

# Persistent Liquidity Effects and Long Run Money Demand\*

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## **Abstract**

We present a monetary model in the presence of segmented asset markets that implies a persistent fall in interest rates after a once and for all increase in liquidity. The gradual propagation mechanism produced by our model is novel in the literature. We provide an analytical characterization of this mechanism, showing that the magnitude of the liquidity effect on impact, and its persistence, depend on the ratio of two parameters: the long-run interest rate elasticity of money demand and the intertemporal substitution elasticity. At the same time, the model has completely classical long-run predictions, featuring quantity theoretic and Fisherian properties. The model simultaneously explains the short-run “instability” of money demand estimates as-well-as the stability of long-run interest-elastic money demand.

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# 1 Summary and Introduction

This paper unifies two main views, or theories, on money demand. One is the transactions-based money demand that emerges in the models of e.g. Baumol-Tobin or Sidrausky. This theory predicts a stable downward sloping relationship between real balances and interest rates. This relationship is apparent in the low-frequency data, e.g. those describing decade to decade movements. The second theory is the so called “liquidity effect”, namely that a central bank’s purchase of bonds, which increases the amount of money, creates a transitory but persistent decrease in interest rates. These patterns are apparent in high-frequency data, such as those used in the VAR literature for the identification of monetary shocks. This paper presents a simple segmented asset market model that unifies both ideas, displaying a stable long-run money demand, and explaining the short-term instability in terms of the liquidity effect.

We develop the model building on the ideas proposed by Rotemberg (1984), Lucas (1990), Fuerst (1992), Christiano and Eichenbaum (1992), and others. To keep the analysis tractable the previous literature ignored the lingering distributional effects that follow an injection of liquidity.<sup>1</sup> Our model is simple enough to include these effects and yet to allow us to completely characterize its solution, including a full analysis of the dynamic transmission of monetary shocks, in terms of 3 primitive parameters: a measure of the segmentation of the bond markets, and two parameters related to the households’ intertemporal and intra-temporal elasticity of substitution between real balances and consumption. As long as only a fraction of agents has access to open market operations, i.e. the bond market is segmented, the model displays a persistent liquidity effect after a once and for all increase in the money supply for any value of the parameters. The basic mechanism for the persistent liquidity effect is the redistribution of wealth that is created by the operation. At the same time, and also novel in the segmented market models applied to monetary shocks, we show that if the changes in money growth are nearly permanent, the model delivers a standard money demand, with a positive relationship between velocity and interest rates. These three parameters dictate the long run elasticity of money demand, the strength and persistence of the liquidity effect, and the amplitude of the interest rates response to monetary shocks.

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<sup>1</sup>See for instance Lucas (1990): “In [2] and [13], (referring to Grossman and Weiss, and Rotemberg) an open market operation that induces a liquidity effect will also alter the distribution of wealth, since agents who participate in the trade will have different post-trade portfolios than those who were absent. These distributional effects linger on indefinitely (as they no doubt do in reality), a fact that vastly complicates the analysis, effectively limiting both papers to the study of a one-time, unanticipated bond issue in an otherwise deterministic setting. This paper studies this same liquidity effect using a simple device that abstracts from these distributional effects.” Fuerst (1992) uses the same assumption and comments about it: “This methodology is not without cost. By entirely eliminating these wealth effects, the model loses the persistent and lingering effects of a monetary injection captured, for example, in Grossman and Weiss (1983).”

The model results are useful to interpret several empirical facts about money demand. First, several authors have used high frequency data to document a persistent liquidity effect as well as a small interest elasticity after a monetary shock, see for example section 4.2.2 in [Christiano, Eichenbaum, and Evans \(1999\)](#) (see [Appendix D.1](#) for a review of the empirical literature). In [Section 5](#) we use our model to reproduce these stylized facts.

Second, combining the results for the dynamics of monetary expansions, inflation and interest rates provides a novel interpretation of the money demand data across low and high frequencies. Assuming that the money growth rate is driven by both permanent and transitory shocks our theory implies that the low-frequency changes in monetary growth rates (e.g. decade to decade) will generate a time series revealing an interest-elastic money demand such as the one estimated by [Lucas \(2000\)](#) and [Ireland \(2009\)](#) (see [Appendix D.1](#)). Instead the high-frequency changes of money growth will generate data-points that do not lie on this money demand curve because of the liquidity effect. This assumption allows us to reconcile low-frequency US data on interest rate and velocity, displaying an interest-elastic money demand, with the high-frequency data, featuring persistent deviations from it and very small (even positive) interest elasticities. Indeed the reconciliation between the high and low frequency fluctuations of money demand is one of the challenges discussed by [Lucas \(2000\)](#).<sup>2</sup> In [Section 6](#) we evaluate the predictions of the model quantitatively by simulating a process for the growth rate of M1 compounding both a low and a high frequency component, and comparing the model predictions with the US data over the period 1900-2006.

