Theory of Income, Fall 2008
Fernando Alvarez, U of C

Problem Set 3

1 Arbitrage Opportunities

Consider a one period pure exchange economy with uncertainty. Let there be m states of the world and one physical commodity. Assume that there are security markets as described in class note 4 with K securities, each with price q_k, and dividends d_{ks} in state s. Agent i is endowed with θ_k^i units of security k and with a vector $\tilde{e}^i \in R_+^m$ of goods.

Definition. Arbitrage Opportunities.
We say that D, q admits arbitrage opportunities if there is a portfolio h such that:

$$\sum_{k=1}^{K} q_k h_k \leq 0 \text{ and } \sum_{k=1}^{K} d_{ks} h_k \geq 0 \text{ for all } s = 1, ..., m,$$

with at least one of the $m+1$ inequalities being strict.

Notice that we can write the conditions for an arbitrage opportunity in vector form as

$$q^T h \leq 0 \text{ and } D^T h \geq \bar{0},$$

with at least one strict inequality, where $\bar{0}$ is a $m-$dimensional vector of zeroes.

Exercise 1. Assume that $u^i : R_+^m \to R$ is strictly increasing and that there exists an asset that pays positive in all states of nature and strictly positive in at least one of them (i.e. one of the rows of D is positive with at least one entry being strictly positive, for example a risk free bond). Show that if (q, D) admits arbitrage opportunities then the agent problem in the security market economy has no solution.

Ans: Without loss of generality let the asset that pays positive in all states of nature and strictly positive in at least one of them be asset number 1. The agent's problem is

$$\max_{\{x_s, h_k\}_{s=1}^{m}, {}_{k=1}^{K}} u^i (x_1, x_2, ..., x_s),$$

subject to

$$\sum_{k=1}^{K} q_k h_k \leq \sum_{k=1}^{K} q_k \theta_k^i, \tag{1}$$

$$x_s = \sum_{k=1}^{K} d_{ks} h_k + \tilde{e}_s^i, \ s = 1, ..., m. \tag{2}$$

By contradiction, suppose that there exists a solution to the above problem and that there are arbitrage opportunities. Let (x^*, h^*) denote the candidate solution to the problem. u^i being strictly increasing and the existence of asset 1 imply that the constraint (1) will hold with equality at any optimal choice. Then (x^*, h^*) satisfies

$$x_s^* = \sum_{k=1}^{K} d_{ks} h_k^* + \tilde{e}_s^i, \ s = 1, ..., m, \tag{3}$$

$$\sum_{k=1}^{K} q_k h_k^* = \sum_{k=1}^{K} q_k \theta_k^i. \tag{4}$$

We will show that there exists some feasible $(\bar{x}, \bar{h})$ such that $u^i(\hat{x}) > u^i(x^*)$ (i.e. (x^*, h^*) is not optimal). By the existence of arbitrage opportunities, there exists some portfolio $\tilde{h}$ such that

$$\sum_{k=1}^{K} q_k \tilde{h}_k \leq 0 \text{ and } \sum_{k=1}^{K} d_{ks} \tilde{h}_k \geq 0, \ s = 1, ..., m,$$

with at least one strict inequality. First notice that the portfolio $\bar{h} = h^* + \tilde{h}$ is feasible since

$$\sum_{k=1}^{K} q_k \bar{h}_k = \sum_{k=1}^{K} q_k h_k^* + \underbrace{\sum_{k=1}^{K} q_k \tilde{h}_k}_{\leq 0} \leq \sum_{k=1}^{K} q_k \theta_k^i. \tag{5}$$

We also have that the consumption vector associated with $\bar{h}$ satisfies

$$\bar{x}_s = \sum_{k=1}^{K} d_{ks} \bar{h}_k + \tilde{e}_s^i = \sum_{k=1}^{K} d_{ks} h_k^* + \underbrace{\sum_{k=1}^{K} d_{ks} \tilde{h}_k}_{\geq 0} + \tilde{e}_s^i \geq x_s^*, \tag{6}$$

for $s = 1, .., m$, with at least one of the $m + 1$ inequalities (5) and (6) being strict.

If the strict inequality is in any of the m inequalities (6), then we are done: $\bar{h} = h^* + \tilde{h}$ is feasible and gives at least as much consumption in all states and strictly more in at least one state of nature, a contradiction to (x^*, h^*) being optimal. If the strict inequality is in (5) we have to work a little bit harder. Let $h^1 = (1, 0, 0, ...0)'$ be a portfolio that consists of buying one unit of asset 1. The strict inequality in (5) implies that, for α sufficiently small, the following portfolio is feasible

$$\hat{h} = h^* + \bar{h} + \alpha h^1,$$

so that $\hat{h}$ is budget feasible (we can always do this). But since the portfolio h^1 gives strictly positive payoff in at least one state and non-negative payoffs in all states we are done, since

$$\hat{x}_s = \sum_{k=1}^{K} d_{ks} h_k^* + \underbrace{\sum_{k=1}^{K} d_{ks} \tilde{h}_k}_{=0} + \alpha \underbrace{\sum_{k=1}^{K} d_{ks} h_k^0}_{\geq 0} + \tilde{e}_s^i \geq x_s^*,$$

with at least one s with strict inequality. Since the utility function is strictly increasing, then $u^i(\hat{x}_1, \hat{x}_2, ..., \hat{x}_m) > u^i(x_1^*, x_2^*, ..., x_m^*)$. In other words, we have found a feasible allocation that is strictly preferred to the candidate optimal allocation. Therefore there is no solution to the consumer's problem.

Exercise 2. Show that if (D, q) can be written as

$$q = D\ \eta,$$

for a strictly positive vector $\eta \in R_{++}^m$, i.e.,

$$q_k = \sum_{s=1}^{m} d_{ks} \eta_s; \text{ for all } k = 1, 2, .., K, \tag{7}$$

then there are no arbitrage opportunities.

Ans: By contradiction, assume that there exists a $\eta \in \mathbb{R}_{++}^m$ such that $q = D\eta$ but there are arbitrage opportunities. Then there exists some h such that $q^T h \leq 0$ and $D^T h \geq \bar{0}$ with at least one strict inequality. But since $q = D\eta$ then $q^T = \eta^T D^T$, therefore the existence of arbitrage opportunities implies that for some h,

$$\begin{aligned} \eta^T D^T h &\leq 0, \\ D^T h &\geq \bar{0}, \end{aligned}$$

with at least one strict inequality. Multiplying the second inequality by η^T we obtain

$$\eta^T D^T h \geq 0.$$

Since η is a vector with strictly positive entries, then the two inequalities can hold if and only if each element of the vector $D^T h$ is zero. Therefore we have

$$q^T h = \eta^T D^T h = 0,$$

and

$$D^T h = \bar{0},$$

contradicting the hypothesis that there are arbitrage opportunities.

Exercise 3. Consider the following corollary to the Separating Hyperplane Theorem:

Theorem: Linear Separation of closed convex cones*: Suppose that A and B are closed convex cones in R^n that intersect precisely at zero. If A does not contain a linear subspace other than $\{\bar{0}\}$, then there is a non-zero vector $\lambda \in R^n$ such that $\lambda y < \lambda z$ for all y in B and non-zero z in A. (i.e. the two sets can be strictly separated by a linear function.)*

Comment 1: a set X is a cone if $x \in X$ implies that $\alpha x \in X$ for all $\alpha \geq 0$ (i.e. if a point x belongs to the set, then any ray passing from zero through x for α positive belongs to the set).

Comment 2: Suppose $L \subseteq A$ is a linear subspace of A other than $\{\bar{0}\}$. By definition, if $x, y \in L$ then $\alpha x + \beta y \in L$ for any pair of real numbers α, β. If A does not contain a linear subspace other than $\{\bar{0}\}$ then there is no such a set as L (informally, A contains no lines).

Assume that (q, D) admits no arbitrage opportunities. Show that there must exist a strictly positive vector $\eta \in R^m_{++}$ such that $q = D\eta$, i.e., (7) must hold.

Hints:

1. Consider the sets $A = R^{m+1}_+$ (i.e. the non-negative orthant) and

$$\begin{aligned} B &= \Big\{(y_0, y_1, ..., y_m) : \text{ there is a } h \in R^K \text{ such that} \\ &\quad y_0 = -q^T h \text{ and } (y_1, y_2, ..., y_m)^T = D^T h\Big\}. \end{aligned}$$

Show that both are closed, convex cones.

i) A is closed: it is evident that if we take a convergent sequence of non-negative vectors, the limit point must also be non-negative, so A is closed.

ii) A is a cone: pick $x \in A$, then x is a non-negative vector. Since $\alpha \geq 0$, αx is also a non-negative vector, so $\alpha x \in A$.

iii) A is convex: pick $x_1, x_2 \in A$ and some $\alpha \in [0, 1]$. Then it obvious that $\alpha x_1 + (1 - \alpha) x_2$ is a non-negative vector, so A is convex.

iv) B is closed: take any convergent sequence of elements $\{y_n\}_{n=1}^{\infty}$ entirely contained in B with limit point y^*. This means that there exists a sequence of $\{h_n\}_{n=1}^{\infty}$ converging to some $h^* \in \mathbb{R}^K$ (because y is a linear function of h). Evidently the limit point satisfies $y_0^* = -q^T h^*$ and $(y_1^*, y_2^*, ..., y_m^*)^T = D^T h^*$, hence $y^* \in B$ and B is closed.

v) B is a cone: Take $y \in B$, then there exists an h in $\mathbb{R}^K$ such that $y_0 = -q^T h$ and $(y_1, ..., y_m)^T = D^T h$. Let α be any non-negative number, and consider $y\alpha$. Taking αh in $\mathbb{R}^K$

we immediately see that $\alpha y \in B$, since

$$-q^T \alpha h = \alpha y_0 \text{ and } D^T \alpha h = (\alpha y_1, ..., \alpha y_m)^T,$$

that is, B is a cone.

vi) B is convex: Take y_0 and y_1 in B and $\alpha \in [0, 1]$. Then there are h^0 and h^1 in $\mathbb{R}^K$ such that

$$\begin{aligned} y_0^0 &= -q^T h^0 \text{ and } \left(y_1^0, ..., y_m^0\right)^T = D^T h^0, \\ y_0^1 &= -q^T h^1 \text{ and } \left(y_1^1, ..., y_m^1\right)^T = D^T h^1. \end{aligned}$$

We want to show that $y^\alpha \equiv \alpha y^0 + (1-\alpha) y^1 \in B$. To that end consider $h^\alpha = \alpha h^0 + (1-\alpha) h^1$ in $\mathbb{R}^K$. Then

$$\begin{aligned} y_0^0 &= -q^T h^0 \rightarrow \alpha y_0^0 = -q^T \alpha h^0, \\ y_0^1 &= -q^T h^1 \rightarrow (1-\alpha) y_0^1 = -q^T (1-\alpha) h^1. \end{aligned}$$

Adding the two expressions we obtain

$$y_0^\alpha = \alpha y_0^0 + (1-\alpha) y_0^1 = -q^T \left[\alpha h^0 + (1-\alpha) h^1\right] = -q^T h^a. \tag{8}$$

We also have

$$\begin{aligned} \left(y_1^0, ..., y_m^0\right)^T &= D^T h^0 \rightarrow \left(\alpha y_1^0, ..., \alpha y_m^0\right)^T = D^T \alpha h^0, \\ \left(y_1^1, ..., y_m^1\right)^T &= D^T h^1 \rightarrow \left((1-\alpha) y_1^1, ..., (1-\alpha) y_m^1\right)^T = D^T (1-\alpha) h^1. \end{aligned}$$

Adding the two expression we obtain

$$(y_1^\alpha, ..., y_m^\alpha)^T = D^T h^\alpha, \tag{9}$$

therefore (8) and (9) imply $y^\alpha \in B$.

2. Argue (show) that if there are no arbitrage opportunities, then $A \cap B = \{\bar{0}\}$ (the sets A and B meet precisely at the vector zero).

Ans: Assume to the contrary that there are no arbitrage opportunities but $A \cap B \neq \{\bar{0}\}$. Then there exists a vector $y = (y_0, y_1, ..., y_m)$ with $y_i \geq 0$ $i = 0, 1, ..., m$ with at least one strict inequality that belongs to B. This in turn implies that there is some $h \in \mathbb{R}^K$ such that

$$-q^T h = y_0 \geq 0,$$

and

$$D^T h = (y_1, ..., y_m)^T \geq (0, 0, ..., 0),$$

with at least one strict inequality. In other words, with this h we created an arbitrage opportunity, contradicting our initial hypothesis. Hence $A \cap B = \{\bar{0}\}$.

3. Use the above corollary to the Separating Hyperplane Theorem to show that there is a vector $\lambda \in R^{m+1}$ such that

$$\lambda \; y < \lambda \; z,$$

for all $y \in B$ and all $z \in \bar{A}$, where $\bar{A} = A - \{\bar{0}\}$ (i.e. all non-zero elements in A).

Ans: The corollary to the Separating Hyperplane Theorem together with steps 1 and 2 imply that there is a non-zero vector $\lambda \in \mathbb{R}^{m+1}$ such that

$$\lambda y < \lambda z,$$

for all $y \in B$ and $z \in \bar{A}$.

4. Use the definitions of $\bar{A}$ and B to argue that $\lambda y \leq 0$ for all $y \in B$.

Ans: By contradiction assume that there exists some $y \in B$ such that $\lambda y > 0$. From step 3, this implies that $0 < \lambda y < \lambda z$ for all $z \in \bar{A}$. Notice that taking $z = (\varepsilon, 0, ..., 0) \in \bar{A}$ and making ε sufficiently small we can make λz as small as we want, violating the condition $\lambda y < \lambda z$. Thus $\lambda y \leq 0$ for all $y \in B$.

5. Use the definition of B to show that if $y \in B$ then $(-y) \in B$.

Ans: $y \in B$ implies that there is some h in $\mathbb{R}^K$ such that

$$y_0 = -q^T h \text{ and } (y_1, ..., y_m)^T = D^T h.$$

Taking the vector $-h \in \mathbb{R}^K$ we immediately see that $-y \in B$ since

$$-q^T(-h) = -\left[-q^T h\right] = -y_0,$$

and

$$D^T(-h) = -\left[D^T h\right] = -(y_1, ..., y_m)^T.$$

6. Use 4 and 5 to show that $\lambda y = 0$ for all $y \in B$.

Ans: Steps 4 and 5 imply $\lambda y = 0$, for if there is some y such that $\lambda y < 0$ then step 5 implies that $-y \in B$, hence $\lambda(-y) > 0$, violating step 4.

7. Use 3 and 6 to argue that $\lambda z > 0$ for all $z \in \bar{A}$.

Ans: From step 3 and 6 we have $0 = \lambda y < \lambda z$ for all $z \in \bar{A}$.

8. Use 7 to show that λ is strictly positive, i.e. $\lambda \in R^{m+1}_{++}$.

Ans: Taking the element $(0, 0, ..0, 1, 0.., 0) \in \bar{A}$ with a 1 in the i^{th} entry and zero in all the other entries, we have $\lambda z = \lambda_i > 0$ for $i = 0, 1, 2, ..., m$. That is, $\lambda \in R^{m+1}_{++}$.

9. Use 6 and 8 to show that η defined as

$$\eta_s = \frac{\lambda_s}{\lambda_0} \text{ for } s = 1, 2, ..., m,$$

satisfies

$$q_k = \sum_{k=1}^{K} d_{ks}\ \eta_s,$$

for all $k = 1, 2, ..., K$.

