Theory of Income, Fall 2007

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Class Note 3:

Aggregation, Social Planner Problems and CE

In the first part of this note we will show that under some regularity conditions, any PO allocation can be obtained by maximizing an object that can be interpreted as the utility function of a representative agent. This object is a weighted sum of the utility functions of all the households in the economy where the weights are non-negative.

In the second part of the note we will concentrate on a pure exchange economy (i.e. an economy without production) with differentiable utility functions. We will analyze PO as well as CE allocations and we will prove the welfare theorems in a different way (that uses heavily the differentiability assumption). Finally, we will discuss an aggregation result. In general, in any CE the equilibrium prices and allocation depend on the distribution of wealth. If preferences are homothetic and endowments are proportional, we show that this is not the case: prices will be independent of the distribution of wealth.

Representative Agent

Definition. The utility possibility set U is defined as the set of utilities that are achievable for a feasible allocation, i.e.

$$U = \left\{u \in R^I : u_i \leq u^i\left(x^i\right) \text{ all } i \in I \right.$$
$$\left. \text{for some feasible allocation } \left\{x^i, y^j\right\}\right\}$$

Question. Show that if the aggregate possibility set Y and all the consumption possibility sets X^i are convex, then the set of feasible allocations is convex.

Assumption CC. u^i are concave.

Question. Is concavity of u^i a cardinal or an ordinal property of u^i? (recall that an ordinal property is not altered if u^i is replaced by a strictly increasing transformation, and hence only ordinal properties of u^i determine the characteristics of the demand functions).

Question. Show that if the convexity assumptions HH and FF are satisfied and if furthermore, all u^i are concave (assumption CC), then U is a convex set.

Theorem. Assume that u^i are strictly increasing, and that assumptions HH, CC and FF are satisfied, then $\{\bar{x}^i, \bar{y}^i\}$ is a PO allocation if and only if there is a vector $\lambda \in R_+^I$ such that $\{\bar{x}^i, \bar{y}^j\}$ solves the problem W:

$$W: \max_{\{x^i, y^i\}} \sum_{i \in I} \lambda_i u^i \left(x^i\right)$$

subject to $\{x^i, y^j\}$ being a feasible allocation.

Proof.

Sufficiency. If $\{\bar{x}^i, \bar{y}^j\}$ solves problem W, then it must be a PO allocation. To see this, notice that, by contradiction, if it were not PO, there must be a feasible allocation $\{\hat{x}^i, \hat{y}^j\}$ that will Pareto dominate $\{\bar{x}^i, \bar{y}^j\}$. If the strict inequality in the utility correspond to those agent with strictly positive weights λ_i, then

$$\sum_{i \in I} \lambda_i u^i \left(\bar{x}^i\right) < \sum_{i \in I} \lambda_i u^i \left(\hat{x}^i\right)$$

contradicting that $\{\bar{x}^i, \bar{y}^j\}$ solves problem W. Thus, it must be the case that

$$u^i \left(\bar{x}^i\right) < u^i \left(\hat{x}^i\right)$$

for those agents with $\lambda_i = 0$ and

$$u^i \left(\bar{x}^i\right) = u^i \left(\hat{x}^i\right)$$

for those with $\lambda_i > 0$. But in this case, consider a feasible allocation $\{\tilde{x}^i\}$ with $\tilde{x}^i = \hat{x}^i + \varepsilon$ for those with $\lambda_i > 0$ and $\tilde{x}^i = x^i - \delta$ for those with $\lambda_i = 0$, where ε and δ are positive vectors. Since u^i are strictly increasing,

$$\sum_{i \in I} \lambda_i u^i\left(\bar{x}^i\right) < \sum_{i \in I} \lambda_i u^i\left(\tilde{x}^i\right)$$

contradicting that $\{\bar{x}^i, \bar{y}^j\}$ solves problem W.

Necessity. If $\{\bar{x}^i, \bar{y}^j\}$ is a PO allocation, then it must solve problem W for some weights λ. To see this, define the set

$$A = \left\{\bar{u} \in R^I : \bar{u}_i = u^i\left(\bar{x}^i\right), \text{ all } i \in I\right\}$$

$$B = int\left(U\right)$$

so that A is a singleton, with the utility values of the PO allocation, and B has the interior of the utility possibility set. Since $\{\bar{x}^i, \bar{y}^j\}$ is a PO allocation, then

$$A \cap B = \emptyset$$

otherwise, if these sets were to have a common element, the resulting allocation will be feasible and Pareto dominate $\{\bar{x}^i\}$, a contradiction with the hypothesis that $\{\bar{x}^i, \bar{y}^j\}$ is a PO allocation.

