Economics 251 — Midterm

April 28, 2008

Solution

1. **Forward rates.** Suppose there are three dates: T_0, T_1, T_2, and the corresponding zero rates are R_1 and R_2. We want to find the interest rate we can lock at T_0 for an investment between T_1 and T_2. Suppose you buy at T_0 1 bond maturing at T_2. In order to have no cash flow at T_0, you decide to short (sell) at T_0 $\mathbf{x}$ bonds maturing at T_1. `For this question assume continuous compounding.`

   (a) What is the value of $\mathbf{x}$?

   Assume a notional principal of 100. The price at T_0 of the bond maturing at T_2 is

   $$P_2 = 100e^{-R_2(T_2-T_0)}$$

   The price at T_0 of the bond maturing at T_1 is

   $$P_1 = 100e^{-R_1(T_1-T_0)}$$

   You buy 1 bond maturing at T_2, thus you pay P_2. If you short x bonds maturing at T_1, you receive xP_1. Therefore, in order to have no cash flow at T_0 it must be the case that

   $$P_2 = xP_1$$

   or,

   $$100e^{-R_2(T_2-T_0)} = x100e^{-R_1(T_1-T_0)}$$
   $$\Longrightarrow x = e^{[R_1(T_1-T_0)-R_2(T_2-T_0)]}$$

   (b) What is the cash flow at T_1?

   Since you shorted $e^{[R_1(T_1-T_0)-R_2(T_2-T_0)]}$ bonds maturing at T_1, your cash flow at T_1 is

   $$CF_1 = -100e^{[R_1(T_1-T_0)-R_2(T_2-T_0)]}$$

   (c) What is the cash flow at T_2?

   Since you bought 1 bond maturing at T_2, your cash flow at T_2 is

   $$CF_2 = 100$$

(d) What is the forward rate? Write down an expression for the forward rate as a function or R_1, R_2, T_1 and T_2.

The forward rate is the future zero rate implied by today's term structure of interest rates.

Suppose in T_1 *you borrow* CF_1 *at the forward rate* $F_{1,2}$*. Then, at* T_2 *your cash flow is*

$$100 - 100e^{[R_1(T_1-T_0)-R_2(T_2-T_0)]}e^{F_{1,2}(T_2-T_1)}$$

Since your cash flow in T_0 *and* T_1 *is zero, in order to avoid arbitrage opportunities, the cash flow in* T_2 *must also be zero. Therefore,*

$$100 - 100e^{[R_1(T_1-T_0)-R_2(T_2-T_0)]}e^{F_{1,2}(T_2-T_1)} = 0$$

$$1 = e^{[R_1(T_1-T_0)-R_2(T_2-T_0)]+F_{1,2}(T_2-T_1)}$$

Taking logs in both sides, and solving for $F_{1,2}$ *we get,*

$$F_{1,2} = \frac{R_2\,(T_2 - T_0) - R_1\,(T_1 - T_0)}{(T_2 - T_1)}$$

2. **Futures vs. Forwards**. Denote S_2 the price of oil at date 2; we consider trading futures or forwards on oil at date 0 (and date 1 as well sometimes). For futures, the *net* interest rate that will apply to our margin account is r_0 between dates 0 and 1, and r_1 between dates 1 and 2. We will consider both the case where r_1 is known and the one where r_1 is *not known* at date 0, and only known at date 1. We will use $F_{i,2}$ for future prices and $f_{i,2}$ for forward prices where $i = 0, 1$ are the dates at which the prices are known for contracts with delivery at date 2. `(For this problem interest is compounded at each date, not continuously.)`

   (a) First consider **forwards**: at date 0, we enter into 1 forward contract on oil for date 2, the forward price being $f_{0,2}$. At date 1, the forward price is $f_{1,2}$. We also short one forward at date 1. Then at date 2 we have to honor our long position in the date 0 forward, and our short position in the date 1 forward: what is our cash-flow c_2 at date 2?