Ans: From step 6 we know that $\lambda y = 0$ for all $y \in B$. Equivalently, $\lambda_0 y_0 + \sum_{s=1}^{m} \lambda_i y_i = 0$. Since $\lambda_0 > 0$ (this follows from step 8), then

$$y_0 + \sum_{s=1}^{m} \frac{\lambda_s}{\lambda_0} y_s = 0.$$

Now let $\eta_s \equiv \lambda_s/\lambda_0 > 0$ (the inequality follows from step 8) and use that $y_0 = -q^T h$ and $y_s = \sum_{k=1}^{K} d_{ks} h_k$ for any h in $\mathbb{R}^K$ to obtain

$$-q^T h + \sum_{s=1}^{m} \eta_s \sum_{k=1}^{K} d_{ks} h_k = 0,$$

or

$$-\sum_{k=1}^{K} h_k q_k + \sum_{k=1}^{K} h_k \sum_{s=1}^{m} \eta_s d_{ks} = 0$$

or

$$\sum_{k=1}^{K} h_k \left[\sum_{s=1}^{m} \eta_s d_{ks} - q_k \right] = 0,$$

for all h in $\mathbb{R}^K$. In particular, taking $h_k = (0, 0, ..0, 1, 0.., 0)$ with zeros in all elements except in the k^{th} entry, the last equation becomes

$$\sum_{s=1}^{m} \eta_s d_{ks} - q_k = 0.$$

Since this holds for all $k = 1, 2, ..., K$ the proof is finished.

Exercise 4. Is it important that portfolio positions h are unrestricted, i.e. $h \in R^K$, to obtain the previous proposition? What are the interpretations of $h \in R^K$ and of $h \in R^K_+$ in terms of portfolio constraints? What would have happen with the previous result if $h \in R^K_+$, i.e. which step of the proof would have been violated?

Ans: $h \in \mathbb{R}^K$ means that we can choose any portfolio. In particular, there are no short-selling constraints and we can go short (i.e. sell) in any security. For example, suppose that I choose a portfolio with $h_k < 0$ (here k refers to the k^{th} entry of the vector h). Then, when uncertainty is resolved I will have to pay d_{ks} units of the numeraire in state s. If h is restricted to belong to $\mathbb{R}^K_+$, then short-selling is not allowed and agents are restricted to hold only long positions.

The step of the proof that is violated if we impose $h \in \mathbb{R}^K_+$ is step 6 when we show that if y belongs to B then, $-y$ also belongs to B.

Exercise 5. Show that markets are complete if and only if there is a unique η such that (7) holds. (Hint: write (7) as a system of linear equations

$$q = D\ \eta,$$

what do you know about the uniqueness of the solution η?)

- $\Rightarrow$: By definition, if markets are complete, $rank\,(D) = m$. From exercise 3 we know that there exists some vector $\eta \in \mathbb{R}^m_{++}$ (not necessarily unique) such that $q = D\eta$. If D has rank m it is invertible and therefore the system of equations $q = D\eta$ has a unique solution
$$\eta = D^{-1}q > 0.$$
- $\Leftarrow$: Suppose there is a unique $\eta \in \mathbb{R}^m_{++}$ but markets are not complete. Then by definition of market incompleteness, $rank\,(D) < m$, and the D matrix is not invertible. Therefore the system of equations $q = D\eta$ has infinite solutions and η is not unique. The solution will be indexed by a set of vectors in a $(m - K)$ dimensional space, which is a contradiction.

Exercise 6. Consider the case where there are three states $m = 3$ and two securities $K = 2$ with

$$\begin{bmatrix} d_{11} & d_{12} & d_{13} \\ d_{21} & d_{22} & d_{23} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix},$$

and

$$\begin{bmatrix} q_1 \\ q_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \end{bmatrix}.$$

Show that in this case there is a one-dimensional space indexed by $a \in (0,2)$ for the state prices $\eta > 0$ satisfying (7)

$$\begin{aligned} \eta_3 &= a, \\ \eta_1 &= 2 - a, \\ \eta_2 &= 2 - a. \end{aligned}$$

Ans: Using the equality $q = D\eta$ we have

$$\begin{bmatrix} 2 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} \eta_1 \\ \eta_2 \\ \eta_3 \end{bmatrix}.$$

Setting $\eta_3 = a$ and solving the above system we have

$$\begin{aligned} 2 &= \eta_1 + a, \\ 2 &= \eta_2 + a, \end{aligned}$$

or $\eta_1 = 2 - a,\ \eta_2 = 2 - a$ and $\eta_3 = a$. For any $a \in (0,2)$ we have that η so defined is a vector of strictly positive state prices satisfying $q = D\eta$.

Exercise 7. Consider the case where there are three states $m = 3$ and two securities $K = 2$ with

$$\begin{bmatrix} d_{11} & d_{12} & d_{13} \\ d_{21} & d_{22} & d_{23} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix},$$

and

$$\begin{bmatrix} q_1 \\ q_2 \end{bmatrix} = \begin{bmatrix} 2 \\ -2 \end{bmatrix}.$$

Show that (q, D) admits arbitrage opportunities, in particular display a portfolio h for which $q^T h = 0$ and $D^T h > \bar{0}$.

Ans: We will show that $q = [2;\ -2]'$ and the payoff matrix from exercise 6 admits arbitrage opportunities. We will display a portfolio h for which $q^T h = 0$ and $D^T h > 0$. Using our information we have

$$q^T h = 2\,(h_1 - h_2)\,,$$

$$D^T h = \begin{bmatrix} h_1 \\ h_2 \\ h_1 + h_2 \end{bmatrix}.$$

For example setting $h_1 = h_2 > 0$ we have $q^T h = 0$ and $D^T h > \bar{0}$. Those portfolios cost nothing and give a strictly positive payoff in all three states (i.e. they are arbitrage opportunities).

Exercise 8. Assume that (q, D) admits no arbitrage opportunities. Denote by η any of its corresponding state prices. Assume that there is a portfolio h such that it gives a risk-free payoffs, i.e. $D^T h = (1, 1, ..., 1)$. Denote the price of this portfolio by q_0 so that

$$q_0 = \sum_{s=1} \eta_s,$$

and define the interest rate r as satisfying $q_0 = 1/(1+r)$. Define the risk-adjusted probabilities φ_s as

$$\varphi_s = \frac{\eta_s}{\sum_{s'=1}^{m} \eta_{s'}},$$

for all $s = 1, 2, ..., m$. Show that the price of any asset k is given by the expected discounted payoffs, where the expectation is taken using the risk-adjusted probability φ,i.e.

$$q_k = E_\varphi \left[\frac{d_k}{1+r} \right] = \sum_{s=1}^{m} \frac{d_{ks}}{1+r} \varphi_s,$$

for all $k = 1, 2, ..., K$.

Ans: Since (q, D) admits no arbitrage opportunities, there is a vector of state prices η such that for each security $k = 1, 2, ..., K$ we have

$$q_k = \sum_{s=1}^{K} d_{ks} \eta_s.$$

Now we just need to multiply and divide the previous expression in an appropriate way and to use some definition to obtain the desired result:

$$\begin{aligned} q_k &= \sum_{s=1}^{K} \eta_s d_{ks} = \sum_{s=1}^{K} \left(\sum_{s'=1}^{K} \eta_{s'} \right) \frac{\eta_s}{\sum_{s'=1}^{K} \eta_s} d_{ks} \\ &= \sum_{s'=1}^{K} \eta_{s'} \left(\sum_{s=1}^{K} \varphi_s d_{ks} \right) = q_0 \left(\sum_{s=1}^{K} \varphi_s d_{ks} \right) \\ &= \sum_{s=1}^{K} \frac{d_{ks}}{1+r} \varphi_s = E_\varphi \left[\frac{d_k}{1+r} \right], \end{aligned}$$

where the second equality follows from multiplying and dividing by $\sum_{s'=1}^{m} \eta_{s'}$; the third equality uses the definition of φ_s and rearranges the expression; the fourth equality follows from the fact that there is a portfolio h with price q_0 that gives risk free payoffs $D^T h = (1, 1, ..., 1)^T$; and the fifth equality follows by the definition $q_0 \equiv 1/(1+r)$. In other words, the price of any asset k is given by its expected discounted payoffs, where the expectation is taken using the risk-adjusted probabilities φ.

Exercise 9. Consider the agent's i problem in a security market economy. Replace her consumption x by

$$x_s = \sum_{k=1}^{K} d_{ks} h_k + \tilde{e}_s^i,$$

for each $s = 1, 2..., m$ in her utility function u^i, so that we have $u^i \left(D^T h + \tilde{e}^i \right)$ or

$$u^i \left(\sum_{k=1}^{K} d_{k1} h_k + \tilde{e}_1^i, ..., \sum_{k=1}^{K} d_{ks} h_k + \tilde{e}_s^i, ..., \sum_{k=1}^{K} d_{km} h_k + \tilde{e}_m^i \right).$$

The agent's problem is

$$\max_h u^i \left(D^T h + \tilde{e}^i \right),$$

subject to

$$q^T h = q^T \theta^i.$$

1) Write down the first order conditions of this problem for each h_k, $k = 1, 2, ...K$ using μ_i for the multiplier of the budget constraint.

Ans: Attaching a multiplier μ^i on the last constraint, the first order conditions are (differentiating with respect to h_k),

$$\sum_{s=1}^{m} \frac{\partial u^i}{\partial x_s} \left(x^i \right) d_{ks} = \mu^i q_k \text{ for } k = 1, 2, ..., K,$$

where $x^i \equiv \left(x_1^i, x_2^i, ..., x_m^i \right)$.

2) Use these FOCs to show that

$$\begin{aligned} q_k &= \sum_{s=1}^{K} d_{ks} \ \eta_s^i \text{ for all } k = 1, ..., K, \\ \eta_s^i &= \frac{1}{\mu_i} \frac{\partial u^i}{\partial x_s} \left(x^i \right). \end{aligned}$$

Ans: Rearrange the last equation to read

$$q_k = \sum_{s=1}^{m} \frac{\frac{\partial u^i}{\partial x_s}\left(x^i\right)}{\mu^i} d_{ks};\ k = 1, 2, ..., K,$$

or equivalently,

$$q_k = \sum_{s=1}^{m} \eta_s^i d_{ks};\ k = 1, 2, ..., K,$$

for $\eta_s^i \equiv \frac{\partial u^i}{\partial x_s}\left(x^i\right)/\mu^i$.

3) Suppose that the allocation $\left\{x^i\right\}$ is PO. Show that η_s^i are the same for all agents [Hint: use the FOC for the λ weighted planning problem with $\lambda_i = 1/\mu_i$ and $\gamma = \eta$].

Ans: The assumed Pareto optimality of the allocation implies that there exists a vector of $\lambda-$weights such that the allocation solves the following planner's problem

$$\max_{\left\{x_s^i\right\}_{i=1}^{I}} \left\{\sum_{i=1}^{I} \lambda_i u^i\left(x_1^i, x_2^i, ..., x_m^i\right)\right\},$$

subject to

$$\sum_{i=1}^{I} x_s^i = \sum_{i=1}^{I}\left[\tilde{e}_s^i + \sum_{k=1}^{K} d_{ks}\theta_k^i\right] \text{ for } s = 1, ..., m.$$

Attaching a multiplier γ_s on each of the feasibility constraints, we can see that the first order conditions of the last problem are

$$\lambda_i \frac{\partial u^i\left(x^i\right)}{\partial x_s^i} = \gamma_s \text{ for all } i \in I \text{ and } s = 1, 2, .., m.$$

By setting $\mu^i \equiv 1/\lambda_i$ and $\eta_s^i \equiv \gamma_s$ for all i we obtain

$$\frac{1}{\mu^i}\frac{\partial u^i\left(x^i\right)}{\partial x_s^i} = \frac{1}{\mu^j}\frac{\partial u^j\left(x^j\right)}{\partial x_s^j},$$

and hence $\eta_s^i = \eta_s^j = \gamma_s$ for all $i \neq j$.

4) Suppose that markets are not complete and that η^i differ across agents. Is the allocation PO?

Ans: If markets are not complete and η^i differ across agents we immediately see that the condition for Pareto optimality

$$\frac{1}{\mu^i}\frac{\partial u^i\left(x^i\right)}{\partial x_s^i} = \frac{1}{\mu^j}\frac{\partial u^j\left(x^j\right)}{\partial x_s^j},$$

is violated and hence, the allocation is not efficient. Intuitively, $\frac{1}{\mu^i}\frac{\partial u^i(x^i)}{\partial x^i_s} = \frac{\partial wealth}{\partial x^i_s}$ is agent $i's$ willingness to pay for a marginal increase in consumption in state s. Any optimal allocation equates the willingness to pay across agents for otherwise there is room for improvement in the allocation: take goods from agents with low willingness to pay in a particular state and give them to the agents with high willingness to pay for consumption in that state.

Exercise 10. Specialize u^i to expected utility so that

$$u^i(x) = \sum_{s=1}^{m} v^i(x_s)\,\pi_s,$$

and $u^i\left(D^T h + \tilde{e}^i\right)$ is given by

$$u^i\left(D^T h + \tilde{e}^i\right) = \sum_{s=1}^{m} v^i\left(\sum_{k=1}^{K} d_{ks}h_k + \tilde{e}^i_s\right)\pi_s.$$

Repeat the analysis of the previous exercise to show that in this case the η^i solving

$$q_k = \sum_{s=1}^{K} d_{ks}\ \eta^i_s \text{ for all } k = 1, ..., K,$$

are given by

$$\eta^i_s = \frac{1}{\mu_i}\frac{\partial v^i\left(x^i_s\right)}{\partial x}\pi_s,$$

for all $s = 1, 2..., m$. Show that the corresponding risk-adjusted probabilities φ^i_s are given by

$$\varphi^i_s = \frac{\partial v^i\left(x^i_s\right)/\partial x}{\sum_{s'=1}^{m}\partial v^i\left(x^i_{s'}\right)/\partial x\ \pi_{s'}}\ \pi_s.$$

Ans: Specializing the FOC to expected utility we obtain

$$\sum_{s=1}^{m}\frac{\partial v^i}{\partial x_s}\left(x^i_s\right)d_{ks}\pi_s = \mu^i q_k \text{ for } k = 1, 2, ..., K.$$

Thus

$$q_k = \sum_{s=1}^{m}\eta^i_s d_{ks},$$

for $\eta^i_s = \frac{\partial v^i}{\partial x_s}\left(x^i_s\right)\pi_s/\mu^i$. Now, rewrite the FOC as

$$q_k = \sum_{s=1}^{m}\frac{\frac{\partial v^i}{\partial x_s}\left(x^i_s\right)d_{ks}\pi_s}{\mu^i}.$$

Multiply and divide by $\sum_{s'=1}^{m} \frac{\partial v^i}{\partial x_{s'}}\left(x_{s'}^i\right)\pi_{s'}/\mu^i$ to obtain

$$q_k = \left(\sum_{s'=1}^{m} \frac{\partial v^i}{\partial x_{s'}}\left(x_{s'}^i\right)\pi_{s'}/\mu^i\right)\sum_{s=1}^{m} \frac{\frac{\partial v^i}{\partial x_s}\left(x_s^i\right)\pi_s/\mu^i}{\left(\sum_{s'=1}^{m} \frac{\partial v^i}{\partial x_{s'}}\left(x_{s'}^i\right)\pi_{s'}/\mu^i\right)} d_{ks}.$$

If there is a portfolio that pays $(1, 1, 1, ..., 1)$ its price will be

$$\sum_{s'=1}^{m} \frac{\partial v^i}{\partial x_{s'}}\left(x_{s'}^i\right)\pi_{s'}/\mu^i = q_0 \equiv \frac{1}{1+r}.$$

Moreover, let

$$\varphi_s^i = \frac{\frac{\partial v^i}{\partial x_s}\left(x_s^i\right)\pi_s}{\left(\sum_{s'=1}^{m} \frac{\partial v^i}{\partial x_{s'}}\left(x_{s'}^i\right)\pi_{s'}\right)}.$$

Hence,

$$q_k = \sum_{s=1}^{m} \frac{d_{ks}}{1+r}\varphi_s^i.$$

Exercise 11. Assume that

$$\theta_k^i = 0,$$

for all $k = 1, ..., K$ and $i \in I$. Furthermore assume that

$$\text{cov}\left[\frac{\partial v^i\left(\tilde{e}^i\right)}{\partial x}, d_k\right] = 0 \text{ for all } k \text{ and } i,$$

i.e.

$$\sum_{s=1}^{m}\left(\frac{\partial v^i\left(\tilde{e}_s^i\right)}{\partial x}\right)\left(d_{ks} - \sum_{s'=1}^{m} d_{ks'}\pi_{s'}\right)\pi_s = 0,$$

for all $k = 1, 2, ..., K$ and $i \in I$.