Also notice that under assumptions HH, FF and CC, both A and B are convex, so by the hyperplane separation theorem there is a vector $\lambda^* \in R^I,\ \lambda^* \neq 0$, such that

$$\lambda^*\ \bar{u} \geq \lambda^*\ u \text{ for all } u \in int\,(U) \equiv B.$$

Thus, by taking limits to all the elements in the closure of B, i.e. U, we have

$$\lambda^*\ \bar{u} \geq \lambda^*\ u \text{ for all } u \in U \tag{0.1}$$

so that $\{\bar{x}^i, \bar{y}^j\}$ solves problem W. It only rests to show that we can chose the weights to be non-negative.

Denote the set of agents for which $\lambda_i^* < 0$ by I^-. Thus, given the inequality above (0.1), the utility of the agents in the set I^- is being minimized. Now consider the set of weights, $\lambda_i = \max\{0, \lambda_i^*\}$ for all i. Since by assumption, u^i are strictly increasing, then

$$\lambda\ \bar{u} \geq \lambda\ u \text{ for all } u \in U$$

Thus, $\{\bar{x}^i, \bar{y}^j\}$ solves problem W for non-negative weights λ. QED

Question. Draw a picture of a convex utility possibility set U for $I = 2$ with strictly increasing preferences. Pick a particular point in its frontier, say $\bar{u}$. Show, graphically, how to locate a vector λ such that the solution of problem W for such λ is $\bar{u}$.

Question. Draw a picture of a non-convex utility possibility set U for $I = 2$. Pick a particular point in its frontier, say $\bar{u}$. Draw U and pick $\bar{u}$ in such a way that $\bar{u}$ does not solve problem W, regardless of what λ is chosen. Can this happen if all u^i are not convex, but they are strictly quasi-concave?

Representative Agent

By virtue of the previous theorem, as well as of the Welfare theorems, many times we will analyze an economy with only one agent. This representative agent has utility u defined as

$$u(x) = \max_{x^i \in X^i} \sum_{i \in I} \lambda_i u^i \left(x^i\right)$$

subject to

$$\sum_{i \in I} x^i = x \ .$$

Clearly, this utility function depends on the weights λ.

By the 1st welfare theorem, we know that the CE allocations are PO, and hence we can find weights λ for which an economy with one representative agent with utility function u corresponds to the one with many agents. Analogously, a PO allocation in an economy with many agents has a corresponding CE for some ownership structure. This CE also corresponds to the one for an economy with one representative agent for some weight λ.

Since the definition of the representative agent does not involve the production side of the economy, i.e. it does not involve the production possibility set Y^j, then next we will study pure exchange economies, those without production.

Pure Exchange Economy

We will consider the case of a pure exchange economy with smooth (differentiable) concave utility functions. This will give a precise interpretation to the weights λ and connect the welfare theorems with the utility of the representative agent.

Definition. An exchange economy is one where the aggregate production possibility set is given by the negative ortant, so that agent can dispose of goods, but no goods are produced. Feasibility is then given by

$$\sum_{i \in I} x^i \leq \sum_{i \in I} e^i \equiv \bar{e}, \tag{0.2}$$

and $x^i \in X^i$ for all i, where we denote the vector $\bar{e} \in R_+^m \subset L$ as the aggregate endowment.

Under the assumptions HH, CC, the necessary and sufficient first order conditions for problem W are:

$$\lambda_i \frac{\partial u^i\left(\bar{x}^i\right)}{\partial x_l} = \gamma_l \tag{0.3}$$

for goods $l = 1, 2, ..., m$ where $\gamma_l \geq 0$ are the Lagrange multipliers of the feasibility constraints (0.2).