   We make $(S_2 - f_{0,2})$ on the forward we were long, and $(f_{1,2} - S_2)$ on the forward we shorted. Our cash-flow at date 2 is therefore

$$c_2 = f_{1,2} - f_{0,2}$$

   (b) Now for **futures**, margin accounts achieve something similar, by design. At date 0, we enter into $h_0 > 0$ futures contracts on oil for date 2, the futures price being $F_{0,2}$ (notice the difference, capital letters for futures, lower-case for forwards). At date 1, the price on each of these futures contracts is $F_{1,2}$. Therefore our margin account shows a balance of $h_0 (F_{1,2} - F_{0,2})$ at date 1. At date 1 we also short h_0 contracts with delivery at date 2. Given r_1, the net interest rate at which cash flows in the margin account can be rolled over, what is the value of the payoff C_2 at date 2? If the interest rate r_1 was known at date 0, how can we choose h_0 so that the payoff at C_2 be equal to $F_{1,2} - F_{0,2}$?

   Our cash flow in T_2 is,

$$\begin{aligned} C_2 &= h_0 (S_2 - F_{1,2}) + h_0 (F_{1,2} - F_{0,2}) (1 + r_1) + h_0 (F_{1,2} - S_2) \\ &= h_0 (F_{1,2} - F_{0,2}) (1 + r_1) \end{aligned}$$

   If r_1 is known, in order to have $C_2 = F_{1,2} - F_{0,2}$ it must be the case that

$$h_0 = \frac{1}{1 + r_1}$$

(c) Suppose that, in addition to our portfolio with h_0 long contracts entered at time $t = 0$, we enter into h_1 additional long futures at date 1, with price $F_{1,2}$ – but we do NOT go short on h_0 contracts at date 1, as was assumed in part (b). What would be our total payoff at date 2 of all the contracts (i.e. h_0 long entered at date 0, with cash flow from the margin account rolled over at rate r_1, and h_1 long contracts at date 1)?

In this case,

$$C_2 = h_0 (S_2 - F_{1,2}) + h_0 (F_{1,2} - F_{0,2}) (1 + r_1) + h_1 (S_2 - F_{1,2})$$

(d) Using your answer to (c) how would you choose h_1 so that your cash-flow at date 2 does not depend on $F_{1,2}$? [your answer should make h_1 a function of r_1 and h_0]. With h_1 chosen in that manner, what is your final cash-flow?

We can rewrite C_2 as

$$C_2 = (h_0 + h_1) S_2 - [h_0 + h_1 - h_0 (1 + r_1)] F_{1,2} - h_0 F_{0,2} (1 + r_1)$$

Then, our cash flow at date 2 will not depend on $F_{1,2}$ if

$$h_0 + h_1 - h_0 (1 + r_1) = 0$$

or,

$$h_1 = r_1 h_0$$

which implies,

$$C_2 = h_0 (1 + r_1) (S_2 - F_{0,2})$$

(e) First assume that r_1 is known at date 0, so that h_0 can be chosen as a function of r_1 : set h_0 in such a way that the cash flow of the portfolio described in (c), and computed in (d), equals $S_2 - F_{0,2}$. Second, assume that r_1 is NOT known at date 0. Use your answer to (d) to explain why following the trading strategy described in (c), you can NOT create a cash flow equal to $S_2 - F_{0,2}$.

If r_1 is known, we set $h_0 = \frac{1}{1+r_1}$ and therefore

$$C_2 = S_2 - F_{0,2}$$

In general the cash flow computed in (d) gives the cash flow of $(1 + r_1) h_0$ contracts paying $[S_2 - F_{0,2}]$ each of them. Since r_1 is NOT known in advance, we cannot choose h_0 to obtain the cash flow of only one contract.

3. **Forwards on stocks with dividends**. Let's price a forward on the Dow Jones Industrial Average stock index (DJIA) for April 28, 2009, exactly 12 months from now: today is April 28, 2008, and the DJIA closed at 10,000 (we are rounding numbers to simplify the answer). You can borrow at 12% and save at 6%. In addition to this bid-ask spread on the interest rate, you also have to consider transaction costs: your broker requires \$5 per unit of the index he trades on your account, each time he shorts or buys the stock index for you. (For instance, if you short the index now and buy it back later, you pay \$5 now to short the index, and again \$5 to buy it back, in addition to the price of the index itself.) The dividend yield on the DJIA is assumed to be constant, equal to 2%. `You don't need to give any decimal for this question, but give explicit answers, i.e. actual figures. For your calculations you can use` $e^{-q}e^{0.06} \cong 1.0408$ `and` $e^{-q}e^{0.12} \cong 1.1052$ `or use the exact continuous compounding figures.`

   (a) Please find the lower bound on the price of the 12-months DJIA forward such that you can't exploit any arbitrage opportunity by being long this forward.