1) Show that the equilibrium in this case has

$$\begin{aligned} h^i &= 0, \\ x^i &= \tilde{e}^i, \\ q_k &= E\left[d_k\right] = \sum_{s=1}^{m} d_{ks}\pi_s \text{ all } k = 1, 2, ..., K. \end{aligned}$$

Ans: The FOCs for the agent's problem with respect to h_k are

$$\sum_{s=1}^{m} \frac{\partial v^i\left(x_s^i\right)}{\partial x} d_{ks}\pi_s = \mu^i q_k; \quad k = 1, 2, .., K.$$

Impose (guess) the condition $h^i = 0$ (no-trade equilibrium), so $x^i = \tilde{e}^i$. Then the FOCs become

$$\sum_{s=1}^{m} \frac{\partial v^i\left(\tilde{e}_s^i\right)}{\partial x} d_{ks}\pi_s = \mu^i q_k,$$

or, equivalently,

$$E\left(\frac{\partial v^i\left(\tilde{e}^i\right)}{\partial x} d_k\right) = \mu^i q_k.$$

Now using $Cov\left(\frac{\partial v^i(\tilde{e}^i)}{\partial x}, d_k\right) = 0$ and the identity $E\left(\frac{\partial v^i(\tilde{e}^i)}{\partial x} d_k\right) = E\left(\frac{\partial v^i(\tilde{e}^i)}{\partial x}\right) E\left(d_k\right) + Cov\left(\frac{\partial v^i(\tilde{e}^i)}{\partial x}, d_k\right)$ the FOC becomes

$$\left(\sum_{s=1}^{m} \frac{\partial v^i\left(\tilde{e}_s^i\right)}{\partial x}\pi_s\right)\left(\sum_{s=1}^{m} d_{ks}\pi_s\right) = \mu^i q_k,$$

so by picking the right units we can normalize $\mu^i = \sum_{s=1}^{m} \frac{\partial v^i(\tilde{e}_s^i)}{\partial x}\pi_s$ and we obtain

$$q_k = \sum_{s=1}^{m} d_{ks}\pi_s \quad \text{for } k = 1, 2, ..., K.$$

2) Moreover assume that

$$\sum_{i \in I} \tilde{e}_s^i = \sum_{i \in I} \tilde{e}_{s'}^i > 0,$$

for all $s, s' = 1, 2, ..., m$ but that $\tilde{e}^i$ has strictly positive variance. Is the equilibrium allocation PO?

Ans: The resulting allocation need not be Pareto Optimal. To see this consider the example where all agents have identical preferences. In an economy with identical agents and constant aggregate endowment it is optimal to allocate constant consumption across states of nature, but if agents consume their own endowment, which has positive variance, consumption will not be constant across states and the allocation will not be efficient.

3) Are the η^i all the same?

Ans: According to our results

$$\eta^i = \frac{\frac{\partial v^i\left(\tilde{e}_s^i\right)}{\partial x}\pi_s}{\mu^i} = \frac{\frac{\partial v^i\left(\tilde{e}_s^i\right)}{\partial x}\pi_s}{\left(\sum_{s'=1}^m \frac{\partial v^i\left(\tilde{e}_{s'}^i\right)}{\partial x}\pi_{s'}\right)},$$

which need not be equal across agents (part 2 of this exercise illustrates that point).

4) Consider the particular case where

$$\begin{aligned} m &= 4, \pi_s = 1/4 \text{ all } s, \\ v^i(c) &= -\frac{1}{2}(5-c)^2, \quad I = 2, \end{aligned}$$

endowments are

$$\begin{bmatrix} \tilde{e}_1^1 \\ \tilde{e}_2^1 \\ \tilde{e}_3^1 \\ \tilde{e}_4^1 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix},$$

$$\begin{bmatrix} \tilde{e}_1^2 \\ \tilde{e}_2^2 \\ \tilde{e}_3^2 \\ \tilde{e}_4^2 \end{bmatrix} = \begin{bmatrix} 4 \\ 3 \\ 2 \\ 1 \end{bmatrix},$$

payoffs of securities

$$\begin{bmatrix} d_{11} \\ d_{12} \\ d_{13} \\ d_{14} \end{bmatrix} = \begin{bmatrix} 1/4+\alpha \\ 1/4+\beta \\ 1/4-\alpha \\ 1/4-\beta \end{bmatrix},$$

$$\begin{bmatrix} d_{21} \\ d_{22} \\ d_{23} \\ d_{24} \end{bmatrix} = \begin{bmatrix} 1/4+\gamma \\ 1/4+\delta \\ 1/4 \\ 1/4-(\gamma+\delta) \end{bmatrix},$$

with

$$\begin{aligned} \alpha + 2\beta - 3\alpha - 4\beta &= 0, \\ 4\alpha + 3\beta - 2\alpha - \beta &= 0, \end{aligned}$$

and

$$\begin{aligned} \gamma + 2\delta - 4(\gamma + \delta) &= 0, \\ 4\gamma + 3\delta - (\gamma + \delta) &= 0, \end{aligned}$$

and endowment of securities $\theta^i = 0$ for $i = 1, 2$. Compute q, h^i, and η^1 and η^2 for this example.

Ans: We can verify that under the restrictions

$$\begin{aligned} \alpha + 2\beta - 3\alpha - 4\beta &= 0, \\ 4\alpha + 3\beta - 2\alpha - \beta &= 0, \end{aligned}$$

and

$$\begin{aligned} \gamma + 2\delta - 4(\gamma + \delta) &= 0, \\ 4\gamma + 3\delta - (\gamma + \delta) &= 0, \end{aligned}$$

it is true that

$$Cov\left(\frac{\partial v^i(\tilde{e}^i)}{\partial x}, d_k\right) = 0 \text{ for all } k \text{ and } i,$$

so the no-trade equilibrium analyzed above exists. Furthermore, since this is a no-trade equilibrium we have $h^i = 0$ for $i = 1, 2$. Prices are

$$\begin{aligned} q_1 &= E[d_1] = \frac{1}{4}\left[\frac{1}{4} + \alpha + \frac{1}{4} + \beta + \frac{1}{4} - \alpha + \frac{1}{4} - \beta\right] = \frac{1}{4}, \\ q_2 &= E[d_2] = \frac{1}{4}\left[\frac{1}{4} + \delta + \frac{1}{4} + \gamma + \frac{1}{4} + \frac{1}{4} - (\delta + \gamma)\right] = \frac{1}{4}. \end{aligned}$$

Finally, noting that $\sum_{s=1}^{4} \frac{\partial v^i(\tilde{e}_s^i)}{\partial x}\pi_s = \frac{1}{4}[1 + 2 + 3 + 4] = 10/4$ for $i = 1, 2$ we obtain the values for η^1 and η^2:

$$\eta^1 = \frac{4}{10}\begin{bmatrix} 4/4 \\ 3/4 \\ 2/4 \\ 1/4 \end{bmatrix}; \quad \eta^2 = \frac{4}{10}\begin{bmatrix} 1/4 \\ 2/4 \\ 3/4 \\ 4/4 \end{bmatrix}.$$

So as can be seen, $\eta^1 \neq \eta^2$ which indeed means that the equilibrium is not Pareto Efficient.

2 Equilibrium in the Security Market

We start with the case of no labor market income at all, i.e. $\hat{e}_s^i = 0$ for all i and s. In this case, the budget constraint is given by:

$$\begin{aligned} \sum_{k=1}^{K} q_k h_k^i &= \sum_{k=1}^{K} q_k \theta_k^i \\ x_s^i &= \sum_{k=1}^{K} h_k^i d_{ks} \text{ for } s = 1, 2, ..., m. \end{aligned}$$

a) Write down and expression for the equilibrium aggregate consumption $\bar{e}_s$ in state s. Your answer should imply:

$$\bar{e} = D^T \bar{\theta},$$

where $\bar{\theta} \in R^K$. Give an interpretation of $\bar{\theta}$ in terms of the elements that define the security market economy (one line maximum).

Ans: Market clearing in the goods market requires

$$\sum_{i=1}^{I} x_s^i = \sum_{i=1}^{I} \left(\sum_{k=1}^{K} d_{ks} \theta_k^i \right) \text{ for } s = 1, 2, ..., m,$$

or

$$\bar{e}_s = \sum_{k=1}^{K} d_{ks} \left(\sum_{i=1}^{I} \theta_k^i \right) = \sum_{k=1}^{K} d_{ks} \bar{\theta}_k, \text{ for } s = 1, 2, ..., m,$$

where $\bar{e}_s \equiv \sum_{i=1}^{I} x_s^i$ is the equilibrium aggregate consumption in state s and $\bar{\theta}_k$ is the aggregate endowment of security k. Writing the above expression in matrix form we obtain

$$\bar{e} = D^T \bar{\theta},$$

which is the desired result.

b) Consider an example with two agents: $i = 1, 2$. Agent 1 has subutility function

$$v^1(x) = \log(x),$$

and agent 2,

$$\begin{aligned} v^2(x) &= x - 1 \text{ for } x \leq 1 \text{ and} \\ &= \log x \text{ for } x > 1. \end{aligned}$$

The aggregate endowment takes 4 values:

$$(\bar{e}_1, \bar{e}_2, \bar{e}_3, \bar{e}_4) = (0.5,\ 1.5,\ \ 2.5,\ 3.5)\,,$$

Solve the planning problem for $\lambda_1 = \lambda_2 > 0$. [Hint: Consider the following two problems:

$$\max_{x_s^1} \log\left(x_s^1\right) + \left[\left(\bar{e}_s - x_s^1\right) - 1\right],$$

and

$$\max_{x_s^1} \log\left(x_s^1\right) + \log\left(\bar{e}_s - x_s^1\right).$$

Write the solution of each of these problems, obtaining x_s^2 as a function of $\bar{e}_s$. Analyze which of the problems applies as a function of $\bar{e}_s$, that is, for which solution we are using the relevant expression for the utility function of agent 2. Note that we allow the consumption of agent 2 to be negative].

Ans: The FOC of the first problem is

$$\frac{1}{x_s^1} = 1,$$

which implies

$$x_s^1 = 1, \qquad \text{and} \qquad x_s^2 = \bar{e}_s - 1.$$

The FOC of the second problem is

$$\frac{1}{x_s^1} = \frac{1}{\bar{e}_s - x_s^1},$$

which implies

$$x_s^1 = \frac{\bar{e}_s}{2}, \qquad \text{and} \qquad x_s^2 = \frac{\bar{e}_s}{2}.$$

Thus,

$$\begin{aligned} x^1 &= \left(1, 1, \frac{5}{4}, \frac{7}{4}\right), \\ x^2 &= \left(-\frac{1}{2}, \frac{1}{2}, \frac{5}{4}, \frac{7}{4}\right). \end{aligned}$$

c) Continue with the example developed in part *b*. Suppose there is a security market with two securities, $K = 2$, with payoffs given by

$$d_{1s} = 0.5 \text{ for } s = 1, 2, 3, 4,$$

for the first security and

$$(d_{21},\ d_{22},\ d_{23},\ d_{24}) = (0,\ 1,\ 2,\ 3)\,,$$

for the second security. The aggregate supply of these securities satisfies:

$$\begin{aligned} \theta_1^1 + \theta_1^2 &= 1, \\ \theta_2^1 + \theta_2^2 &= 1. \end{aligned}$$

Show that there is an endowment of securities θ_k^i for $i = 1, 2$ and $k = 1, 2$ satisfying the two restrictions on their aggregate supply such that the security market equilibrium reproduces the P.O. allocations obtained in the A-D economy with $\lambda_1 = \lambda_2$. [Hint: use your answer to parts *a*) and *b*)].

Ans: The budget constraint of each agent requires

$$x^i = D^T h^i.$$

Thus, for agent 1 we must have that

$$\begin{bmatrix} 1 \\ 1 \\ 5/4 \\ 7/4 \end{bmatrix} = \begin{bmatrix} 0.5 & 0 \\ 0.5 & 1 \\ 0.5 & 2 \\ 0.5 & 3 \end{bmatrix} \begin{bmatrix} h_1^1 \\ h_2^1 \end{bmatrix}.$$

The first two equations imply that $h_1^1 = 2$ and $h_2^1 = 0$. Evidently, this portfolio holdings do not satisfy the last two equations. Thus, the security market economy is not able to reproduce the PO allocations obtained in the A-D economy. To solve this problem, we would need to *i*) complete markets by increasing the number of securities to four or reducing the number of states to two, or *ii*) change the particular allocation that we are trying to decentralize by varying the pattern of endowments (recall that the fact that markets are incomplete does not necessarily imply that the security market equilibrium is not PO).

d) Consider now the case where $L = R^m$ and $X = L$ (so we are allowing negative consumption) and the subutility functions $v^i : R \to R$ are exponential with parameter ρ_i :

$$v^i(x) = (-1/\rho_i)\exp(-\rho_i x).$$

The parameter ρ_i measures the curvature of v^i, defined as $-v''(x)/v'(x)$, and is referred to as the coefficient of "absolute risk aversion". Its reciprocal, $1/\rho_i$, is called "risk tolerance".

$d.0$) To help interpreting the results below, show the following: if x is normally distributed with $x \sim N(\mu, \sigma^2)$, then the insurance premium $\hat{p}$ defined implicitly by

$$v^i(Ex - \hat{p}) = E\left[v^i(x)\right],$$

satisfies:

$$\hat{p} = \frac{1}{2}\rho^i \sigma^2.$$

Give a (one line) intuitive explanation of this formula. [Hint: use the formula for the expected value of the exponential of a normal variable in the notes, as well as the definition of the insurance premium above].

Ans: By definition of the utility function we have that

$$v^i(E[x] - \hat{p}) = (-1/\rho_i)\exp(-\rho_i(\mu - \hat{p})).$$

Moreover, using the fact that $\exp(x)$ is lognormally distributed we obtain

$$E\left[v^i(x)\right] = (-1/\rho_i)\exp\left(-\rho_i\mu + \rho_i^2\frac{1}{2}\sigma^2\right).$$

Equating these two expressions we arrive at

$$\hat{p} = \frac{1}{2}\rho_i\sigma^2,$$

as was to be shown. Notice that $\hat{p}$ is the maximum amount that an individual is willing to pay to avoid the lottery associated with the random variable x. The above result implies that this willingness to pay is proportional to the individual's coefficient of risk aversion and to the variance of the lottery he would otherwise face.

Now you should solve the planner's problem for the A-D economy with weights λ_i. In particular:

$d.1$) Obtain an expression for x_s^i as a function of λ_i and γ_s using the FOC w.r.t. x_s^i and letting γ_s be the multiplier of the resource constraint at state s.

Ans: The planner's problem consists of

$$\max_{\{x^i\}_{i\in I}} \sum_{i\in I} \lambda_i u^i\left(x^i\right) = \sum_{i\in I} \lambda_i \sum_{s=1}^{m} (-1/\rho_i) \exp\left(-\rho_i x_s^i\right) \pi_s,$$

$$\text{s.t. : } \sum_{i\in I} x_s^i = \bar{e}_s \text{ for } s = 1, 2, ..., m.$$

The FOC associated with this problem is

$$x_s^i : \lambda_i \exp\left(-\rho_i x_s^i\right) \pi_s = \gamma_s \text{ for } s = 1, 2, ..., m.$$

Thus,

$$x_s^i = -\frac{1}{\rho_i} \log\left(\frac{\gamma_s}{\pi_s \lambda_i}\right),$$

or

$$x_s^i = \frac{1}{\rho_i}\left(\log \pi_s + \log \lambda_i - \log \gamma_s\right). \tag{10}$$

d.2) Use *d*.1 and market clearing $\left(\sum_{i\in I} x_s^i = \bar{e}_s\right)$ to obtain an expression for $\log \gamma_s$ as a function of $\bar{e}_s,\ \pi_s,\ \{\rho_i\}_{i\in I}$ and $\{\lambda_i\}_{i\in I}$. Use the following normalization for λ_i:

$$\sum_{i'\in I} \frac{1}{\rho_{i'}} \log \lambda_{i'} = 0.$$

Give an economic interpretation of the effects of $\bar{e}_s,\ \pi_s$ and $\{\rho_i\}$ in your formula for γ_s (three lines maximum).