Third, the predictions concerning the effects of transitory and persistent monetary shocks are useful to interpret the evidence in [Sargent and Surico \(2011\)](#) and [Cogley, Sargent, and Surico \(2011\)](#). They estimate that the correlation between interest rates and money growth is smaller than the one between money growth and inflation, and also show that these correlations vary from decade to decade.<sup>3</sup> They also provide an interpretation of these facts in terms of changes in the systematic response of monetary policy. Our model provides a complementary explanation of why, due to the liquidity effect, the correlation between money growth and interest rates is systematically smaller than the correlation between money growth and inflation.

At a more abstract level the paper relates to the literature, both empirical and theoretical, where shocks to different “segments” of the asset market have mean reverting but persistent

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<sup>2</sup> Quoting from page 250: “The interest elasticity needed to fit the long-term trends (and very sharply estimated by these trends) is much too high to permit a good fit on a year-to-year basis. Of course, it is precisely this difficulty that has motivated much of the money demand research of the last 30 years, and has led to distributed lag formulations of money demand that attempt to reconcile the evidence at different frequencies. In my opinion, this reconciliation has not yet been achieved [...]”.

<sup>3</sup>For the full sample averages see the first row of Table 1 and the solid lines in Figure 5 in [Sargent and Surico \(2011\)](#).

effects on relative prices because, due to frictions, capital moves slowly between them, e.g. [Duffie \(2010\)](#), [Duffie and Strulovici \(2011\)](#), [Mitchell, Pedersen, and Pulvino \(2007\)](#), [Acharya, Shin, and Yorulmazer \(2009\)](#) or [Edmond and Weill \(2011\)](#). Our model describes an economy with slow-moving capital and provides an analytical characterization of the link between the long-run behavior and the speed of adjustment.

## The setup and main results

Our model is a version of the segmented market model in [Alvarez, Lucas, and Weber \(2001\)](#) where, instead of using a binding cash in advance constraint, we use a Sidrauski money in the utility function set-up. Markets are segmented in the sense that only a fraction  $\lambda$  of households (the “traders”) participates to the open market operations implemented by the central bank to control the money supply. The household preferences are defined over a bundle  $h(c, m)$  made of consumption  $c$  and real money balances  $m$ . The intra-temporal substitution elasticity between  $c$  and  $m$  is  $1/(1 + \rho)$ , the intertemporal substitution elasticity for  $h$  is  $1/\gamma$ . We show analytically how the three parameters  $\lambda, \rho, \gamma$  determine the properties of the model steady-state and the economy’s response, in terms of interest rates and inflation, to money supply shocks.

We begin by showing that velocity is a function of an appropriately expected discounted value of future monetary expansions, as occurs in the Sidrauski model as well as in related monetary models (i.e. cash-credit, shopping time, etc). This implies that the price level (and hence the inflation rate) are also functions of future expected path of money (and hence of the growth of money). Surprisingly, the relationship between inflation and money is, up to a first order, exactly the same regardless of the degree of segmentation  $\lambda$ , and hence it is the same as in an otherwise standard monetary model. The equilibrium nominal interest rate is determined by the  $m/c$  ratio of traders, hence the interest elasticity of the money demand is  $-1/(1 + \rho)$ .

The economic effects of monetary shocks depend on whether the shocks are temporary or permanent. We illustrate this result considering two extreme cases. The first is a permanent increase in the growth rate of money supply. This immediately increases expected inflation and produces a persistent increase in nominal interest rates and no liquidity effect. The second is a once-and-for-all increase of the money supply. In this case there is a jump on impact in the price level, and no effect on expected inflation. The model with segmented markets necessarily produces a persistent decrease in nominal interest rates: since the shock temporarily increases the  $m/c$  ratio of traders, they will only absorb the increase in money holdings at a lower interest rate. The magnitude of this effect at impact is inversely proportional to  $\lambda$ , the fraction of agents participating in the open market operation. The persistence of the liquidity effect

is increasing in  $\gamma/(1 + \rho)$ , the ratio of the interest-elasticity of the money demand relative to the intertemporal substitution elasticity. This ratio governs how long it takes to the  $m/c$  ratio of traders to return to its steady state value. Intuitively, the liquidity effect is short lived if the  $m/c$  ratio of traders is not responsive to monetary shocks: this happens when  $h$  is very substitutable intertemporally (small  $\gamma$ ) or  $m$  and  $c$  are poor substitutes (large  $\rho$ ).