   Suppose first that the price of the forward is too low, so that you want to take a long position in this contract: you short e^{-q} units of the DJIA (which generates transaction costs $e^{-q} \times \$5$), save the proceeds (at the saving interest rate of 6%), and give back one unit of the DJIA to your broker (this includes the dividends which would have been continuously reinvested: your broker agreed to exchange e^{-q} now against 1 in 12 months) in April 2009. That is, you save $\$e^{-q}(10,000-5)$ *at 6% for 12 months, then you have to expend* $\$(\underline{F}_0+5)$ *in 12 months, to comply with the forward contract, and give back the stock index. This is an arbitrage opportunity if*

$$\begin{aligned} e^{-q}(10,000-5)\,e^{0.06} &\geq 5+\underline{F}_0 \\ e^{-q}e^{0.06}10,000 &\geq 5\left(1+e^{-q}e^{0.06}\right)+\underline{F}_0 \\ 1.0408\times 10,000 &\geq 5\,(1+1.0408)+\underline{F}_0 \\ 10,408-10 &\geq \underline{F}_0 \\ 10,398 &\geq \underline{F}_0 \end{aligned}$$

   (b) Please find the upper bound on the price of the 12-months DJIA forward such that you can't exploit any arbitrage opportunity by shorting this forward.

   Now suppose that the price of the forward is too high, so that you want to take a short position in this contract: you buy e^{-q} units of the DJIA and pay an additional Transaction Cost $\$(e^{-q}\times 5)$ *to your broker for executing the transaction, borrowing the total cost (at the* *borrowing interest rate of* 12%*). In 12 months, to comply with the forward contract, you sell the unit of the DJIA for* $\overline{F}_0$ *and reimburse the loan. This is an arbitrage opportunity if*

$$\begin{aligned}
e^{-q}\left(10,000+5\right)e^{0.12} &\leq \overline{F}_0 - 5 \\
e^{-q}e^{0.12}10,000 &\leq \overline{F}_0 - 5\left(1+e^{0.12}e^{-q}\right) \\
1.1052 \times 10,000 &\leq \overline{F}_0 - 5\left(1+1.1052\right) \\
11,052+11 &\leq \overline{F}_0 \\
11,063 &\leq \overline{F}_0
\end{aligned}$$

We find $\left[\underline{F}_0, \overline{F}_0\right] = [10,398\ ,\ 11,063]$. *This is a large band of admissible forward prices, reflecting large bid-ask spreads and large transaction costs.*

4. **Forward Rate Agreement (FRA).** Suppose you can trade zero-coupon bonds of maturity 2 and 2.5 years. The zero rates are such that the forward rate is 5%. You have been offered a FRA at a (fixed) 6% rate and reference (variable) rate equals to LIBOR (determined 2 year from now, to be applied for 6 months). Since the forward rate is different than the rate of the FRA, you know that there is an arbitrage opportunity and you would like to exploit it. Suppose that at $t = 2$ (year 2) you can lend or borrow at LIBOR for six months. Assume a notional amount of $1,000,000. In order to exploit the arbitrage opportunity:

(a) What position would you take in the zero-bond of maturity 2 at $t = 0$? (long, short, and by how much)

Long 1 bond.

(b) What position would you take in the zero-bond of maturity 2.5 at $t = 0$? Long, short, and by how much (Hint: your cash flow has to be equal to zero at $t = 0$).

Short x bonds, where

$$x = \frac{P_2}{P_{2.5}} = e^{R_{2.5}2.5 - R_2 2}$$

Note that,

$$0.05 = F_{2,2.5} = \frac{R_{2.5}2.5 - R_2 2}{2.5 - 2}$$

Then,

$$x = 1.025$$

(c) What woud you do at $t = 2$? (specify whether you will lend or borrow at LIBOR, and if so where will you get the principal to do so).

Receive 1,000,000 from the long position in the bond maturing in two years. Lend at LIBOR and enter in the FRA.

(d) According to your answers in (a),(b) and (c), what would be your cash flow at $t = 2.5$?