Ans: Adding our expression for x_s^i across all agents we obtain

$$\sum_{i'\in I} x_s^i = \sum_{i'\in I} \frac{1}{\rho_i}\left(\log \pi_s + \log \lambda_i - \log \gamma_s\right),$$

or

$$\bar{e}_s = \left(\log \pi_s - \log \gamma_s\right) \sum_{i'\in I} \frac{1}{\rho_i} + \underbrace{\sum_{i'\in I} \frac{1}{\rho_i} \log \lambda_i}_{=0},$$

or

$$\log\left(\gamma_s\right) = \log\left(\pi_s\right) - \frac{1}{\left(\sum_{i'\in I} \frac{1}{\rho_i}\right)} \bar{e}_s. \tag{11}$$

Thus, the value of an additional unit of endowment of the good in state s, γ_s, is higher when: *i*) state s is more likely to occur (i.e., π_s is high), *ii*) the good is more scarce in state s (i.e., $\bar{e}_s$ is low), and *iii*) the economy is more risk tolerant (i.e., $\sum_{i'\in I} \frac{1}{\rho_i}$ is high), so the demand for risky consumption is higher.

$d.3$) Use your answer to $d.2$ and $d.1$ to write the value of x_s^i as a function of $\bar{e}_s$, $\{\rho_i\}_{i\in I}$ and $\{\lambda_i\}_{i\in I}$. Use the following normalization for λ_i:

$$\sum_{i'\in I} \frac{1}{\rho_{i'}} \log \lambda_{i'} = 0.$$

In particular, you have to show that

$$x_s^i = \frac{1/\rho_i}{\left(\sum_{i'\in I} \frac{1}{\rho_{i'}}\right)} \bar{e}_s + \frac{1}{\rho_i} \log \lambda_i.$$

Give an economic interpretation of the terms and form of this equation (two lines maximum).

Ans: Plugging (11) into (10) yields

$$x_s^i = \frac{1}{\rho_i} \left(\log \pi_s + \log \lambda_i - \left(\log (\pi_s) - \frac{1}{\left(\sum_{i'\in I} \frac{1}{\rho_i}\right)} \bar{e}_s \right) \right),$$

or

$$x_s^i = \frac{1/\rho_i}{\left(\sum_{i'\in I} \frac{1}{\rho_i}\right)} \bar{e}_s + \frac{1}{\rho_i} \log \lambda_i,$$

as desired. This expression implies that the consumption of agent i in state s moves with the aggregate endowment $\bar{e}_s$ in proportion to the relative risk tolerance of agent i. Thus, the consumption of an agent with a relatively high (low) risk tolerance will move more (less) with the aggregate endowment, as expected from an optimal-insurance perspective. Moreover, the consumption of an agent moves with the planner's weight of that agent, λ_i, but in proportion to the agent's willingness to tolerate risk.

$d.4$) Consider now a security market equilibrium. Assume that security $k = 1$ is a riskless bond, so it pays $d_{1s} = 1$ for all $s = 1, 2, ..., m$. Describe a portfolio h_s^i for each agent $i \in I$ and each security $k = 1, 2, ..., K$, so that the consumption of each agent in the security market economy will coincide with the consumption that solves the planner's problem found in $d.3$). [Hint: you can write the holdings for agent i of each security $k = 2, 3, ..., K$ as a fraction of the aggregate portfolio of each security in proportion to the relative risk tolerance of agent i, and include an additional adjustment to the amount of security $k = 1$ held according to the weight of agent i].

Ans: Using the hint, let us propose the following holdings of security $k = 1$ for each agent

i:

$$h_1^i = \frac{1/\rho_i}{\left(\sum_{i' \in I} \frac{1}{\rho_i}\right)} \bar{\theta}_1 + \frac{1}{\rho_i} \log \lambda_i,$$

and of security $k = 2, 3, ..., K$ for each agent i:

$$h_k^i = \frac{1/\rho_i}{\left(\sum_{i' \in I} \frac{1}{\rho_i}\right)} \bar{\theta}_k.$$

First, notice that

$$\sum_{i' \in I} h_1^i = \underbrace{\sum_{i' \in I} \frac{1}{\rho_i} \log \lambda_i}_{=0} + \sum_{i' \in I} \frac{1/\rho_i}{\left(\sum_{i' \in I} \frac{1}{\rho_i}\right)} \bar{\theta}_1 = \bar{\theta}_1,$$

and

$$\sum_{i' \in I} h_k^i = \sum_{i' \in I} \frac{1/\rho_i}{\left(\sum_{i' \in I} \frac{1}{\rho_i}\right)} \bar{\theta}_k = \bar{\theta}_k,$$

which implies that our proposed portfolios are feasible. Moreover,

$$\begin{aligned} h_1^i + \sum_{k=2}^{K} h_k^i d_{ks} &= \frac{1}{\rho_i} \log \lambda_i + \frac{1/\rho_i}{\left(\sum_{i' \in I} \frac{1}{\rho_i}\right)} \bar{\theta}_1 + \frac{1/\rho_i}{\left(\sum_{i' \in I} \frac{1}{\rho_i}\right)} \sum_{k=2}^{K} d_{ks} \bar{\theta}_k \\ &= \frac{1}{\rho_i} \log \lambda_i + \frac{1/\rho_i}{\left(\sum_{i' \in I} \frac{1}{\rho_i}\right)} \sum_{k=1}^{K} d_{ks} \bar{\theta}_k \\ &= \frac{1}{\rho_i} \log \lambda_i + \frac{1/\rho_i}{\left(\sum_{i' \in I} \frac{1}{\rho_i}\right)} \bar{e}_s = x_s^i, \end{aligned}$$

so the consumption of each agent in the security market economy coincides with the consumption that solves the planner's problem found in $d.3$), which is the desired result.

$d.5$) Use the value of γ_s solved out in $d.2$) to find q_k, the value of a security k in the CE that gives the same allocation than the planning problem in the A-D economy with weights $\{\lambda_i\}$. The expression for q_k should be a function of the probabilities $\{\pi_s\}_{s \in S}$, the risk aversion parameters $\{\rho_i\}_{i \in I}$, the aggregate endowment $\{\bar{e}_s\}_{s \in S}$, and the payoffs $\{d_{ks}\}_{s \in S}$.

Ans: Recall that the prices q and payoffs D are consistent with state price p iff

$$q_k = \sum_{s=1}^{m} p_s d_{ks}.$$

This follows from the fact that we can replicate the payoffs of security k by buying an amount d_{ks} of Arrow-Debreu securities in each state s. Since this operation would cost $\sum_{s=1}^{m} p_s d_{ks}$, the absence of arbitrage opportunities implies that this must also be the price of security k, q_k. Now, as usual, prices are equal to the marginal social value of the final good in each state: $p_s = \gamma_s$. Moreover, from (11) we know that

$$\gamma_s = \exp\left(-\frac{1}{\left(\sum_{i' \in I} \frac{1}{\rho_i}\right)} \bar{e}_s\right) \pi_s.$$

Thus,

$$q_k = \sum_{s=1}^{m} \exp\left(-\frac{1}{\left(\sum_{i' \in I} \frac{1}{\rho_{i'}}\right)} \bar{e}_s\right) \pi_s d_{ks}. \tag{12}$$

$d.6$) Use your answers to $d.4$ and $d.5$ to find the weights $\{\lambda_i\}$ that correspond to an equilibrium for an arbitrary endowment of securities $\left\{\theta^i\right\}_{i \in I}$. [Hint: replace the h_k^i obtained in $d.4$ and the q_k obtained in $d.5$ into the budget constraint $q^T\ h^i = q^T\ \theta^i$ and solve for λ_i].

Ans: Using the h_k^i obtained in $d.4$ into the agent's budget constraint $\sum_{k=1}^{K} q_k h_k^i = \sum_{k=1}^{K} q_k \theta_k^i$, we obtain

$$\begin{aligned} q_1 \left(\frac{1}{\rho_i} \log \lambda_i + \frac{1/\rho_i}{\left(\sum_{i' \in I} \frac{1}{\rho_i}\right)} \bar{\theta}_1\right) + \sum_{k=2}^{K} q_k \frac{1/\rho_i}{\left(\sum_{i' \in I} \frac{1}{\rho_i}\right)} \bar{\theta}_k &= \sum_{k=1}^{K} q_k \theta_k^i \\ q_1 \left(\frac{1}{\rho_i} \log \lambda_i\right) + \sum_{k=1}^{K} q_k \frac{1/\rho_i}{\left(\sum_{i' \in I} \frac{1}{\rho_i}\right)} \bar{\theta}_k &= \sum_{k=1}^{K} q_k \theta_k^i, \end{aligned}$$

or

$$\log \lambda_i = \frac{\rho_i}{q_1} \sum_{k=1}^{K} q_k \left(\theta_k^i - \frac{1/\rho_i}{\left(\sum_{i' \in I} \frac{1}{\rho_i}\right)} \bar{\theta}_k\right),$$

where q_k for $k = 1, 2, ..., K$ is defined by expression (12).

Now we turn to the case of positive labor risk. Assume that $\hat{e} = \left(\hat{e}^1, \hat{e}^2, .., \hat{e}^I\right)$ and $d = (d_1, d_2, ..., d_K)$. We let asset $k = 1$ be the riskless asset, so it has zero variance and $d_1 = 1$. We let the remaining $K - 1$ asset have expected values of dividends given by μ_d. We use μ_e for the vector of expected values of labor earnings for all agents with $\mu_e(i)$ denoting the one for agent i. We use $\hat{d} = (d_2, d_1, ..., d_K)$ to denote the risky assets payoffs.

Assume that all agents use expected utility and have exponential utility function with absolute risk aversion coefficient ρ, so that $u^i\left(x^i\right) = E\left[v\left(x^i\right)\right]$, where

$$v(x) = -\left(1/\rho_i\right)\exp\left(-\rho_i\ x\right).$$

$e.1$) In an equilibrium, agents maximize

$$E\left[v^i\left(\hat{e}^i + \sum_{k=1}^{K} h_k^i d_k\right)\right],$$

subject to

$$q^T h^i = q^T \theta^i.$$

Assume that $\hat{e}^i$ and $\hat{d}$ are independent. Show that with this assumption and with the exponential utility function v^i this problem is equivalent to

$$\max_{h^i} E\left[v^i\left(\sum_{k=1}^{K} h_k^i d_k\right)\right],$$

subject to $q^T h^i = q^T \theta^i$.

Ans: Let $A \equiv \hat{e}^i$ and $B \equiv \sum_{k=1}^{K} h_k^i d_k$. Then,

$$E\left[v^i\left(A+B\right)\right] = E\left[-\left(1/\rho_i\right)\exp\left(-\rho_i\left(A+B\right)\right)\right]$$

But

$$\begin{aligned}
E\left[\exp\left(-\rho_i\left(A+B\right)\right)\right] &= \exp\left(-\rho_i E\left[A+B\right] + \rho_i^2 \frac{1}{2}\operatorname{var}\left[A+B\right]\right) \\
&= \exp\left(-\rho_i E\left[A\right] + \rho_i^2 \frac{1}{2}\operatorname{var}\left[A\right] - \rho_i E\left[B\right] + \rho_i^2 \frac{1}{2}\operatorname{var}\left[B\right]\right) \\
&= \exp\left(-\rho_i E\left[A\right] + \rho_i^2 \frac{1}{2}\operatorname{var}\left[A\right]\right)\exp\left(-\rho_i E\left[B\right] + \rho_i^2 \frac{1}{2}\operatorname{var}\left[B\right]\right) \\
&= E\left[\exp\left(-\rho_i A\right)\right] E\left[\exp\left(-\rho_i B\right)\right]
\end{aligned}$$

Thus,

$$\begin{aligned}
E\left[v^i\left(\hat{e}^i + \sum_{k=1}^{K} h_k^i d_k\right)\right] &= -\rho_i E\left[-\left(1/\rho_i\right)\exp\left(-\rho_i\ \left(\hat{e}^i\right)\right)\right] E\left[-\left(1/\rho_i\right)\exp\left(-\rho_i\ \left(\sum_{k=1}^{K} h_k^i d_k\right)\right)\right] \\
&= -\rho_i E\left[v^i\left(\hat{e}^i\right)\right] E\left[v^i\left(\sum_{k=1}^{K} h_k^i d_k\right)\right].
\end{aligned}$$

Since $-\rho_i E\left[v^i\left(\hat{e}^i\right)\right]$ is a constant, maximizing the objective function $E\left[v^i\left(\hat{e}^i + \sum_{k=1}^K h_k^i d_k\right)\right]$ is equivalent to maximizing the objective function $E\left[v^i\left(\sum_{k=1}^K h_k^i d_k\right)\right]$, as was to be shown. Thus, we can neglect random labor income and all of our results in part d) will apply here as well.

e.2) Given your result in question d), are equilibrium prices of securities q affected by the the presence of random labor income that is independent of the asset payoffs and returns?

Ans: No, prices will still be determined by expression (12), which is independent of labor income $\hat{e}$.

e.3) Describe the consumption allocation x_s^i in the equilibrium with independent labor income [Hint: use your answer to d)]. Is this equilibrium allocation Pareto Optimal in the A-D (contingent claims) economy?