For temporary but persistent shocks to the growth rate of the money supply the impulse responses of the nominal interest rate ranges between the two extreme cases described above, depending on the assumed persistence of the growth rate of money. We show analytically that impulse response of the nominal interest rate to a persistent shock is characterized by two eigenvalues. One is inherited from the persistence of the shock process itself. This component creates the classic “fisherian” response of the interest rate to changes in money growth and inflation. The other eigenvalue is related to the gradualism of the traders’ adjustment rule for  $m/c$ , which was discussed above and is determined by  $\gamma/(1 + \rho)$ . This component creates the “liquidity” effect which can, at least temporarily, dominate the fisherian effect.

Our model displays a liquidity effect for both unanticipated and anticipated monetary shocks, although the effect of anticipated monetary shocks may differ from that of unanticipated shocks. In the model in [Lucas \(1990\)](#) and in versions of [Christiano and Eichenbaum \(1992\)](#) and [Fuerst \(1992\)](#) only unanticipated shocks display liquidity effects. On the other hand, in [Alvarez, Lucas, and Weber \(2001\)](#), [Occhino \(2004\)](#), [Alvarez, Atkeson, and Kehoe \(2002\)](#), there is no distinction between the effect of expected and unexpected monetary shocks on interest rates. The fact that both anticipated and unanticipated monetary shocks have an effect on interest rates, and that these effects differ, can be used in future applied work to estimate the impulse responses of interest rates by applying the identification strategy proposed by [Cochrane \(1998\)](#).

## Related literature on the liquidity effect

The model in this paper is a descendent of the monetary models with segmented markets of [Grossman and Weiss \(1983\)](#) and [Rotemberg \(1984\)](#). These models are motivated by the hypothesis that, as described in Friedman’s presidential address, and in contrast with the working of simpler neoclassical monetary models, an open market operation that increases the quantity of money once-and-for-all produces a protracted decrease of the nominal interest rate.<sup>4</sup> In the models in these two papers agents are subject to a cash in advance constraint,

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<sup>4</sup>For instance, pages 5-6 of Friedman’s 1968 presidential address: “Let the Fed set out to keep interest rates down. How will it try to do so? By buying securities. This raises their prices and lowers their yields. In the process, it also increases the quantity of reserves available to banks, hence the amount of bank credit, and, ultimately the total quantity of money. That is why central bankers in particular, and the financial community more broadly, generally believe that an increase in the quantity of money tends to lower interest

but access to asset markets where open market operation takes place is restricted to a fraction of the agents. This fraction of agents, who holds only half of the money stock, have to absorb the entire increase on the money supply associated with the open market operation, which will be absorbed only with a lower real interest rate. Moreover, if this effect is large enough, the nominal interest rate decreases also. Mostly for tractability, these two papers assume that agents have access to the asset markets every-other period. Not surprisingly, with this pattern of visits to the asset market, the effect of a once and for all increase in money supply on interest rates are short lived: the largest effect is in the first two periods, after which there are small lingering echo effects.

Several monetary models of segmented asset markets have been written to analyze a variety of related questions since the seminal work of Grossman and Weiss, and Rotemberg. In all of them some carefully chosen assumptions are used to avoid keeping track of the lingering effects on the cross section distribution of asset holdings produced by an open market operation. The simplifications have the advantage of allowing a sharper analytical characterization of the equilibrium. Examples of these are Lucas (1990), Alvarez and Atkeson (1997), Fuerst (1992), Christiano and Eichenbaum (1992), Alvarez, Atkeson, and Kehoe (2002), Alvarez, Lucas, and Weber (2001) and Occhino (2004), which are discussed briefly in Appendix D.2. All these models have in common that an open market operation that produces a once and for all increase in the money supply decreases interest rate on impact, but then the interest rate returns immediately to its previous level. These models also have in common that they produce a version of the quantity theory in which different permanent values of the growth rate of money supply are associated with different inflation rates, and hence -via a Fisher equation- different nominal interest rates, but with the same level of velocity. In this sense, these models have an interest rate inelastic long run money demand. In this paper, we introduce a modification of the set-up in which a once and for all increase in the money supply produces a persistent liquidity effect, due to the persistent redistribution effects of the open market operation, and additionally it implies a long-run interest-elastic money demand.