At $t = 2.5$:

$$\begin{aligned} C_{2.5} &= 1,000,000e^{0.06(2.5-2)} - 1,000,000x \\ &= 1,000,000\,(1.031 - 1.025) \\ &= 6,000 \end{aligned}$$

5. **Swaps 1.** You consider trading swaps. You can borrow at a variable interest rate of $(LIBOR + 0.4\%)$ and save at a variable interest rate of $(LIBOR - 0.6\%)$. You can also lock in a fixed interest rate by trading 5 years Treasury bonds: these bonds pay annual coupons of 5% for the bonds you can buy, and 6% coupons for the bonds you can issue. Both of these bonds trade at par. For instance, the 5% coupon bond that you can buy has price \$100 at date 0, pays \$5 at the end of each year 1, 2, 3, 4 and pays \$105 at the end of year 5. And you can sell the 6% coupon bond for \$100 at date 0, pay \$6 each year, and finally pay \$106 at the end of year 5. `For this question, compounding is annual, not continuous and payments are annual, not semi-annual.`

   (a) Suppose you meet a trader that offers you 5-year swaps where he pays LIBOR (at the same date as the coupons from the fixed rate bonds) and receives a fixed 5.5%? Can you find an arbitrage opportunity by dealing with this trader? [The answer of this part is either yes or no.] If there is an arbitrage opportunity, outline the relevant portfolio, with notional amount \$100. (For instance, this may include buying or selling the fixed coupond bond, and borrowing or lending at the variable libor interest rate, taking into account the spreads at several dates).

   Let's enter into the swap, with notional amount P. This means that we receive $P \times LIBOR$ each year and have to pay $P \times 5.5\%$. To pay the fixed rate to the trader, we buy a bond, whose 5% coupons we pay the trader (losing $P \times (5.5 - 5)\%$ on that variable part) and receive LIBOR. To buy the bond we borrowed P on the LIBOR, and therefore have to pay $P \times (LIBOR + 0.4\%)$ each date: we would lose $(5.5 - 5 + 0.4)\,\% = 0.9\%$ on such a portfolio. There's no arbitrage opportunity.

   (b) Now suppose you meet another trader that offers 5 year swaps where he pays a fixed 6.2% and receives LIBOR (at the same date as the coupons from the fixed rate bonds)? Can you find an arbitrage opportunity by dealing with this trader? [The answer of this part is either yes or no.] If there is an arbitrage opportunity, outline the relevant portfolio, with notional amount \$100. (For instance, this may include buying or selling the fixed coupond bond, and borrowing or lending at the variable libor interest rate taking into account the spreads at several dates).

   Let's enter into the swap, with notional amount P. This means that we receive $P \times 6.2\%$ each year and have to pay $P \times LIBOR$. By depositing P into a variable rate account, we obtain $P \times (LIBOR - 0.6\%)$ each year. To deposit this \$$P$ at date 0, we issue a note at 6%, so we have to repay $P \times 6\%$. In total, each period we pay $P \times (LIBOR + 6\%)$

and we receive $P \times (LIBOR - 0.6\% + 6.2\%)$, *hence we lose money (i.e., 0.4%). There's no arbitrage opportunity.*

6. **Swaps 2.** Suppose you can borrow or lend at a variable interest rate of LIBOR and that you can also trade a 4 years Treasury bonds that pay annual coupons of 6% at the end of each year. These bonds trade at par. In addition to this, suppose you meet a trader that offers you a 4-year swaps where he pays LIBOR (at the same date as the coupons from the fixed rate bonds) and receives a fixed 5%. After meeting the trader, you realize there is an arbitrage opprotunity. Assume a notional amount of $100. `For this question, compounding is annual, not continuous and payments are annual, not semi-annual.`

   (a) In order to exploit this arbitrage opportunity, what position would you take in the 4-years Treasury bonds?.

   Long 1 bond.

   (b) Would you borrow or lend at the variable interest rate?

   Borrow at LIBOR.

   (c) According to your answers in part (a) and (b), specify the cash flow for each year (i.e., specify for $t = 0, 1, 2, 3, 4$ how much you will have to pay and how much you will receive).

	Long position in the bond	*Borrow at LIBOR*	*SWAP*	*Total CF*
$t = 0$	-100	100	–	0
$t = 1$	6	$-100 * LIBOR$	$100 * LIBOR - 5$	1
$t = 2$	6	$-100 * LIBOR$	$100 * LIBOR - 5$	1
$t = 3$	6	$-100 * LIBOR$	$100 * LIBOR - 5$	1
$t = 4$	106	$-100 - 100 * LIBOR$	$100 * LIBOR - 5$	1