Ans: The equilibrium consumption allocation of agent i in state s satisfies

$$x_s^i = \tilde{e}_s^i + \sum_{k=1}^K d_{ks} h_k^i,$$

or, using our answer to d.3,

$$x_s^i = \tilde{e}_s^i + \frac{1/\rho_i}{\left(\sum_{i' \in I} \frac{1}{\rho_i}\right)} \sum_{k=1}^K d_{ks} \bar{\theta}_k + \frac{1}{\rho_i} \log \lambda_i.$$

However, from d.3 we know that Pareto optimality in the A-D economy requires that

$$\left(x_s^i\right)^* = \frac{1/\rho_i}{\left(\sum_{i' \in I} \frac{1}{\rho_i}\right)} \underbrace{\left(\tilde{e}_s + \sum_{k=1}^K d_{ks} \bar{\theta}_k\right)}_{\bar{e}_s} + \frac{1}{\rho_i} \log \lambda_i.$$

Thus, unless it happens that $\tilde{e}_s^i = \frac{(1/\rho_i)}{\left(\sum_{i' \in I} \frac{1}{\rho_i}\right)} \tilde{e}_s$ for all i and s (which implies that the PO allocation requires no risk-sharing of random labor income), it follows that the securities market equilibrium is not PO in the A-D economy. This result is due to the fact that markets are incomplete, in the sense that it is not possible to hedge against the fluctuations in random labor income.

e.4) From now on assume that asset and labor income are normally distributed. Let the variance $\text{var}\left(\hat{d}\right) = \Omega_{dd}$, $\text{var}\left(\hat{e}\right) = \Omega_{ee}$, and $\text{cov}\left(\hat{d}, \hat{e}\right) = \Omega_{ed}$, with typical elements

$\mu_e(i)\,,\ \mu_d(k)\,,\ \Omega_{ee}(i,j)\,,\ \Omega_{dd}(k,s)\,,$ etc.

$$\begin{bmatrix} \hat{e} \\ \hat{d} \end{bmatrix} \sim N\left(\begin{bmatrix} \mu_e \\ \mu_d \end{bmatrix}, \begin{bmatrix} \Omega_{ee} & \Omega_{ed} \\ \Omega_{ed} & \Omega_{dd} \end{bmatrix}\right).$$

Let $x^i(h) = \sum_{k=1}^{K} h_k^i\ d_k + \hat{e}^i$, so that

$$\begin{aligned} E\left(x^i(h)\right) &= \sum_{k=1}^{K} h_k\ E(d_k) + E\left(\hat{e}^i\right) \\ &= h_1 + \left(\hat{h}\right)^T \mu_d\ + \mu_e(i)\,, \end{aligned}$$

where we let $\hat{h} = (h_2, h_3, ..., h_K)$ denote the $K-1$ vector of portfolio holdings for risky assets and

$$\operatorname{var}\left(x^i(h)\right) = \hat{h}^T\ \Omega_{dd}\ \hat{h} + \Omega_e(i,i) + 2\sum_{k=1}^{K-1} \hat{h}_k\ \Omega_{ed}(i,k)\,.$$

Show that

$$E\left[v\left(x^i(h)\right)\right] = -(1/\rho_i)\exp\left(-\rho_i E\left[x^i(h)\right] + \frac{1}{2}\rho_i^2 \operatorname{var}\left(x^i(h)\right)\right),$$

or

$$-\frac{1}{\rho_i}\log\left(\ (-\rho_i)\ \ E\left[v\left(x^i(h)\right)\right]\ \right) = E\left[x^i(h)\right] - \frac{1}{2}\rho_i \operatorname{var}\left(x^i(h)\right).$$

Ans: Notice that $x^i(h)$ is a linear combination of random variables that are jointly normally distributed. Hence, $x^i(h)$ is itself normally distributed. We can then use the properties of a lognormally distributed random variable to obtain the desired result.

e.5) Consider the increasing transformation $f^i : R_- \to R$,

$$f^i(u) = -\frac{1}{\rho_i}\log\left(-\rho_i\ u\right),$$

where $u^i(h) = E\left[v\left(x^i(h)\right)\right]$. Show that

$$w^i(h) = f\left(u^i(h)\right) = E\left[x^i(h)\right] - \frac{1}{2}\rho_i \operatorname{var}\left(x^i(h)\right).$$

Ans: Using the definition of $f^i(u)$ we can write

$$w^i(h) = f\left(u^i(h)\right) = -\frac{1}{\rho_i}\log\left(-\rho_i E\left[v\left(x^i(h)\right)\right]\right).$$

But from $e.4$ we know that

$$-\frac{1}{\rho_i}\log\left(-\rho_i E\left[v\left(x^i\left(h\right)\right)\right]\ \right)=E\left[x^i\left(h\right)\right]-\frac{1}{2}\rho_i\operatorname{var}\left(x^i\left(h\right)\right),$$

so the result is immediate. Thus, since f^i is a monotone transformation of $u^i\left(h\right)$, we may as well use $w^i\left(h\right)$ as our (simplified) objective function.

$e.6$) Consider the problem

$$\max_h\ E\left[x^i\left(h\right)\right]-\frac{1}{2}\rho_i\operatorname{var}\left(x^i\left(h\right)\right),$$

subject to

$$q^T h=q^T\theta,$$

where the expressions for $E\left[x^i\left(h\right)\right]$ and $\operatorname{var}\left(x^i\left(h\right)\right)$ are given above.

Normalizations. From now on we normalize $q_1=1$ (which, since $d_1=1$ means that we normalize the rate of interest to zero). We also normalize $\mu_d\left(k\right)=1$ for each $k=2,...,K$. We denote the vector μ_d by ι, a $K-1$ vector of ones. This means that $1/q_k$ is the gross expected return of asset k, and since the interest rate is zero, it is also the gross risk premium. We can think of $1-q_k$ as a measure of the risk premium of asset k.

Use the budget constraint to write

$$h_1=q^T\theta-\hat{q}^T\hat{h},$$

where $\hat{q}$ denotes the $K-1$ vector of prices of risky assets $\hat{q}=\left(q_2,q_3,...,q_K\right)$. Thus, the problem is equivalent to

$$\max_{\hat{h}}\ \left[q^T\theta+\hat{h}^T\left(\iota-\hat{q}\right)\ +\mu_e\left(i\right)\ \right]-\frac{\rho_i}{2}\left[\hat{h}^{\ T}\ \Omega_{dd}\ \hat{h}+\Omega_e\left(i,i\right)+2\sum_{k=1}^{K}\hat{h}_k\ \Omega_{ed}\left(i,k\right)\right].$$

Write down the FOC for this problem w.r.t. $\hat{h}_k$ for $k=2,...,K$. Collect these FOCs in vector form. Use $\Omega\left(i,\cdot\right)$ to denote the vector of the $K-1$ covariances of $\hat{e}^i$. Derive an expression for $\hat{h}^i$ as a function of Ω_{dd}^{-1}, ρ_i, $\hat{q}$, ι and $\Omega_{ed}\left(i,\cdot\right)$. Derive an expression for $h_1^i\left(\hat{q}\right)$ [Hint: your expression should be a function of Ω_{dd}^{-1}. ρ_i, $\hat{q}$, ι and $\Omega_{ed}\left(i,\cdot\right)$]. Assume (only for this last part of $e.6$) that Ω_{dd} is diagonal, solve for $\hat{h}_k^i$ explicitly, and give an economic explanation of each of the terms on it.

Ans: The FOC associated with this problem is

$$\hat{h}_k : (1 - \hat{q}_K) = \rho_i \left[\sum_{s=1}^{K} \Omega_{dd}(k,s)\, \hat{h}_s + \ \Omega_{ed}(i,k) \right].$$

Collecting these FOCs in vector form we obtain

$$(\iota - \hat{q}) = \rho_i \left[\Omega_{dd}\ \hat{h} + \ \Omega_{ed}(i,\cdot) \right], \tag{13}$$

or

$$\hat{h}^i(\hat{q}) = \Omega_{dd}^{-1} \left[\frac{1}{\rho_i}(\iota - \hat{q}) - \Omega_{ed}(i,\cdot) \right]. \tag{14}$$

Moreover,

$$h_1^i(\hat{q}) = q^T\theta^i - \hat{q}^T\Omega_{dd}^{-1} \left[\frac{1}{\rho_i}(\iota - \hat{q}) - \Omega_{ed}(i,\cdot) \right]. \tag{15}$$

When Ω_{dd} is diagonal, from (14) we can write $\hat{h}_k^i$ as

$$\hat{h}_k^i = \frac{1}{\operatorname{var}\left(\hat{d}_k\right)} \left[\frac{1}{\rho_i}(1 - \hat{q}_K) - \operatorname{cov}\left(\hat{d}_k, \hat{e}^i\right) \right].$$

Thus, the demand for the risky asset k is higher when: *i*) the variance -which is a measure of the risk- of that asset is lower (var $\left(\hat{d}_k\right)$ is low), *ii*) the individual is more risk tolerant (ρ_i is low), *iii*) the risk premium -which is the excess return for holding a risky asset- is higher ($1 - \hat{q}_K$ is high), and *iv*) the covariance between the asset payoffs and labor income is more negative (since this implies that this asset would be helpful in insuring against fluctuations in random labor income).

e.7) What is the effect on the demands for risky assets $\hat{h}^i$ and for the riskless asset h_1^i if an agent has higher wealth (say a higher value of θ^i)?

Ans: From (14) we see that the demand for risky assets $\hat{h}^i$ is invariant to changes in θ^i (and hence the demand for the riskless asset h_1^i must increase). This result is specific to this particular example, an arises from the fact that the agent's attitude toward risk is independent of her wealth level. Indeed, from the setup of the problem we know that the coefficient of absolute risk aversion in this example equals ρ, which is invariant to changes in wealth. Those preferences that satisfy this property are said to exhibit Constant Absolute Risk Aversion (CARA). In contrast, if preferences exhibited Decreasing Absolute Risk Aversion (DARA), so that an individual's risk aversion would decrease with her wealth level, the demand for risky assets should rise with the wealth of the agent (that is, risky assets would be normal goods).

e.8) Use that in equilibrium the following $K-1$ equations have to be solved

$$\sum_{i=1}^{I} \hat{\theta}^{i} = \sum_{i=1}^{I} \hat{h}^{i}(\hat{q}),$$

where $\hat{h}^{i}(\hat{q})$ denotes the optimal demand of risky asset, and $\hat{\theta}^{i}$ denotes the endowment of the $K-1$ risky assets for agent i. Write an explicit solution for $\hat{q}$. Your solution should be a function of the reciprocal of the average risk tolerance, $1/\left(\sum_i 1/\rho_i\right)$, the variance Ω_{dd} of assets, the vector of the endowment of risky assets, $\sum_i \hat{\theta}^{i}$, and the sum of the covariances, $\sum_i \Omega_{ed}(i,\cdot)$. Give an economic explanation of the effect of each of these terms (four lines maximum).

Ans: Adding expression (13) across agents we obtain

$$\sum_{i=1}^{I} \frac{1}{\rho_i}(\iota - \hat{q}) = \sum_{i=1}^{I} \left[\Omega_{dd}\ \hat{h}^{i}(\hat{q}) + \ \Omega_{ed}(i,\cdot)\right],$$

or, using the fact that in equilibrium $\sum_i \hat{\theta}^{i} = \sum_i \hat{h}^{i}(\hat{q})$ and solving for $\hat{q}$,

$$\hat{q} = \iota - \frac{1}{\left(\sum_{i=1}^{I} \frac{1}{\rho_i}\right)} \left[\Omega_{dd}\left(\sum_{i=1}^{I} \hat{\theta}^{i}\right) + \sum_{i=1}^{I}\ \Omega_{ed}(i,\cdot)\right].$$

Thus, the price of risky asset is higher when: i) the variance -which is a measure of the risk- of risky assets is lower ($|\Omega_{dd}|$ is low), ii) the economy is more risk tolerant ($\sum_{i=1}^{I} \frac{1}{\rho_i}$ is high), iii) the endowment of risky assets is lower ($\sum_i \hat{\theta}^{i}$ is low), and iv) the covariance between the asset payoffs and labor income is more negative (since this implies that this asset would be helpful in insuring against fluctuations in random labor income).

Remark 1 *In general, the right measure of risk is that of second order stochastic dominance (you will later see this concept in detail). It turns out, however, that if two random variables X and Y are normally distributed and Y has a higher variance, then X second-order stochastically dominates Y. Thus, between two portfolios with identical mean, a risk averter will always choose the portfolio with minimum variance, irrespective of his particular utility function. That is why in this particular problem we can equate riskiness with variance.*

Theory of Income, Fall 2007

Fernando Alvarez, U of C

Midterm

In this exercise we will study the risk sharing implications for equilibrium allocations and Pareto Optimal allocations in the context of a pure exchange economy, with one good, two periods, S states of nature, and preferences given by expected utility. We will first specialize to the case of a one period economy and then solve the two period version.

3 Uncertainty

Consider a one period pure exchange economy with uncertainty. Let there be S states of the world and one physical commodity. We will index a commodity by the state, so there are $m = S$ goods, or $L = R^m$. Thus, we can interpret the vector x as the consumption of the good in each of the different states. We will write x_s for the good in state s. Thus the utility function u^i is a function of a vector on R^m. The endowment of agent $i \in I$, denoted by e^i are also indexed by state s. We let $\bar{e}_s$ be the aggregate endowment in state s.

Assume u^i is given by

$$u^i(x_1, x_2, ..., x_s) = \left(v^i\right)^{-1}\left(\sum_{s=1}^{m} v^i(x_s)\,\pi_s\right) \tag{16}$$

where $\left(v^i\right)^{-1}(\cdot)$ is the inverse function of v^i, $v^i : R \to R$ is the sub-utility function of agent i and $\pi_s \in R^m_+$ is the common probability of state s, with $\sum_{s=1}^{m} \pi_s = 1$. We assume that v^i are differentiable, strictly increasing and strictly concave. Notice that utility is measured in consumption equivalent units. To note this, by the definition of certainty equivalence ($CE(x)$) we have that:

$$\begin{aligned} v^i(CE(x)) &= \sum_{s=1}^{m} v^i(x_s)\,\pi_s \\ &\Rightarrow \\ CE(x) &= \left(v^i\right)^{-1}\left(\sum_{s=1}^{m} v^i(x_s)\,\pi_s\right) \end{aligned}$$

To understand these preferences even more, consider a bundle in which $x_1 = x_2 = ...x_S = x'$.

Question 1. [5 points] What is $\left(v^i\right)^{-1}\left(\sum_{s=1}^{S} v^i(x)\,\pi_s\right)$ evaluated at $x = x'$?

Answer:

$$\left(v^i\right)^{-1}\left(\sum_{s=1}^{S} v^i(x')\,\pi_s\right) = x'$$

Question 2. [10 points] State, without proof, a result that characterizes the Pareto Optimal allocation, i.e., how x^i_s depends on $\bar{e}_s$. State carefully all the characteristics of the functions that are mentioned (for instance, what is the domain of the functions).

Hint: This comes straight from the notes.

Answer.Fix and arbitrary vector of λ weights. The Pareto optimal allocation can be described by a set of strictly increasing functions g^i, $g^i : R_+ \rightarrow R_+$, of the aggregate endowment, i.e. the optimal allocation can be written as $x_s^i = g^i(\bar{e}_s)$ for all $i \in I$, $s = 1, .., S$.

Question 3. [5 points] Take two arbitrary states, s and s', write the Marginal Rate of Substitution (MRS) between them using (16)

Answer.

$$MRS_{s,s'} = \frac{\partial u^i(x)/\partial x_s}{\partial u^i(x)/\partial x_{s'}} = \frac{\pi_s \partial v^i(x_s)/\partial x_s}{\pi_{s'} \partial v^i(x_{s'})/\partial x_{s'}}$$

Question 4. [15 points] Assume that $v^i(x) = x^{1-\gamma}/(1-\gamma)$. What is the form of the function $g^i(\bar{e}_s)$ that describes the PO allocation described in your answer to Question 2?

Answer.

We can solve the following program

$$\max_{\{x_s^i\}} \sum \lambda_i v^i\left(x_s^i\right) \text{ s.t. } \sum_{i=1}^{I} x_s^i = \bar{e}_s$$

by the fact that v^{-1} is a monotonous, increasing function. FOC:

$$\begin{aligned} \lambda_i \frac{\partial v^i\left(x_s^i\right)}{\partial x} &= \mu_s \\ \lambda_i \left(x_s^i\right)^{-\gamma} &= \mu_s \end{aligned}$$

solving for x_s^i,

$$x_s^i = \left(\frac{\mu_s}{\lambda_i}\right)^{-\frac{1}{\gamma}}$$

Then,

$$\sum_i x_s^i = \bar{e}_s = \mu_s^{-\frac{1}{\gamma}} \sum_i \left(\frac{1}{\lambda_i}\right)^{-\frac{1}{\gamma}} = \mu_s^{-\frac{1}{\gamma}} \sum_i \lambda_i^{\frac{1}{\gamma}}$$

or,

$$\mu_s^{-\frac{1}{\gamma}} = \bar{e}_s \left[\sum_i \lambda_i^{\frac{1}{\gamma}}\right]^{-1}$$

Finally,

$$x_s^i = \bar{e}_s \left[\sum_i \lambda_i^{\frac{1}{\gamma}}\right]^{-1} \lambda_i^{\frac{1}{\gamma}}$$

so,

$$x_s^i = g(\bar{e}_s) = \delta_i \bar{e}_s$$

where

$$\delta_i \equiv \left[\sum_i \lambda_i^{\frac{1}{\gamma}}\right]^{-1} \lambda_i^{\frac{1}{\gamma}}$$

Consider the security market economy as the one described in your lecture notes. Recall that the budget constraint is written in two (set of) equations. The first equation is given by

$$\sum\nolimits_{k=1}^{K} h_k^i q_k = \sum\nolimits_{k=1}^{K} \theta_k^i q_k$$

The second equation is one for each state $s = 1, .., m$:

$$x_s^i = \sum\nolimits_{k=1}^{K} h_k^i d_{ks} + \hat{e}_s^i$$

Let security $k = 1$ be a risk-less bond, i.e. it pays $d_{ks} = 1$ in all states of nature. Security $k = 2$ is similar to a stock, it pays $d_{ks} = \bar{e}_s$. Assume complete markets, that $v^i(x) = x^{1-\gamma}/(1-\gamma)$ and that $\log \bar{e} \sim N(\mu, \sigma^2)$.