There are a few models where a once and for all increase in the money supply produces a transitory, but *persistent*, decline in interest rates. These models feature a different mechanism to generate persistence. An early example is the model in Christiano and Eichenbaum (1992). In their basic set up firms face a CIA constraint and, given the assumption on when the household have to decide their cash holdings, asset markets are segmented. In this set-

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rates. [...] The initial impact of increasing the quantity of money at a faster rate than it has been increasing is to make interest rates lower for a time than they would otherwise have been. [...] after a somewhat longer interval, say, a year or two, [they will tend] to return interest rates to the level they would otherwise have had.”

up, as well in the closely related set-up of [Fuerst \(1992\)](#), liquidity effects are short lived.<sup>5</sup> [Christiano and Eichenbaum \(1992\)](#) add to this basic-set up a convex adjustment cost applied to changes in the households' cash holdings. The adjustment cost naturally retards the adjustments, producing a persistent liquidity effect. Another example is [Williamson \(2008\)](#), who combines a segmented asset market model similar to [Alvarez, Lucas, and Weber \(2001\)](#), with persistent but mean reverting segmentation in the goods markets. In this set-up money spent by those connected to asset market leaks slowly to the rest of the economy, spreading the effect of a once and for all increase in money. A different mechanism proposed in the literature for persistent liquidity effects is sticky prices, a' la Calvo. Several papers focus on the conditions under which liquidity effects emerge in simple versions of these models, such as [Gali \(2002\)](#) and [Andres, David Lopez-Salido, and Valles \(2002\)](#). A shortcoming of these models is that they are unable to generate a liquidity effect *and* to be consistent with the features of long run money demand such as a unit income elasticity and an interest elasticity around  $-1/2$  (see [Appendix C](#) for a more detailed discussion).

## 2 The Environment

Let  $U(c, m)$  be the period utility function, where  $c^i$  is consumption  $m^i$  are beginning-of-period real balances. We assume that  $U$  is strictly increasing and concave. Let  $i = T, N$  be the type of agents, trader, or which there are  $\lambda$ , and non-traders, of which there are  $(1 - \lambda)$ , respectively. While the focus of the paper is on the analysis of the behavior of interest rates due to the unequal access to asset market at the time of open market operations, we first describe the equilibrium in a model without a bond market, and hence without open market operations. We do so because it is easier, and it highlights the logic of the determination of different equilibrium variables.

Time is discrete and starts at  $t = 0$ . The timing within a period is as follows: agent start with nominal beginning-of-period cash balances  $M_t^i$ , they receive nominal income  $P_t y^i$ , choose real consumption  $c_t^i$  and end-of-period nominal balances  $N_t^i$ . Next period nominal cash balances  $M_{t+1}^i$  are given this period nominal balances plus  $P_{t+1} \tau_{t+1}^i$  the nominal lump-sum

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<sup>5</sup>Fuerst explains the lack of propagation in terms of his model very clearly: “As for the failures, the most glaring is that the model is lacking a strong propagation mechanism. The real effects of monetary injections are a result of (serially uncorrelated) forecast errors. These effects will therefore be strongest during the initial period of the shock.” ...“This failure to achieve persistence is more a criticism of my particular modeling strategy than of this class of models as a whole. If it takes more than one period for the economy to re-balance its portfolio and ‘undo’ the monetary injection, then the effects of monetary shocks will of course persist. While this assumption may be the most natural, it is also somewhat intractable...”

transfer from the government. Their budget constraint at  $t \geq 0$  are:

$$N_t^i + P_t c_t^i = P_t y^i + M_t^i \quad , \quad M_{t+1}^i = N_t^i + P_{t+1} \tau_{t+1}^i \quad (1)$$

Our choice of the timing convention for this problem is consistent with an interpretation of  $U$  as a cash-credit good, as in [Lucas and Stokey \(1987\)](#). We use a Sidrauski money-in-the-utility function specification because, relative to a cash-in-advance model, it allows more flexibility to accommodate a stock  $m$  and a flow  $c$ , albeit in a mechanical way. We will return to the discussion of the specification, and the relation between stocks and flows, below.

