Integrals of a log-normal

Assume that the stock log-return during period T is normally distributed with mean μT and variance $T\sigma^2$:

$$\log \frac{S_T}{S_0} = N\left(\mu T, \sigma^2 T\right)$$

Denote

$$F(x; \mu, \sigma) = \Pr\left\{ z \le x \ ; \ z \sim N\left(\mu, \sigma^2\right) \ \right\}$$

We'll show that

$$E\left[S_T \mid S_T \ge K\right] = S_0 e^{\mu T - \frac{1}{2} T \sigma^2} F\left(\log K; T\mu, T\sigma^2\right)$$

Proof. Without loss of generality assume that $T = 1$. By definition

$$E\left[S \ge a\right] = \frac{1}{\sqrt{2\pi\sigma^2}} \int_{\log a}^{\infty} e^{\log S} \ e^{-\frac{1}{2}\left(\frac{\log S - \mu}{\sigma^2}\right)^2} d \log S$$

where we used the fact that

$$e^{\log S} = S$$

and that the density of a normal r.v. with mean and variance $\left(\mu, \sigma^2\right)$ is

$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$$

Let $\log S = x$

$$E\left[S \ge a\right] = \frac{1}{\sqrt{2\pi \ \sigma^2}} \int_{\log a}^{\infty} e^{x} \ e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx$$

developing the square

$$E\left[S \ge a\right] = \frac{1}{\sqrt{2\pi \ \sigma^2}} \int_{\log a}^{\infty} e^{x} \ e^{-\frac{1}{2}\frac{x^2 - 2x\mu + \mu^2}{\sigma^2}} dx$$

collecting terms in x

$$E\left[S \ge a\right] = \frac{1}{\sqrt{2\pi \ \sigma^2}} \int_{\log a}^{\infty} e^{-\frac{1}{2}\frac{x^2 - 2x\mu - 2x\sigma^2 + \mu^2}{\sigma^2}} dx$$

completing the square

$$\begin{aligned} E\left[S \ge a\right] &= \frac{1}{\sqrt{2\pi \ \sigma^2}} \int_{\log a}^{\infty} e^{-\frac{1}{2}\frac{\mu^2}{\sigma^2}} e^{\frac{1}{2}\frac{\left(\mu+\sigma^2\right)^2}{\sigma^2}} e^{-\frac{1}{2}\frac{x^2 - 2x\left(\mu+\sigma^2\right) + \left(\mu+\sigma^2\right)^2}{\sigma^2}} dx \\ &= e^{-\frac{1}{2}\frac{\mu^2 - \left(\mu+\sigma^2\right)^2}{\sigma^2}} \frac{1}{\sqrt{2\pi \ \sigma^2}} \int_{\log a}^{\infty} e^{-\frac{1}{2}\frac{x^2 - 2x\left(\mu+\sigma^2\right) + \left(\mu+\sigma^2\right)^2}{\sigma^2}} dx \end{aligned}$$

developing the square

$$E\left[S \geq a\right] = e^{-\frac{1}{2}\frac{\mu^2-\mu^2-2\mu\sigma^2-\sigma^4}{\sigma^2}} \frac{1}{\sqrt{2\pi\ \sigma^2}} \int_{\log a}^{\infty} e^{-\frac{1}{2}\frac{\left(x-\left(\mu+\sigma^2\right)\right)^2}{\sigma^2}} dx$$

simplifying

$$E\left[S \geq a\right] = e^{\mu+\frac{1}{2}\sigma^2} \frac{1}{\sqrt{2\pi\ \sigma^2}} \int_{\log a}^{\infty} e^{-\frac{1}{2}\frac{\left(x-\left(\mu+\sigma^2\right)\right)^2}{\sigma^2}} dx$$

and using the fact that

$$1 - F\left(\log a; \left(\mu+\sigma^2\right); \sigma^2\right) = \frac{1}{\sqrt{2\pi\ \sigma^2}} \int_{\log a}^{\infty} e^{-\frac{1}{2}\frac{\left(x-\left(\mu+\sigma^2\right)\right)^2}{\sigma^2}} dx =$$

then

$$E\left[S \geq a\right] = e^{\mu+\frac{1}{2}\sigma^2}\left(1-N\right)$$