For future reference,

Question 5 [10 points] Compute the Certainty Equivalence of a boundle $\bar{x}$ when $\log \bar{e} \sim N(\mu, \sigma^2)$.

Hint: recall that if X is lognormally distributed, then

$$E\left[X^{1-\gamma}\right] = \exp\left((1-\gamma)\mu + (1-\gamma)^2 \frac{1}{2}\sigma^2\right)$$

Answer.

$$CE(\bar{x}) = \exp\left(\mu + (1-\gamma)\frac{1}{2}\sigma^2\right)$$

Question 6. [10 points] Write down and expression for the risk premium.

Hint: recall that the multiplicative risk premium is given by $(1+r_2)/(1+r_1)$. With this particular utility form, this is given by

$$\frac{1+r_2}{1+r_1} = \frac{E\left[\bar{e}^{-\gamma}\right] E\left[\bar{e}\right]}{E\left[\bar{e}^{-\gamma}\bar{e}\right]}$$

Also, recall that if X is lognormally distributed, then

$$\begin{aligned} E[X] &= \exp\left(\mu + \frac{1}{2}\sigma^2\right) \\ E\left[X^{-\gamma}\right] &= \exp\left(-\gamma\mu + \gamma^2\frac{1}{2}\sigma^2\right) \end{aligned}$$

Answer.

$$\frac{1+r_2}{1+r_1} = \exp\left(\gamma\sigma^2\right)$$

and for small $\gamma\sigma$, the risk premium is given by:

$$r_2 - r_1 \simeq \gamma\sigma^2$$

4 A Two Period Economy

Consider a version of an endowment economy in which there is no uncertainty and there are only two periods, then $L = R^2$. Let $u^i\left(x_1, x_2\right) = F^i\left(x_0, x_1\right)$, where $F^i : R_+^2 \to R$

Question 7 .[5 points] Write down the Arrow-Debreu budget set.

Hint: One equation

Answer.

$$p_0 x_0^i + p_1 x_1^i \leq p_0 e_0^i + p_1 e_1^i$$

Now we consider an alternative representation of the economy. Let $d = 1$ be the payoff, in period 1, of the security. Denote by q be the price of the security (measured in units of the good x_0), h^i the purchases of this security by agent i, θ^i the endowment of this security by agent i, and by $\hat{e}_0^i$ and $\hat{e}_1^i$ the endowment of goods in period 0 and 1.

Question 8. [10 points] Write down the budget set of the corresponding security market economy. This consists of two equations, one that applies to period 0,when the sale and purchase of the security takes place, and the other for period 1.

Answer.

$$\begin{aligned} x_0^i + q h^i &= q\theta^i + \hat{e}_0^i \\ x_1^i &= h^i + \hat{e}_1^i \end{aligned}$$

Question 9. [10 points] How does the net interest rate relate to q?

Answer.

$$q = \frac{1}{1+r}$$

Question 10. [5 points] How does the Arrow Debreu prices p relate to the price of the security q? (One line)

Answer.

$$q = \frac{p_1}{p_0}$$

Question 11. .[5 points] In this context, what will be the endowments in the Arrow Debreu economy e^i that correspond to the endowments in the security market θ^i, and $\hat{e}^i$?

Answer.

$$\begin{aligned} \hat{e}_0^i &= e_0^i \\ \hat{e}_1^i + \theta^i &= e_1^i \end{aligned}$$

Question 12. [10 points] What is the MRS in this case? Interpret your results and relate them to the interest rate.

Answer.

$$MRS_{0,1} = \frac{\partial F^i(x_0, x_1)/\partial x_1}{\partial F^i(x_0, x_1)/\partial x_0} = \frac{1}{1+r} = \frac{p_1}{p_0}$$

The marginal rate of substitution is equal to the relative price of goods. The relative price of consumption goods at date one in terms of the price of consumption goods at date zero is equal to the inverse of the gross interest rate.

Specialize to the following Constant Elasticity of Substitution form:

$$F^i(x_0, x_1) = \left((1-\beta)\, x_0^{1-\rho} + \beta x_1^{1-\rho}\right)^{\frac{1}{1-\rho}}, \quad \text{for } \rho > 0$$

Question 13. [20 points] Calculate the equilibrium interest rate only as a function of β, ρ and $\bar{e}_o/\bar{e}_1$ (the ratio of the aggregate endowments in period 0 over period 1).

Answer.

$$1 + r = \frac{1-\beta}{\beta}\left(\frac{\bar{e}_o}{\bar{e}_1}\right)^{-\rho}$$

Question 14. [15 points] Give an intuitive explanation of how changes in $\bar{e}_o/\bar{e}_1$ affect the interest rate.

Answer.

$$\frac{\partial\left(1+r\right)}{\partial\frac{\bar{e}_o}{\bar{e}_1}}=\left(-\rho\right)\frac{1-\beta}{\beta}\left(\frac{\bar{e}_o}{\bar{e}_1}\right)^{-\rho-1}<0$$

Intuitively, you will like to smooth consumption over time, higher relative scarcity in period 1 will make your demand for savings to increase and hence preasure the interest rate to decrease.

Question 15. [10 points] Why the equilibrium interest rate does not depend on the distribution of initial endowments?

Answer: Since p_0/p_1 is independent of the λ -weights, relative prices are independent of the distribution of wealth and there is aggregation. Alternative: identical and homothetic preferences (as in problem set 2)

4.1 A Two Period Economy with Uncertainty

Consider a two period pure exchange economy with uncertainty. In the current period, commodity "0" is consumed, while in the next period, commodities "1", ..."S" are consumed in state s. This means that there are $m = S+1$ commodities and $L = R^{S+1}$,. u^i is given by $u^i\left(x_0, x_1, ..., x_S\right)$, hence you have preferences over your consumption today, commodity "0", and consumption tomorrow according to what state you are in. The endowment of agent i, denoted by e^i also belong to R^{S+1}. We let $\bar{e}$ be the aggregate endowment.

Question 16.[5 points] Consider the following securities. Denote by d_{ks} be the payoff of security $k \in K$ in sate s. Denote by q_k be the price of security k (measured in units of commodity "0"), h^i_k the purchases of this security by agent i, θ^i_k the endowment of this security by agent i, and by $\hat{e}^i$ the endowment of goods.

Write the budget constraint for the security market as a set of two equations, one that applies to period 0 where consumption of x_0 and trade take place, and a set of equations for each state in period s.

Answer:

$$x^i_0+\sum\nolimits_{k=1}^{K}h^i_kq_k=\sum\nolimits_{k=1}^{K}\theta^i_kq_k+\hat{e}^i_0$$

The second equation is one for each state $s = 1, .., m$:

$$x^i_s=\sum\nolimits_{k=1}^{K}h^i_kd_{ks}+\hat{e}^i_s$$

Question 17. [10 points] Assume that prices and payoffs in the security market are consistent with state prices of the Arrow-Debreu economy, and that the endowments of both economy are equivalent. Which Budget Set is larger?

Answer: If (x, h) is budget feasible in the security market economy, then x is budget feasible in the Arrow-Debreu economy. This means that the budget set of the Arrow-Debreu economy is larger.

Question 18. [5 points] When are the Budget Sets of the security market and the Arrow Debreu economy the same?

Answer: The budget sets of both economies are equal provided that D (the matrix with the payoffs of the K securities in the m states) has full rank.

From now on specialize the utility to be separable, so that we can highlight both how agents substitute consumption across the different states of the world as well as intertemporally. Let preferences by given by

$$u^i(x_0, x_1, ..., x_S) = F^i\left(x_0, \psi^i(x_1, ..., x_S)\right) \tag{17}$$

where

$$\psi^i(x_1, ..., x_S) = \left(v^i\right)^{-1}\left(\sum_{s=1}^{S} v^i(x_s)\,\pi_s\right) \tag{18}$$

the subutility function $\psi^i : R_+^s \to R$, π_s is the common probability of state s. We assume that v^i is strictly increasing and concave and that F^i is strictly increasing and concave in both arguments.

Question 19. [5 points] Compute F_1^i/F_2^i, where F_j^i is the partial derivative of the function F^i with respect to argument j. Give an intuitive interpretation of your result.

Answer:

$$F_1^i/F_2^i = \frac{\partial F^i(x_0, \psi(x_1, ..., x_S))/\partial x_0}{\partial F^i(x_0, \psi(x_1, ..., x_S))/\partial \psi^i(x_1, ..., x_S)}$$

This is the marginal rate of substitution between consumption in period zero and period 1.

Question 20. [20 points] Take two arbitrary states, s and s', where $s > 0$. What is the MRS between these two states? Be very explicit in all the terms you write. Does it depend on x_0? Does it depend on F^i? Why? (1 line).

Answer:

$$
\begin{aligned}
\frac{\partial u^i(x)}{\partial x_s} &= \frac{\partial F^i\left(x_0, \psi^i(x_1, ..., x_S)\right)}{\partial \psi^i(x_1, ..., x_S)} \frac{\partial \psi^i(x_1, ..., x_S)}{\partial x_s} \\
\frac{\partial u^i(x)/\partial x_s}{\partial u^i(x)/\partial x_{s'}} &= \frac{\partial \psi^i(x)/\partial x_s}{\partial \psi^i(x)/\partial x_{s'}} = \frac{\pi_s \partial v^i(x_s)/\partial x_s}{\pi_{s'} \partial v^i(x_{s'})/\partial x_{s'}}
\end{aligned}
$$

Does it depend on x_0? NO, Does it depend on F^i? No.
Why? Separability.

It is easy to show that the following theorem holds for this economy:

Theorem 2 *Fix and arbitrary vector of λ weights. If x_s^i is a Pareto optimal allocation, then there are strictly increasing functions g^i, $g^i : R_+ \to R_+$, of the aggregate endowment, so that $x_s^i = g^i(\bar{e}_s)$ for all $i \in I$, $s = 1, .., S$.*

Proof. *We will use the theorem that we developed in the lectures notes (Uncertainty, page 6) and separability of the utility function.*

We will aim to prove the theorem by contradiction.

Assume $X = \{x_s^i\}_{i \in I, s \in S}$ is a PO allocation. We can also define this allocation as $X = \{x_0^i, y^i\}_{i \in I}$ where $y^i = \{x_s^i\}_{i \in I, s \geq 1}$. Say there is another feasible allocation X' such that $(x_0^i)' = x_0^i$ but $(y^i)' \succ y^i$ (this is analogous to say that $(y^i)'$ has higher certainty equivalent). Then $(y^i)'$ pareto dominates y^i (here we are using the theorem of the notes). Furthermore, by separability $X' \succ X$. In other words, X' pareto dominates X. This contradicts the fact that X was PO. Finally, note that we proved that separability implies that a PO allocation for the whole program implies a PO allocation for the sub-program in period 1. Then, the theorem that we have in the lectures notes applies there and thus the theorem stated here holds. ■

Question 21.[10 points] You need to answer True or False to the following questions about the function g^i on the previous theorem:

1) $g^i(\cdot)$ depends on the λ (weights).
True
2) $g^i(\cdot)$ depends on F^i.
False
3) $g^i(\cdot)$ depends on $\bar{e}_0$.
False
4) $g^i(\cdot)$ depend on $v^i(\cdot)$.
True
5) $g^i(\cdot)$ depend on π_s.
False

Question 22. [15 points] Write down an expression for first order conditions for h_1^i, h_2^i and x_0^i. Use μ_i for the Lagrange multiplier of the budget constraint.

Answer:

$$\begin{aligned} \left[x_0^i\right] &: F_1^i = \mu^i \\ \left[h_1^i\right] &: F_2^i \sum_{s=1}^{S} \frac{\partial \psi^i\left(x_s^i\right)}{\partial x_s} d_{1s} = \mu^i q_1 \\ \left[h_2^i\right] &: F_2^i \sum_{s=1}^{S} \frac{\partial \psi^i\left(x_s^i\right)}{\partial x_s} d_{2s} = \mu^i q_2 \end{aligned}$$

Question 23. [20 points] Define the expected gross return of the securities 1 and 2, i.e $(1+r_1) = E\left[d_1\right]/q_1$ and $(1+r_2) = E\left[d_2\right]/q_2$. What is the relationshipt between r_1 and the interest rate? Using your answer to Question 22, do $1+r_1$ and $1+r_2$ depend on the choice of numeraire?

Answer:

We know that the interest rate is given by

$$\frac{F_2^i}{F_1^i} = \frac{1}{1+r}$$

using the answers to question 22 we get that:

$$\begin{aligned} \frac{F_2^i}{F_1^i} \sum_{s=1}^{S} \frac{\partial \psi^i\left(x_s^i\right)}{\partial x_s} d_{1s} &= q_1 \\ \frac{1+r_1}{1+r} &= \frac{\sum_{s=1}^{S} d_{1s}\pi_s}{\sum_{s=1}^{S} \frac{\partial \psi^i\left(x_s^i\right)}{\partial x_s} d_{1s}} \end{aligned}$$

Clearly $1+r_1$ and $1+r_2$ depend on the numeraire since q_1 and q_2 are measured in terms of units of goods at time zero consumption.

Question 24. [20 points] Define the multiplicative risk premium as $(1+r_2)/(1+r_1)$, the ratio of the expected gross return of the securities 2 to 1. Write down an expression for $(1+r_2)/(1+r_1)$. Does your answer depend on F^i or x_0^i ? Is your expression the same as the one for an economy with uncertainty but with only one period (like the one discussed in the class notes)?

Answer:

$$
\begin{aligned}
\frac{1+r_2}{1+r_1} &= \frac{E\left[d_2\right]}{E\left[d_1\right]} \frac{\sum_{s=1}^{S} \frac{\partial \psi^i\left(x_s^i\right)}{\partial x_s} d_{1s}}{\sum_{s=1}^{S} \frac{\partial \psi^i\left(x_s^i\right)}{\partial x_s} d_{1s}} \\
&= \frac{E\left[d_2\right]}{E\left[d_1\right]} \frac{\sum_{s=1}^{S} \frac{\partial v^i\left(x_s\right)}{\partial x_s} d_{1s} \pi_s}{\sum_{s=1}^{S} \frac{\partial v^i\left(x_s\right)}{\partial x_s} d_{2s} \pi_s} \\
&= \frac{E\left[d_2\right]}{E\left[d_1\right]} \frac{E\left[\frac{\partial v^i\left(x_s\right)}{\partial x_s} d_{1s}\right]}{E\left[\frac{\partial v^i\left(x_s\right)}{\partial x_s} d_{2s}\right]}
\end{aligned}
$$

Does your answer depend on F^i or x_0^i ? NO

Is your expression the same as the one for an economy with uncertainty but with only one period (like the one discussed in the class notes)? YES, the separability property of the utility function garanties this.

Now, we further specialize the utility function. Let $F^i\left(x_0, \psi^i\left(x_1, ..., x_S\right)\right) = \Big(\left(1-\beta\right) x_0^{1-\rho} + \beta\left(\psi^i\left(x_1, ..., x_S\right)\right.$ for $\rho > 0$, $\psi^i\left(x_1, ..., x_S\right)$ given by (18) and $v^i\left(x\right) = x^{1-\gamma}/\left(1-\gamma\right)$.

Question 25 .[5 points] Are the preferences described above identical and homothetic? (yes or no answer)

Answer: Yes

Let security $k=1$ be a risk-less bond, i.e. it pays $d_{ks}=1$ in all states of nature. Assume complete markets, and that $\log \bar{e}_s/\bar{e}_0 \sim N\left(\mu, \sigma^2\right)$ and $\bar{e}_0 = 1$. Recall that i X is lognormally distributed, i.e. if $\log X$ is $N\left(\mu, \sigma^2\right)$ then

$$
\begin{aligned}
E\left[X\right] &= \exp\left(\mu + \frac{1}{2}\sigma^2\right) \\
E\left[X^{-\gamma}\right] &= \exp\left(-\gamma\mu + \gamma^2 \frac{1}{2}\sigma^2\right)
\end{aligned}
$$

Question 26. Use the answer to your previous question to find an expression for $1 + r_1$. Your answer should be a function of the preference parameters ρ, β, γ and the parameters describing the distribution of the aggregagte endowment σ and μ. How does this compare with your answer for the economy with two periods and no uncertainty?