The problem of an agent of type  $i$  is

$$\max_{\{N_t^i\}_{t=0}^{\infty}} \mathbb{E}_0 \left[ \sum_{t=0}^{\infty} \beta^t U(c_t^i, M_t^i/P_t) \right]$$

subject to (1) given  $M_0$ . Notice that while we labeled the agents traders and non-traders, the budget constraint in (1) indicates that neither type of agent is allowed to trade in bonds or any other security. Their only intertemporal choice is the accumulation of cash balances. Yet in [Section 2.1](#) we show that the equilibrium of the model will be the same as one in which traders and the government participate in a market for nominal bonds.

Market clearing of goods and money is given by

$$\begin{aligned} \lambda c_t^T + (1 - \lambda) c_t^N &= \lambda y^T + (1 - \lambda) y^N, \\ \lambda M_t^T + (1 - \lambda) M_t^N &= M_t \end{aligned}$$

for all  $t \geq 0$ . The government budget constraint is given by

$$M_t - M_{t-1} = P_t [\lambda \tau_t^T + (1 - \lambda) \tau_t^N] \quad ,$$

for all  $t \geq 1$ . Notice that the government budget constraint does not apply for  $t = 0$ , since our timing convention is that we start the period with the cash after transfers. To simplify the notation, in this section we assume that the government does not trade in bonds, an assumption that we remove in [Section 2.1](#). We note for future reference that the budget constraint of the agents and market clearing imply that aggregate beginning-of period and end-of-period money balances are the same:

$$M_t = N_t \equiv \lambda N_t^T + (1 - \lambda) N_t^N$$

for all  $t \geq 0$ . Notice that using the definition of  $N_t$  the budget constraint of the government

follows from the budget constraint of the households:

$$M_{t+1} = M_t + P_{t+1} [\lambda \tau_{t+1}^T + (1 - \lambda) \tau_{t+1}^N] = N_t + P_{t+1} [\lambda \tau_{t+1}^T + (1 - \lambda) \tau_{t+1}^N] .$$

We define inflation  $\pi_t$ , growth rate of money,  $\mu_t$ , beginning-of-period real balances,  $m_t$ , and end-of-period real balances,  $n_t$  as:

$$\pi_{t+1} = \frac{P_{t+1}}{P_t}, \mu_{t+1} = \frac{M_{t+1}}{M_t}, m_t = \frac{M_t}{P_t}, m_t^i = \frac{M_t^i}{P_t}, n_t^i = \frac{N_t^i}{P_t},$$

for  $i = T, N$  and  $t \geq 0$ . With these definitions we write the budget constraints as

$$c_t^i + n_t^i = y^i + m_t^i \quad \text{and} \quad m_{t+1}^i = n_t^i / \pi_{t+1} + \tau_{t+1}^i, \quad t \geq 0 \quad (2)$$

market clearing as

$$\lambda m_t^T + (1 - \lambda) m_t^N = m_t, \quad , \quad \lambda c_t^T + (1 - \lambda) c_t^N = \lambda y^T + (1 - \lambda) y^N, \quad t \geq 0 \quad (3)$$

the money-growth identity

$$\pi_t = \mu_t \frac{m_{t-1}}{m_t}, \quad t \geq 0 \quad (4)$$

and the government budget constraint as

$$m_t - \frac{m_{t-1}}{\pi_t} = \lambda \tau_t^T + (1 - \lambda) \tau_t^N, \quad t \geq 1. \quad (5)$$

The exogenous random processes for this economy are given by  $s_t \equiv (\mu_t, \tau_t^T, \tau_t^N)$ . We use  $s^t$  for the histories of such shocks, but we avoid this notation when it is clear from its context. The f.o.c. for the agent problem w.r.t.  $n_t^i$ ,  $i = T, N$  are:

$$U_1(c_t^i, m_t^i) = \mathbb{E}_t \left\{ \frac{\beta}{\pi_{t+1}} [U_1(c_{t+1}^i, m_{t+1}^i) + U_2(c_{t+1}^i, m_{t+1}^i)] \right\}. \quad (6)$$

for  $t \geq 0$ , where the expectation is with respect to the realization of inflation and the lump sum subsidy  $\tau_{t+1}^i$ . We can now define an equilibrium:

**DEFINITION 1.** Given initial conditions  $\{M_0^i\}$  and a monetary and fiscal policy described by stochastic processes  $\{\mu_{t+1}, \tau_{t+1}^T, \tau_{t+1}^N\}_{t=0}^\infty$  an *Equilibrium* is an initial price level  $P_0$ , inflation rates  $\{\pi_{t+1}\}_{t=0}^\infty$ , and stochastic processes  $\{n_t^i, m_t^i, c_t^i, m_t\}_{t=0}^\infty$  for  $i = T, N$  such that: the budget constraints (2), market clearing (3), identity (4), the government budget constraint (5), and the Euler equations (6) hold.

