds.

Find i) an expression for the price of the three, Q^e as a function of δ and β, ii) an expression for the gross expected return of the tree, $E_t\left[\left(q_{t+1}^e + d_{t+1}\right)/q_t^e\right]$, as a function of δ, β and $E[z]$, and iii) an expression for the gross return on the discount bond $1/Q^b$ as a function of β, δ and $E[1/z]$.

Hints: use the foc's obtained in the previous question, as well as $d_{t+1} = \delta Y_{t+1}$, $w_t = (1-\delta) Y_t$, $z_{t+1} = Y_{t+1}/Y_t$, $q_{t+1}^e = Y_{t+1} \ Q^e$, $q_t^e = Y_t Q^e$.

Question 7. [10 points] To double check your answer to Question 6.i), consider the following alternative reasoning, which uses that with log utility the share of expenditure is proportional to the share parameter in preferences. i) Using that the total wealth of a young agent is given by w_t, express the value of a young agent's consumption c_t^t as a function of $\beta,\ \delta$ and Y_t. ii) Assuming that $b_t = 0$ and $s_t = 1$ use the answer of i) and the budget constraint of the young agents to derive and expression for Q^e. Compare this expression with your answer to i) in the previous question. iii) Give a short explanation of why Q^e is increasing in β and decreasing in δ (maximum two lines for each parameter).

Assume from now on that $x = \log z$ is normally distributed with expected value μ and variance σ^2. Recall that

$$Ee^x = e^{\mu+(1/2)\sigma^2}$$

Let g be the expected continuously compounded growth rate of Y_t, so $g = \mu + (1/2)\,\sigma^2$.

Question 8. [15 points] Under the log normal assumption for z find: i) an expression for the expected return of the trees as a function of $\beta,\ \delta,\ g$ and σ^2, ii) an expression for the level of the gross return on the discount bond as a function of $\beta,\ \delta,\ g$ and σ^2, and iii) an expression for the multiplicative risk premium, i.e. the ratio of the gross expected return on the tree and the gross return on the discount bond as a function of only one of the parameters of the model (β, δ, g and σ^2).

Hint: To compute $E\,(1/z)$ use the formula displayed above for $(1/z) = e^{-x}$.

Question 9 . [10 points] Is the equilibrium value of aggregate consumption -using the corresponding Arrow-Debreu prices- finite?

Hint. To answer this, write down the present value of aggregate consumption as a function of the price of a tree. Notice that by paying the price of the tree an agent can obtain the right to the fruits d_t at all dates $t \geq 2$. Recall that feasibility can be written as $c_t^t + c_t^{t-1} = Y_t$, and that we assume that $d_t = \delta Y_t$.

Question 10 . [5 points] Show that for σ^2 large enough, $R^b\ < e^g$, so that net interest rates are smaller than the expected growth rate of fruits, and hence of GDP.

Question 11. [20 points] Given your answer to the previous questions, i) is the competitive allocation Pareto Optimal?, ii) Can the introduction of social security improve the welfare of all generations?, iii) Does your answer depends on the value of interest rates being smaller, equal or higher than the (expected) growth rate of the economy?, iv) Recall that in the last 100 years the average for the US economy the real net return on US government bonds (the risk-free rate) is about 3% per year, the average net growth rate of total GDP is about 3%, and the difference between the net real return of a well diversified equity portfolio such as the SP500 and the net risk-free rate is about 5% per year. Using this simple model as a guide, what you conclude about the presence (or lack of) dynamic inefficiency in the US economy?