Answer:

First note that $\psi^i\left(x_1, ..., x_S\right) = \left(v^i\right)^{-1}\left(\sum_{s=1}^{S} v^i\left(x_s\right)\pi_s\right) = \left(\sum_{s=1}^{S}\left(x_s^i\right)^{1-\gamma}\pi_s\right)^{\frac{1}{1-\gamma}}$, recall that the solution to the PO are increasing functions of the aggregate endowment. These functions take the form $g^i(\bar{e}_s) = \frac{\lambda_i^{1/\gamma}}{\sum_i \lambda_i^{1/\gamma}} \bar{e}_s$. Using this result, that $\bar{e}_0 = 1$ and that $\log \bar{e}_s / \bar{e}_0$ has a normal distribution, we get:

$$
\begin{aligned}
\psi^i(x_1, ..., x_S) &= \left(v^i\right)^{-1}\left(\sum\nolimits_{s=1}^S v^i(x_s)\,\pi_s\right) = \left(\sum_{s=1}^S \left(x_s^i\right)^{1-\gamma}\pi_s\right)^{\frac{1}{1-\gamma}} \\
&= \left(\sum_{s=1}^S g^i(\bar{e}_s)^{1-\gamma}\pi_s\right)^{\frac{1}{1-\gamma}} \\
&= \left(\sum_{s=1}^S \left(\delta^i\right)^{1-\gamma}(\bar{e}_s)^{1-\gamma}\pi_s\right)^{\frac{1}{1-\gamma}} \\
&= \delta^i\left(\sum_{s=1}^S (\bar{e}_s)^{1-\gamma}\pi_s\right)^{\frac{1}{1-\gamma}}
\end{aligned}
$$

Then,

$$
\begin{aligned}
1 + r_1 &= \frac{1}{q_1} = \left[\left(\frac{\beta}{1-\beta}\right)\frac{\psi(x_1, ..., x_S)^{\gamma-\rho}\sum_{s=1}^S \left(x_s^i\right)^{-\gamma}\pi_s}{\left(x_0^i\right)^{-\rho}}\right]^{-1} \\
&= \left[\left(\frac{\beta}{1-\beta}\right)\frac{\left(\delta^i\right)^{-\rho}\left(\sum_{s=1}^S (\bar{e}_s)^{1-\gamma}\pi_s\right)^{\frac{\gamma-\rho}{1-\gamma}}\sum_{s=1}^S \bar{e}_s^{-\gamma}\pi_s}{\left(x_0^i\right)^{-\rho}}\right]^{-1} \\
&= \left[\left(\frac{\beta}{1-\beta}\right)\frac{\left(\delta^i\right)^{-\rho}E\left(\bar{e}_s^{1-\gamma}\right)^{\frac{\gamma-\rho}{1-\gamma}}E\left(\bar{e}_s^{-\gamma}\right)}{\left(x_0^i\right)^{-\rho}}\right]^{-1} \\
&= \left[\delta^i\right]^\rho\left(\frac{1-\beta}{\beta}\right)\left[\frac{\left[\exp\left([1-\gamma]\,\mu + (1-\gamma)^2\frac{1}{2}\sigma^2\right)\right]^{\frac{\gamma-\rho}{1-\gamma}}\exp\left(-\gamma\mu + \gamma^2\frac{1}{2}\sigma^2\right)}{\left(x_0^i\right)^{-\rho}}\right]^{-1} \\
&= \left[\delta^i\right]^\rho\left(\frac{1-\beta}{\beta}\right)\left[\frac{\exp\left([\gamma-\rho]\,\mu + [\gamma-\rho]\,(1-\gamma)\,\frac{1}{2}\sigma^2\right)\exp\left(-\gamma\mu + \gamma^2\frac{1}{2}\sigma^2\right)}{\left(x_0^i\right)^{-\rho}}\right]^{-1} \\
&= \left[\delta^i\right]^\rho\left(\frac{1-\beta}{\beta}\right)\left[\frac{\exp\left(-\rho\mu + (\gamma-\rho+\rho\gamma)\,\frac{1}{2}\sigma^2\right)}{\left(x_0^i\right)^{-\rho}}\right]^{-1}
\end{aligned}
$$

Using that $g^i(\bar{e}_s) = \frac{\lambda_i^{1/\gamma}}{\sum_i \lambda_i^{1/\gamma}} \bar{e}_s$, we can define $\delta \equiv \frac{\lambda_i^{1/\gamma}}{\sum_i \lambda_i^{1/\gamma}}$ and thus,

$$1 + r_1 = \left[\frac{\lambda_i^{1/\gamma}}{\sum_i \lambda_i^{1/\gamma}} \right]^{\rho} \left(\frac{1-\beta}{\beta} \right) \left[\frac{\exp\left(-\rho\mu + (\gamma - \rho + \rho\gamma) \frac{1}{2}\sigma^2\right)}{\left(x_0^i\right)^{-\rho}} \right]^{-1}$$

For future reference,

$$\begin{aligned}
1 + r_2 &= E[\bar{e}_s] q_2^{-1} \\
&= E[\bar{e}_s] \left(\frac{1-\beta}{\beta} \right) \left[\frac{\psi(x_1, ..., x_S)^{\gamma-\rho} \sum_{s=1}^{S} \bar{e}_s g^i(\bar{e}_s)^{-\gamma} \pi_s}{\left(x_0^i\right)^{-\rho}} \right]^{-1} \\
&= E[\bar{e}_s] \left(\frac{1-\beta}{\beta} \right) \left[\frac{\left[\delta^i\right]^{-\rho} \left(\sum_{s=1}^{S} \bar{e}_s^{1-\gamma} \pi_s \right)^{\frac{\gamma-\rho}{1-\gamma}} \sum_{s=1}^{S} \bar{e}_s^{1-\gamma} \pi_s}{\left(x_0^i\right)^{-\rho}} \right]^{-1} \\
&= E[\bar{e}_s] \left(\frac{1-\beta}{\beta} \right) \left[\frac{\left[\delta^i\right]^{-\rho} E\left[\bar{e}_s^{1-\gamma}\right]^{\frac{\gamma-\rho}{1-\gamma}} E\left[\bar{e}_s^{1-\gamma}\right]}{\left(x_0^i\right)^{-\rho}} \right]^{-1} \\
&= E[\bar{e}_s] \left(\frac{1-\beta}{\beta} \right) \left[\frac{\left[\delta^i\right]^{-\rho} E\left[\bar{e}_s^{1-\gamma}\right]^{\frac{\gamma-\rho}{1-\gamma}+1}}{\left(x_0^i\right)^{-\rho}} \right]^{-1} \\
&= E[\bar{e}_s] \left(\frac{1-\beta}{\beta} \right) \left[\frac{\left[\delta^i\right]^{-\rho} E\left[\bar{e}_s^{1-\gamma}\right]^{\frac{1-\rho}{1-\gamma}}}{\left(x_0^i\right)^{-\rho}} \right]^{-1} \\
&= E[\bar{e}_s] \left(\frac{1-\beta}{\beta} \right) \left[\frac{\left[\delta^i\right]^{-\rho} \left[\exp\left([1-\gamma]\mu + (1-\gamma)^2 \frac{1}{2}\sigma^2\right)\right]^{\frac{1-\rho}{1-\gamma}}}{\left(x_0^i\right)^{-\rho}} \right]^{-1} \\
&= E[\bar{e}_s] \left(\frac{1-\beta}{\beta} \right) \left[\frac{\left[\frac{\lambda_i^{1/\gamma}}{\sum_i \lambda_i^{1/\gamma}} \right]^{-\rho} \left[\exp\left((1-\rho)\mu + (1-\rho)(1-\gamma) \frac{1}{2}\sigma^2\right)\right]}{\left(x_0^i\right)^{-\rho}} \right]^{-1}
\end{aligned}$$

Question 27. Use the answer to your previous question and the formulas for the expected value of a lognormal, to find expression for the mulitiplicative risk premium $(1 + r_2)/(1 + r_1)$ in this economy. Your expression should be a function of the preference parameters ρ, β, γ and the parameters describing the distribution of the aggregate endowment σ and μ. How does this compare with the expression for an economy with uncertainty and only one period, like the one in the class notes?

Answer.

Note that

$$\begin{aligned} q_1 &= \left(\frac{\beta}{1-\beta}\right) \frac{\psi\left(x_1, ..., x_S\right)^{\gamma-\rho} \sum_{s=1}^{S} \left(x_s^i\right)^{-\gamma} \pi_s}{\left(x_0^i\right)^{-\rho}} \\ & \left(\frac{\beta}{1-\beta}\right) \frac{\psi\left(x_1, ..., x_S\right)^{\gamma-\rho} \sum_{s=1}^{S} g^i\left(\bar{e}_s\right)^{-\gamma} \pi_s}{\left(x_0^i\right)^{-\rho}} \end{aligned}$$

and

$$q_2 = \left(\frac{\beta}{1-\beta}\right) \frac{\psi\left(x_1, ..., x_S\right)^{\gamma-\rho} \sum_{s=1}^{S} \bar{e}_s g^i\left(\bar{e}_s\right)^{-\gamma} \pi_s}{\left(x_0^i\right)^{-\rho}}$$

We want

$$\begin{aligned} \frac{1+r_2}{1+r_1} &= \frac{\sum_{s=1}^{S} \bar{e}_s \pi_s q_1}{q_2} = \frac{\sum_{s=1}^{S} \bar{e}_s \pi_s \sum_{s=1}^{S} g^i\left(\bar{e}_s\right)^{-\gamma} \pi_s}{\sum_{s=1}^{S} \bar{e}_s g^i\left(\bar{e}_s\right)^{-\gamma} \pi_s} \\ &= \frac{E\left[\bar{e}_s\right] E\left[g^i\left(\bar{e}_s\right)^{-\gamma}\right]}{E\left[\bar{e}_s g^i\left(\bar{e}_s\right)^{-\gamma}\right]} \end{aligned}$$

We discussed in detail that $g^i\left(\bar{e}_s\right)$ is a linear function of $\bar{e}_s$. Using this fact,

$$\frac{1+r_2}{1+r_1} = \frac{E\left[\bar{e}_s\right] E\left[\bar{e}_s^{-\gamma}\right]}{E\left[\bar{e}_s^{1-\gamma}\right]} = e\left(\gamma\sigma^2\right) \simeq \gamma\sigma^2$$

Alternative: Using the results of question 26,

$$
\begin{aligned}
\frac{1+r_2}{1+r_1} &= \frac{E\left[\bar{e}_s\right]\left(\frac{1-\beta}{\beta}\right)\left[\frac{\left[\delta^i\right]^{-\rho}\left[\exp\left((1-\rho)\mu+(1-\rho)(1-\gamma)\frac{1}{2}\sigma^2\right)\right]}{\left(x_0^i\right)^{-\rho}}\right]^{-1}}{\left[\delta^i\right]^{\rho}\left(\frac{1-\beta}{\beta}\right)\left[\frac{\exp\left(-\rho\mu+(\gamma-\rho+\rho\gamma)\frac{1}{2}\sigma^2\right)}{\left(x_0^i\right)^{-\rho}}\right]^{-1}} \\
&= \exp\left(\mu+\frac{1}{2}\sigma^2\right)\frac{\left[\exp\left((1-\rho)\mu+(1-\rho)(1-\gamma)\frac{1}{2}\sigma^2\right)\right]^{-1}}{\left[\exp\left(-\rho\mu+(\gamma-\rho+\rho\gamma)\frac{1}{2}\sigma^2\right)\right]^{-1}} \\
&= \exp\left(\mu+\frac{1}{2}\sigma^2\right)\frac{\exp\left(-\rho\mu+(\gamma-\rho+\rho\gamma)\frac{1}{2}\sigma^2\right)}{\exp\left((1-\rho)\mu+(1-\rho)(1-\gamma)\frac{1}{2}\sigma^2\right)} \\
&= \exp\left(\mu+\frac{1}{2}\sigma^2\right)\exp\left(-\rho\mu-(1-\rho)\mu+\left[(\gamma-\rho+\rho\gamma)-(1-\rho)(1-\gamma)\right]\frac{1}{2}\sigma^2\right) \\
&= \exp\left(\mu+\frac{1}{2}\sigma^2\right)\exp\left(-\mu+\gamma\sigma^2-\frac{1}{2}\sigma^2\right) \\
&= \exp\left(\gamma\sigma^2\right)\simeq\gamma\sigma^2
\end{aligned}
$$

Remark (after the midterm). In the first part of the exam we used the CRRA specification for the utility function, while in the last part we used the CES specification. In the first specification, risk aversion and intertemporal elasticity of substitution coincide (they are both defined by γ!) while in the second one, they differ (γ is risk aversion while ρ is intertemporal elasticity of substitution). Conceptually it is nice to distinguish them (even though in a vast majority of papers the CRRA specification is used). For a better understanding pay attention to the following: define

$$F\left(x_0,x_1\right)=(1-\beta)\,v\left(x_0\right)+\beta v\left[\psi^i\left(x_1,...,x_S\right)\right]$$

(key: separability!). This is the usual specification.

using the definition of ψ^i,

$$F\left(x_0,x_1\right)=(1-\beta)\,x_0^{1-\gamma}+\beta\sum_{s=1}^{S}\left(x_s^i\right)^{1-\gamma}\pi_s$$

it is straightforawrd to show that if we set $\rho=\gamma$ in $F^i\left(x_0,\psi^i\left(x_1,...,x_S\right)\right)$ we will get the same result.

This was the core question for 2006-2007.

Time is discrete and starts at $t=1$. Agents live for 2 periods, and have a labor endowment only when they are young. There is mass one of agents of each age, or if you prefer one agent per cohort. We index agents by the year in which they were born.

There is one unit a perishable "tree" in the economy. This tree gives a fruit, in consumption units, of d_t each period. The labor endowment of an agent born at time t is denoted by w_t. The young born at $t = 1, 2,$ have no endowment of trees. The old born at $t = 0$ have, at time $t = 1$, an endowment of 1 tree. Labor endowment and fruits are both proportional to the same random variable, Y_t :

$$w_t = (1 - \delta)\ Y_t \text{ and } d_t = \delta\ Y_t$$

where $\delta\ \in (0, 1)$ is a constant, and where Y_t satisfy

$$Y_{t+1} = Y_t\ z_{t+1}$$

with $Y_1 = 1$ and $\{z_{t+1}\}$ i.i.d. and $z_{t+1} > 0$.

There is one unit of the tree in the economy. Feasibility is then given by

$$c_t^t + c_t^{t-1} = w_t + d_t = Y_t \text{ for all } t \geq 1.$$

The preferences for the young born at $t \geq 1$ are:

$$(1 - \beta)\, u\left(c_t^t\right) + \beta E_t\left[u\left(c_{t+1}^t\right)\right]$$

where $E_t[\cdot]$ is the conditional expectation. The preferences for the old born at $t = 0$ are c_1^0 .

The budget constraint of agent born at t when young is:

$$c_t^t + s_t q_t^e + b_t q_t^b = w_t$$

where s_t are his purchases of the tree at price q_t^e, b_t are the purchases of a one period discount bond- a bond that pays one unit of consumption at $t+1$, with current price q_t^b. At time $t+1$, when this agent becomes old his consumption is financed by the fruits and sale of the trees, as well as the purchases of the discount bonds:

$$c_{t+1}^t = s_t\ \left[q_{t+1}^e + d_{t+1}\right] + b_t\ .$$

Question 1. [5 points] Using the budget constraint of the agent, replace c_{t+1}^t into the preferences and write down the objective function of the young agent born at time t as a function of s_t and b_t, taking current prices q_t^e, q_t^b and the distribution of future prices q_{t+1}^e and fruits d_{t+1} as given.