We conclude this section with two comments on the set-up. The first is that our convention for the timing of the model is one where the initial conditions  $M_0^i$  contain the period zero monetary injection, and thus  $\tau_0^i$  and  $\mu_0$  are not part of the set-up. For instance, our convention entails that  $\{\mu_1, \tau_1^i\}$  are random variables, whose realization is not known as of time  $t = 0$ . Second, in the monetary-fiscal policy described by (5), the resources obtained by monetary expansions can be used for redistribution across agents, since the lump sum transfers  $\tau_t^T$  and  $\tau_t^N$  are allowed to differ across types.

## 2.1 Interest Rates in an Equilibrium with an Active Bond Market

In this section we introduce a bond market where traders participate and where the open market operation takes place. We show an equivalence result for the equilibrium with and without an active bond market, and analyze how the interest rate depends on the equilibrium allocation.

We assume that traders at time  $t$  have access to a bond market and a set of Arrow securities that pay contingent on the realization of  $s_{t+1}$  which opens at the beginning of the period, before consumption takes place, at the same time of the money transfer  $\tau_t^T$ . Thus, if a trader buys one nominal zero coupon bond in period  $t$ , he reduces his money holdings  $M_t^T$  by  $Q_t$  dollars (the bond price), and increases next period holding of money in all states by one dollar. Let  $W_t$  be the beginning-of-period money stock, after the current period lump-sum transfer from the government but before participating in the bond market. Let  $B_t^T$  be the number of nominal bonds purchased, and  $A_t(s_{t+1})$  the quantity of Arrow securities that pay one dollar contingent on the realization of  $s_{t+1}$ . The price of each of these securities at the beginning of period  $t$  is denoted by  $q_t(s_{t+1})$ . In this case we can write the budget constraint of the trader at time  $t$  and history  $s^t$  –which we omit to simplify the notation–

$$\begin{aligned} \sum_{s_{t+1}} A_t^T(s_{t+1})q_t(s_{t+1}) + Q_t B_t^T + M_t^T &= W_t, \\ N_t^T + P_t c_t^T &= M_t^T + P_t y^T, \\ W_{t+1}(s_{t+1}) &= N_t^T + B_t^T + A_t^T(s_{t+1}) + P_{t+1}(s_{t+1}) \tau_{t+1}^T(s_{t+1}) \end{aligned}$$

for  $t \geq 0$ . The interpretation is that during the period the agent chooses bond holdings  $B_t^T$  and consumption  $c_t^T$ , and given the budget constraint, this gives next period cash balances before transfers  $N_t^T$ . The last line shows the beginning-of-next-period cash balances, which include the cash from the bonds purchased this period and the cash transfer from the

government. Equivalently, we can write the budget constraint in real terms as:

$$\begin{aligned} \sum_{s_{t+1}} a_t^T(s_{t+1})q_t(s_{t+1}) + Q_t b_t^T + m_t^T &= w_t, \\ n_t^T + c_t^T &= m_t^T + y^T, \\ w_{t+1}(s_{t+1}) &= \frac{n_t^T + b_t^T + a_t^T(s_{t+1})}{\pi_{t+1}(s_{t+1})} + \tau_{t+1}^T, \end{aligned} \quad (7)$$

where  $w_t = W_t/P_t$ ,  $a_t = A_t/P_t$ ,  $b_t = B_t/P_t$  and  $t \geq 0$ . The government budget constraint is:

$$\sum_{s_{t+1}} A_t(s_{t+1})q_t(s_{t+1}) + Q_t B_t + M_t - M_{t-1} = B_{t-1} + A_{t-1} + P_t [\lambda \tau_t^T + (1 - \lambda) \tau_t^N], \quad (8)$$

for  $t \geq 1$  or in real terms:

$$\sum_{s_{t+1}} a_t(s_{t+1})q_t(s_{t+1}) + Q_t b_t + m_t = \frac{b_{t-1} + a_{t-1} + m_{t-1}}{\pi_t} + \lambda \tau_t^T + (1 - \lambda) \tau_t^N, \quad (9)$$

for  $t \geq 1$ . The trader's problem is to maximize utility by choice of  $\{n_t^T, b_t^T\}_{t=0}^\infty$  subject to (7), and given an initial condition  $w_0$ .