Notice that this implies that the marginal rate of substitutions are the same for all agents:

$$\frac{\partial u^i(\bar{x}^i)}{\partial x_l} / \frac{\partial u^i(\bar{x}^i)}{\partial x_k} = \frac{\partial u^j(\bar{x}^j)}{\partial x_l} / \frac{\partial u^j(\bar{x}^j)}{\partial x_k}$$

for all goods l and k and agents i and j. Notice that this expression does not contain the $\lambda-$ weights. The $\lambda-$ weights determine the consumption across agents:

$$\frac{\partial u^i(\bar{x}^i)}{\partial x_l} / \frac{\partial u^j(\bar{x}^j)}{\partial x_l} = \frac{\lambda_j}{\lambda_i}$$

for all goods l and agents i and j.

Since these conditions are necessary and sufficient, they characterize the set of PO optimal allocations. Furthermore, the vector γ has the interpretation of the marginal value of an extra unit of the aggregate endowment. As it is obvious, the solution of problem W, as well as the vector γ, are a function of λ. Thus, the marginal value of an extra unit of the aggregate endowment depends on the weights λ.

The maximization of the households problem is the key part of a competitive equilibrium for the pure endowment case. When the assumptions HH and CC hold, this maximization problem is characterized by the following necessary and sufficient conditions:

$$\frac{\partial u^i\left(\bar{x}^i\right)}{\partial x_l} = \mu_i \, p_l \tag{0.4}$$

where $\mu_i \geq 0$ are the Lagrange multipliers of the budget constraint

$$p \, \bar{x}^i \leq p \, e^i$$

>From the envelope theorem we know that μ_i measures the change in maximized agent's utility if the agent's income is increased by one unit of the numeraire good, so μ_i is the marginal utility of income.

Notice that these first order conditions imply that the marginal rate of substitution is equated across agents, i.e.

$$\frac{\partial u^i\left(\bar{x}^i\right)}{\partial x_l} \Big/ \frac{\partial u^i\left(\bar{x}^i\right)}{\partial x_k} = \frac{\partial u^j\left(\bar{x}^j\right)}{\partial x_l} \Big/ \frac{\partial u^j\left(\bar{x}^j\right)}{\partial x_k}$$

for all goods l and k and agents i and j. Notice that the multipliers μ_i do not enter in this expression. The relative consumption of agents i and j depend on the multipliers, as

$$\frac{\partial u^i\left(\bar{x}^i\right)}{\partial x_l} \Big/ \frac{\partial u^j\left(\bar{x}^j\right)}{\partial x_l} = \mu_i \, / \, \mu_j$$

for all goods l and all agents i and j.

As the welfare theorems show in the general case, there is a close connection between CE and PO allocations.

Welfare Theorems, λ weights and marginal utility of income.

We first analyze the analogous to the first welfare theorem. Given a CE $\{\bar{x}^i, p\}$, the household problem satisfies the necessary first order conditions stated using the multiplier μ_i and prices p. We will use them to find the $\lambda-$weights and Lagrange multipliers γ of the corresponding planning problem with solution $\{\bar{x}^i\}$. To do this, multiply the foc from the households problem (0.4) of good l by $1 \ / \ \mu_i$

$$\frac{1}{\mu_i} \frac{\partial u^i\left(\bar{x}^i\right)}{\partial x_l} = p_l$$

This is identical to the foc (0.3) for the planner problem with weights given

$$\lambda_i = \frac{1}{\mu_i}$$

and with Lagrange multiplier for good l equal to the price

$$\gamma_l = p_l \ .$$

Since the first order conditions for the planning problem are sufficient, then the

CE $\{\bar{x}^i\}$ is indeed a PO allocation.

Income distribution and $\lambda-$weights

Thus, agents with low marginal utility of their income, i.e. those consuming a lot given the concavity of u^i, are those that are assigned high λ weights. Likewise, goods with high prices p_l are those whose marginal social value is high, i.e. those with high γ_l multiplier.

We now analyze the analogous to the second welfare theorem. Let's start with a PO allocation $\{\bar{x}^i\}$, or equivalently, with an allocation and Lagrange multipliers γ satisfying the necessary first order conditions (0.3) of the problem W for some weights λ. We must now show that the foc for the household problem hold for some choice of μ_i and p. We set prices $p = \gamma$ and marginal utility of income μ_i equal to $\mu_i = 1/\lambda_i$. Then using the necessary conditions for the planning problem ((0.3), we find that

$$\frac{1}{\mu_i}\frac{\partial u^i\left(\bar{x}^i\right)}{\partial x_l} = p_l$$

for all goods l and all agents i. We set the endowments $e^i = \bar{x}^i$ for all i, thus $\bar{x}^i$ is budget feasible for each agent i. Since these first order conditions are sufficient for the household problem, and the allocation $\bar{x}^i$ is budget feasible, then the agent maximize utility by choosing $\bar{x}^i$. Since the allocation is feasible, we have shown that indeed with these prices and endowment it constitutes a competitive equilibrium.