Answer to Q1. Problem of young at t

$$\max_{s_t,\ b_t} (1 - \beta)\, u\left(w_t - s_t q_t^e - b_t q_t^b\right) + \beta E_t\left[u\left(s_t\left[q_{t+1}^e + d_{t+1}\right] + b_t\right)\right]$$

Question 2. [5 points] Write down the first order conditions (foc's) with respect to s_t and b_t of the problem stated above:

Answer Q2.

$$
\begin{aligned}
(s_t) \quad &: \quad -(1-\beta)\, u'\left(c_t^t\right) q_t^e + \beta E_t\left[u'\left(c_{t+1}^t\right)\left(q_{t+1}^e + d_{t+1}\right)\right] = 0 \\
(b_t) \quad &: \quad -(1-\beta)\, u'\left(c_t^t\right) q_t^b + \beta E_t\left[u'\left(c_{t+1}^t\right)\right] = 0
\end{aligned}
$$

We can rewrite them as

$$
\begin{aligned}
(s_t) \quad &: \quad (1-\beta)\, u'\left(c_t^t\right) = \beta E_t\left[\left(\frac{q_{t+1}^e + d_{t+1}}{q_t^e}\right) u'\left(c_{t+1}^t\right)\right] \\
(b_t) \quad &: \quad (1-\beta)\, u'\left(c_t^t\right) = \beta E_t\left[\frac{1}{q_t^b} u'\left(c_{t+1}^t\right)\right]
\end{aligned}
$$

Question 3. [5 points] What has to be the equilibrium holding of trees of for generation $t \geq 1$, i.e. what has to be the equilibrium values of s_t for $t \geq 1$?

Answer Q 3. The initial old will supply their trees inelastically, so generation one will buy then when young, $s_1 = 1$. Generation 1 will then sell them when old, hence $s_2 = 1$. By induction, $s_t = 1$ for all $t \geq 1$.

Question 4. [5 points] Use the answer for the previous question, and the assumption that initial old (born at $t = 0$) have zero initial endowment of the discount bonds ($b_0 = 0$) to find out what should be the equilibrium value of b_t be for all $t \geq 1$?

Answer Q 4. The young at $t = 1$ have nobody to trade the one period bond with, so $b_1 = 0$. By repeating this argument we obtain $b_t = 0$ for all $t \geq 1$.

Question 5 . [10 points] Specialize the foc's obtained in Question 2 replacing the equilibrium values of s_t and b_t obtained in the Questions 3 and 4.

Answer to Q 5.

$$
(1-\beta)\, u'\left(w_t - q_t^e\right) = \beta E_t\left[\left(\frac{q_{t+1}^e + d_{t+1}}{q_t^e}\right) u'\left(q_{t+1}^e + d_{t+1}\right)\right] \tag{19}
$$

$$
(1-\beta)\, u'\left(w_t - q_t^e\right) = \beta E_t\left[\left(\frac{1}{q_t^b}\right) u'\left(q_{t+1}^e + d_{t+1}\right)\right] \tag{20}
$$

Question 6. [10 points] Assume from now on that preferences are given by $u(c) = \log(c)$. Guess that $q_t^e = Y_t\, Q^e$ and that $q_t^b = Q^b$ where Q^e and Q^b are two constants. Define $R_{t+1}^e = \left(q_{t+1}^e + d_{t+1}\right)/q_t^e$ and $R_t^b = 1/q_t^b$ as the gross returns on the trees and discount bonds.

Find i) an expression for the price of the three, Q^e as a function of δ and β, ii) an expression for the gross expected return of the tree, $E_t\left[\left(q_{t+1}^e + d_{t+1}\right)/q_t^e\right]$, as a function of δ, β and $E[z]$, and iii) an expression for the gross return on the discount bond $1/Q^b$ as a function of β, δ and $E[1/z]$.

Hints: use the foc's obtained in the previous question, as well as $d_{t+1} = \delta Y_{t+1}$, $w_t = (1-\delta)Y_t$, $z_{t+1} = Y_{t+1}/Y_t$, $q_{t+1}^e = Y_{t+1}\, Q^e$, $q_t^e = Y_t Q^e$.

Answer to Q 6.

i) Specializing (19) in $u(c) = \log c$,

$$
\begin{aligned}
(1-\beta)\frac{q_t^e}{w_t - q_t^e} &= \beta E_t\left[\frac{q_{t+1}^e + d_{t+1}}{q_{t+1}^e + d_{t+1}}\right] \\
\frac{q_t^e}{w_t - q_t^e} &= \frac{\beta}{1-\beta}
\end{aligned}
$$

using that $w_t = (1-\delta)Y_t$ and $q_t^e = Y_t Q^e$,

$$
\begin{aligned}
\frac{Y_t Q^e}{(1-\delta)Y_t - Y_t Q^e} &= \frac{\beta}{1-\beta} \\
\frac{Q^e}{(1-\delta) - Q^e} &= \frac{\beta}{1-\beta}
\end{aligned}
$$

from where it follows that

$$
Q^e = (1-\delta)\beta
$$

ii) using the definition of d_{t+1}, the conjecture for q_t^e, and the process for income,

$$
\begin{aligned}
E_t\left[\frac{q_{t+1}^e + d_{t+1}}{q_t^e}\right] &= E_t\left[\frac{Y_{t+1}Q^e + d_{t+1}}{Y_t Q^e}\right] \\
&= E_t\left[\frac{Y_{t+1}Q^e + \delta Y_{t+1}}{Y_t Q^e}\right] \\
&= E_t\left[\frac{Y_{t+1}}{Y_t}\frac{Q^e + \delta}{Q^e}\right] \\
&= \frac{(1-\delta)\beta + \delta}{(1-\delta)\beta} E_t\left[z_{t+1}\right] \\
&= \left[1 + \frac{\delta}{(1-\delta)\beta}\right] E[z]
\end{aligned}
$$

iii) specializing (20) in $u(c) = \log c$, and using that $w_t = (1-\delta) Y_t$, $q_t^e = Q^e Y_t$, $q_t^b = Q^b$, $d_{t+1} = \delta Y_t$, and the process for income,

$$
\begin{aligned}
(1-\beta) \frac{q_t^b}{w_t - q_t^e} &= \beta E_t \left[\frac{1}{q_{t+1}^e + d_{t+1}} \right] \\
(1-\beta) \frac{Q^b}{(1-\delta) Y_t - Y_t Q^e} &= \beta E_t \left[\frac{1}{Y_{t+1} Q^e + \delta Y_{t+1}} \right] \\
\frac{Q^b}{(1-\delta) - Q^e} &= \frac{\beta}{1-\beta} E_t \left[\frac{Y_t}{Y_{t+1}} \frac{1}{Q^e + \delta} \right] \\
\frac{Q^b}{(1-\delta) - (1-\delta)\beta} &= \frac{\beta}{1-\beta} E_t \left[\frac{Y_t}{Y_{t+1}} \frac{1}{(1-\delta)\beta + \delta} \right] \\
\frac{Q^b}{(1-\delta)(1-\beta)} &= \frac{\beta}{1-\beta} E_t \left[\frac{Y_t}{Y_{t+1}} \frac{1}{(1-\delta)\beta + \delta} \right] \\
Q^b &= \frac{(1-\delta)\beta}{(1-\delta)\beta + \delta} E_t \left[\frac{Y_t}{Y_{t+1}} \right] \\
Q^b &= \frac{(1-\delta)\beta}{(1-\delta)\beta + \delta} E_t \left[\frac{1}{z_{t+1}} \right]
\end{aligned}
$$

Thus,

$$
\frac{1}{Q^b} = \left[1 + \frac{\delta}{(1-\delta)\beta} \right] \frac{1}{E_t \left[\frac{1}{z_{t+1}} \right]} = \left[1 + \frac{\delta}{(1-\delta)\beta} \right] \frac{1}{E\left[\frac{1}{z} \right]}
$$

Summarizing:

$$
\begin{aligned}
Q^e &= (1-\delta)\beta, \\
E\left[R_{t+1}^e \right] &= \left[1 + \frac{\delta}{(1-\delta)\beta} \right] E[z], \\
R^b &= \left[1 + \frac{\delta}{(1-\delta)\beta} \right] \frac{1}{E[1/z]}.
\end{aligned}
$$

Question 7. [10 points] To double check your answer to Question 6.i), consider the following alternative reasoning, which uses that with log utility the share of expenditure is proportional to the share parameter in preferences. i) Using that the total wealth of a young agent is given by w_t, express the value of a young agent's consumption c_t^t as a function of β, δ and Y_t. ii) Assuming that $b_t = 0$ and $s_t = 1$ use the answer of i) and the budget constraint of the young agents to derive and expression for Q^e. Compare this expression with your answer to i) in the previous question. iii) Give a short explanation of why Q^e is increasing in β and decreasing in δ (maximum two lines for each parameter).

Answer Q 7.

i)

$$c_t^t = (1-\beta)\, w_t = (1-\beta)\,(1-\delta)\, Y_t$$

ii)

$$c_t^t = w_t - q_t^e = (1-\delta)\, Y_t - Q^e Y_t$$

thus

$$(1-\beta)\,(1-\delta)\, Y_t = (1-\delta)\, Y_t - Q^e Y_t$$

or

$$Q^e = (1-\delta)\,\beta$$

iii). For higher β agents care more about the future, so the tree is more valuable. For higher δ the young agents has smaller income to buy the trees, and with log preferences young agents dedicate a constant fraction of their income. Since the trees are inelastically supplied by old agents, their price Q^e must be smaller.

Assume from now on that $x = \log z$ is normally distributed with expected value μ and variance σ^2. Recall that

$$Ee^x = e^{\mu+(1/2)\sigma^2}$$

Let g be the expected continuously compounded growth rate of Y_t, so $g = \mu + (1/2)\,\sigma^2$.

Question 8. [15 points] Under the log normal assumption for z find: i) an expression for the expected return of the trees as a function of $\beta,\ \delta,\ g$ and σ^2, ii) an expression for the level of the gross return on the discount bond as a function of $\beta,\ \delta,\ g$ and σ^2, and iii) an expression for the multiplicative risk premium, i.e. the ratio of the gross expected return on the tree and the gross return on the discount bond as a function of only one of the parameters of the model $(\beta, \delta, g$ and $\sigma^2)$.

Hint: To compute $E\,(1/z)$ use the formula displayed above for $(1/z) = e^{-x}$.

Answer to Q 8.

i)

$$\begin{aligned} R^e &= \left[1 + \frac{\delta}{(1-\delta)\,\beta}\right] E_t\,[z_{t+1}] \\ &= \left[1 + \frac{\delta}{(1-\delta)\,\beta}\right] e^{\mu+(1/2)\sigma^2} \\ &= \left[1 + \frac{\delta}{(1-\delta)\,\beta}\right] e^{g-(1/2)\sigma^2+(1/2)\sigma^2} \\ &= \left[1 + \frac{\delta}{(1-\delta)\,\beta}\right]\ e^{g} \end{aligned}$$

ii)

$$\begin{aligned} R^b &= \left[1+\frac{\delta}{(1-\delta)\beta}\right]/E_t\left[\frac{1}{z_{t+1}}\right] \\ &= \left[1+\frac{\delta}{(1-\delta)\beta}\right]\frac{1}{e^{-\mu+(1/2)\sigma^2}} \\ &= \left[1+\frac{\delta}{(1-\delta)\beta}\right] e^{\mu-(1/2)\sigma^2} \\ &= \left[1+\frac{\delta}{(1-\delta)\beta}\right] e^{g-(1/2)\sigma^2-(1/2)\sigma^2} \\ &= \left[1+\frac{\delta}{(1-\delta)\beta}\right] e^{g-\sigma^2} \end{aligned}$$

iii)

$$\begin{aligned} \frac{E\left[R^e_{t+1}\right]}{R^b} &= \frac{\left[1+\frac{\delta}{(1-\delta)\beta}\right]E_t\left[z_{t+1}\right]}{\left[1+\frac{\delta}{(1-\delta)\beta}\right]/E_t\left[\frac{1}{z_{t+1}}\right]} \\ &= E_t\left[z_{t+1}\right]E_t\left[\frac{1}{z_{t+1}}\right] = e^{\mu+(1/2)\sigma^2}e^{-\mu+(1/2)\sigma^2} \\ &= e^{\sigma^2} \end{aligned}$$

Question 9 . [10 points] Is the equilibrium value of aggregate consumption -using the corresponding Arrow-Debreu prices- finite?

Hint. To answer this, write down the present value of aggregate consumption as a function of the price of a tree. Notice that by paying the price of the tree an agent can obtain the right to the fruits d_t at all dates $t \geq 2$. Recall that feasibility can be written as $c_t^t + c_t^{t-1} = Y_t$, and that we assume that $d_t = \delta Y_t$.

Answer to Q 9.

Suggested Solution.

The present value of aggregate consumption is given by

$$\sum_{t=1}^{\infty} p_t\left(c_t^t + c_t^{t-1}\right)$$

where p_t denote the Arrow-Debreu price at time zero associated with consumption in period t. Market clearing, together with the definition of the dividend process d_t implies

$$\sum_{t=1}^{\infty} p_t\left(c_t^t + c_t^{t-1}\right) = \sum_{t=1}^{\infty} p_t\frac{d_t}{\delta} = \frac{1}{\delta}\sum_{t=1}^{\infty} p_t d_t$$

Note that $\sum_{t=0}^{\infty} p_t d_t$ is the discounted present value of a claim that pays d_t in every period. No arbitrage suggests that the owner of this claim (or tree) should be indifferent between keeping the asset and selling it. It follows that

$$\sum_{t=1}^{\infty} p_t d_t = Y_1 + q_1^e$$

where Y_1 are the direct proceipts from owning the claim and q_1^e are the proceipts of selling it. Finally,

$$\sum_{t=1}^{\infty} p_t \left(c_t^t + c_t^{t-1}\right) = \frac{Y_1 + q_1^e}{\delta} = \frac{1}{\delta}\left[Y_1 + Q^e Y_1\right] = \frac{Y_1}{\delta}\left[1 + (1-\delta)\beta\right] < \infty$$

Question 10 . [5 points] Show that for σ^2 large enough, $R^b < e^g$, so that net interest rates are smaller than the expected growth rate of fruits, and hence of GDP.

Answer to Q 10.

$$\begin{aligned} R^b / \, e^g &= \left[1 + \frac{\delta}{(1-\delta)\beta}\right] e^{g-\sigma^2}/e^g \\ &= \left[1 + \frac{\delta}{(1-\delta)\beta}\right] e^{-\sigma^2} \end{aligned}$$

so that for σ^2 large enough $R^b < e^g$.

Question 11. [20 points] Given your answer to the previous questions, i) is the competitive allocation Pareto Optimal?, ii) Can the introduction of social security improve the welfare of all generations?, iii) Does your answer depends on the value of interest rates being smaller, equal or higher than the (expected) growth rate of the economy?, iv) Recall that in the last 100 years the average for the US economy the real net return on US government bonds (the risk-free rate) is about 3% per year, the average net growth rate of total GDP is about 3%, and the difference between the net real return of a well diversified equity portfolio such as the SP500 and the net risk-free rate is about 5% per year. Using this simple model as a guide, what you conclude about the existence (or lack of) dynamic inefficiency in the US economy?

Answer Q 11.

i) The first welfare theorem applies, so the allocation of the corresponding competitive equilibrium is Pareto Optimal.

ii) By definition of Pareto Optimality, no policy can improve upon it.

iii) The value of the interest rate is not relevant for the case with uncertainty. It is the value of aggregate consumption, which in this case is risky.

iv) From the previous answer it follows that it is not whether the risk-free interest rate is close to the expected growth rate of the economy what is important to determine the possibility of dynamic inefficiency. It is whether the expected return of an asset with dividends equal to the aggregate consumption is higher than the expected growth rate of consumption. Based on the given values for the returns and growth rate, the US economy shows no sign of dynamic inefficiency.