The first order conditions for the choice of nominal bond holdings  $B_t^T$  and Arrow securities  $A_t^T(s_{t+1})$  are:

$$\begin{aligned} Q_t [U_1(c_t^T, m_t^T) + U_2(c_t^T, m_t^T)] &= \mathbb{E}_t \left[ \frac{\beta}{\pi_{t+1}} [U_1(c_{t+1}^T, m_{t+1}^T) + U_2(c_{t+1}^T, m_{t+1}^T)] \right] \\ q_t(s_{t+1}) [U_1(c_t^T, m_t^T) + U_2(c_t^T, m_t^T)] &= \frac{\beta \Pr(s_{t+1}|s^t)}{\pi_{t+1}} [U_1(c_{t+1}^T, m_{t+1}^T) + U_2(c_{t+1}^T, m_{t+1}^T)] \end{aligned} \quad (10)$$

where  $\Pr(s_{t+1}|s^t)$  is the probability of state  $s_{t+1}$  conditional on the history  $s^t$ . Combining this first order condition with the Euler equation for  $n_t^T$  we obtain:

$$Q_t = \frac{U_2(c_t^T, m_t^T)}{U_1(c_t^T, m_t^T) + U_2(c_t^T, m_t^T)}, \quad (11)$$

or, by letting  $r_t$  be the nominal interest rate,  $1 + r_t \equiv Q_t^{-1}$ , we can write:

$$r_t = \frac{U_2(c_t^T, m_t^T)}{U_1(c_t^T, m_t^T)}.$$

Notice that, given the timing assumptions for the bond market, interest rates are functions of the time  $t$  allocation, i.e. they do not involve any expected future values. For instance

consider the utility function

$$U(c, m) = \frac{h(c, m)^{1-\gamma} - 1}{1-\gamma}, \quad \text{where } h(c, m) = [c^{-\rho} + \mathcal{A}^{-1}m^{-\rho}]^{-\frac{1}{\rho}}, \quad (12)$$

which has a constant elasticity of substitution  $1/(1+\rho)$  between  $c$  and  $m$ , a constant intertemporal substitution elasticity  $1/\gamma$  between the consumption-money bundles, and  $\mathcal{A}$  is a parameter. This case yields the constant elasticity, unitary income, money demand:

$$r_t = \frac{U_2(c_t^T, m_t^T)}{U_1(c_t^T, m_t^T)} = \frac{1}{\mathcal{A}} \left( \frac{m_t^T}{c_t^T} \right)^{-(1+\rho)}. \quad (13)$$

Finally, market clearing for Arrow securities and bonds, under the assumption that only traders participate in these markets, is:

$$b_t = \lambda b_t^T \quad \text{and} \quad a_t(s_{t+1}) = \lambda a_t^T(s_{t+1}), \quad \forall s_{t+1} \quad (14)$$

for all  $t \geq 0$ . Next we give an equilibrium definition for the model with an active nominal bond market:

**DEFINITION 2.** Given initial conditions  $\{M_0^N, W_0\}$  and a monetary and fiscal policy described by stochastic processes  $\{\tilde{\mu}_{t+1}, \tilde{\tau}_{t+1}^T, \tilde{\tau}_{t+1}^N, \tilde{b}_t, \tilde{a}_t\}_{t=0}^\infty$ , an *Equilibrium with an active bond market* is an initial price level  $\tilde{P}_0$ , inflation rate process  $\{\tilde{\pi}_{t+1}\}_{t=0}^\infty$ , stochastic processes  $\{\tilde{n}_t^i, \tilde{m}_t^i, \tilde{c}_t^i, \tilde{m}_t^i\}_{t=0}^\infty$  for  $i = T, N$ , and stochastic processes  $\{w_t, b_t^T, a_t^T, q_t, Q_t\}_{t=0}^\infty$  that satisfy: the budget constraints for non-traders (2) and traders (7), identity (4), Euler equations for end-of-period cash balances (6), the government budget constraint (9), the f.o.c. for bonds and Arrow securities (10), and market clearing (3) and (14).