Notice that those agents with high λ_i weight, are agents with low marginal utility of income μ_i, i.e. those with high consumption.

Prices and Income Distribution

In general in a CE, the equilibrium price p and the equilibrium allocation $\{\bar{x}^i\}$ depend on the income distribution measured by the vector of individual endowments $\{e^i\}$. Likewise, in general the shadow value of the social endowment γ and the PO allocation $\{\bar{x}^i\}$ depend on the λ weights.

Aggregation.

For particular utility functions and endowments -or λ-weights- the equilibrium prices p - or the shadow value of the social endowment γ- do not depend on the distribution of the endowment $\{e^i\}$ -or on the distribution of the Pareto weights λ. Consider, for example, the case where the endowments are proportional, and the utility functions u^i are homothetic. In particular all the utility functions are given by

$$u^i(x) = g^i(\ h(x)\ )$$

where g^i is an strictly increasing concave function and where h is homogenous of degree one and independent of i,

$$h(\eta\, x) = \eta\, h(x)$$

for any scalar $\eta > 0$ and for any x. The endowments are proportional: there are some positive fractions $\delta^i > 0$, such that $e^i = \delta^i\ \bar{e}$ for all i, and where $\bar{e}$ is the vector of the aggregate endowment. Notice that in this case the CE allocation (and the PO allocation) are equal to their endowment, in particular

$$\bar{x}^i = \delta^i\ \bar{e}$$

and the equilibrium prices are given by

$$p_l = \kappa\, \frac{\partial h(\bar{e})}{\partial x_l} \tag{0.5}$$

for all goods $l = 1, 2, ..., .m$, and where $\kappa > 0$ is an arbitrary positive constant (notice that this expression is evaluated on the aggregate endowment $\bar{e}$). To check this, we use the properties of the utility function in the foc for the agent (0.4),

$$\frac{\partial g^i\left(h\left(\delta^i\ \bar{e}\right)\right)}{\partial h}\,\frac{\partial h\left(\delta^i\ \bar{e}\right)}{\partial x_l} = \mu_i\ p_l$$

using the expression of equilibrium price (0.5),

$$\frac{\partial g^i\left(h\left(\delta^i\,\bar{e}\right)\right)}{\partial h}\;\frac{\partial h\left(\delta^i\,\bar{e}\right)}{\partial x_l}=\mu_i\;\kappa\;\frac{\partial h\left(\bar{e}\right)}{\partial x_l}$$

and since h is homogeneous of degree one[1],

$$\frac{\partial h\left(\delta^i\,\bar{e}\right)}{\partial x_l}=\frac{\partial h\left(\bar{e}\right)}{\partial x_l}.$$

Thus, letting μ_i

$$\mu_i=\frac{\partial g^i\left(h\left(\delta^i\,\bar{e}\right)\right)}{\partial h}\;/\;\kappa$$

for all i, the foc for (0.5) are satisfied for the proposed prices and allocations.

Likewise, the social shadow value of the aggregate endowment in this case are

$$\gamma_l=\kappa\;\frac{\partial h\left(\bar{e}\right)}{\partial x_l}$$

for all goods l and for some arbitrary positive constant $\kappa>0$. The corresponding λ weights are

$$\lambda_i=\kappa\;/\;\frac{\partial g^i\left(h\left(\delta^i\,\bar{e}\right)\right)}{\partial h}$$

[1] Recall that if a function is homogeneous of degree one, all its partial derivatives are homogeneous of degree zero.

for all $i \in I$.

Thus, the equilibrium prices p are independent of the distribution of the endowments parameterized by δ^i, and the shadow value of the endowment γ is independent of the weights λ. Instead the weights λ of the PO allocation that corresponds to the CE allocation depend on the income distribution given by δ. Notice that the larger the δ^i is, then, the larger λ_i is (as can be seen since $\partial g / \partial h$ is decreasing, and h increasing).

Remark: the proportionality of the endowments is not necessary. Indeed, the same aggregation result can be shown to be valid even for arbitrary endowment patterns.