As in the equilibrium described in **Definition 1**, our convention for the initial conditions  $W_0, M_0^N$  include the time zero money injection, and so neither  $\tau_0^i$  nor  $\mu_0$  are part of the definition. Instead  $B_0^T$  and  $M_0^T$  are choices for the traders, and hence bond prices  $Q_t$  are determined starting from period  $t = 0$  on. The following proposition shows the sense in which the equilibrium with and without an active bond market are equivalent.

**PROPOSITION 1.** Consider an equilibrium in the model without bond market:  $\langle P_0, \{\pi_{t+1}\}_{t=0}^\infty, \{n_t^i, c_t^i, m_t^i, m_t\}_{t=0}^\infty \rangle$ , for initial conditions  $\{M_0^i\}$  and policy  $\{\mu_{t+1}, \tau_{t+1}^T, \tau_{t+1}^N\}_{t=0}^\infty$  for  $i = T, N$ . Then, for any stochastic process of transfers to traders,  $\{\tilde{\tau}_{t+1}^T\}_{t=0}^\infty$ , there is an equilibrium with an active bond market that satisfies:

$$\left\langle \tilde{P}_0, \{\tilde{\pi}_{t+1}\}_{t=0}^\infty, \{\tilde{n}_t^i, \tilde{c}_t^i, \tilde{m}_t^i, \tilde{m}_t\}_{t=0}^\infty \right\rangle = \left\langle P_0, \{\pi_{t+1}\}_{t=0}^\infty, \{n_t^i, c_t^i, m_t^i, m_t\}_{t=0}^\infty \right\rangle,$$

with bond and Arrow prices  $\{Q_t, q_t\}_{t=0}^\infty$  given by equation (10) for all  $t \geq 0$ , the fiscal and monetary policy given by  $\{\tilde{\tau}_t^N, \tilde{\mu}_t\}_{t=1}^\infty = \{\tau_t^N, \mu_t\}_{t=1}^\infty$ , and  $\{a_t, b_t\}_{t=1}^\infty$  satisfying (14) and:

$$\frac{a_{t-1} + b_{t-1}}{\tilde{\pi}_t} - Q_t b_t - \sum_{s_{t+1}} q_t(s_{t+1}) a_t(s_{t+1}) = \lambda (\tau_t^T - \tilde{\tau}_t^T) , \quad t \geq 1.$$

with initial conditions  $\tilde{M}_0^N = M_0^N$ ,  $W_0 = M_0^T - Q_0(a_0 + b_0)/\lambda$ , where  $a_0 + b_0$  satisfies an appropriately chosen present value.

The proposition shows that it is only the combination of monetary and fiscal policy that matters. We use this proposition to analyze an equilibrium where all monetary injections are carried out through open market operations. To see this, first consider an equilibrium without an active bond market and where  $\tau_t^N = 0$ . In this case the budget constraint of the government is:

$$M_t - M_{t-1} = \lambda P_t \tau_t^T .$$

Then, using the previous proposition, we can construct an equilibrium where  $\tilde{\tau}_t^T = 0$  for all  $t \geq 0$ . The government budget constraint is:

$$\sum_{s_{t+1}} q_t(s_{t+1}) A_t(s_{t+1}) + Q_t B_t + M_t - M_{t-1} = B_{t-1} + A_{t-1} ,$$

which in the case of a time varying but deterministic policy is:

$$Q_t B_t + M_t - M_{t-1} = B_{t-1} , \quad t \geq 1 \quad \text{or} \quad \sum_{t=1}^{\infty} (M_t - M_{t-1}) (\prod_{i=1}^t Q_i) = B_0 .$$

In this equilibrium traders start with an initial value of government bonds that enables them to buy the present value of the future seignorage. The equivalence of real allocations, in the equilibrium where transfers differ across agents and in the one where money injections are carried out through open market operations, illustrates the sense in which open market operations with segmented asset markets have redistributive effects.

In Section 5 we will consider a fiscal - monetary policy that is similar to the one described above. We will assume  $\tau_t^N = \tilde{\tau}_t^N = \bar{\tau}$  and  $\tilde{\tau}_t^T = 0$ . In this case non-traders receive a transfer with constant real value  $\bar{\tau}$  and the government budget constraint is:

$$\sum_{s_{t+1}} q_t(s_{t+1}) A_t(s_{t+1}) + Q_t B_t + M_t - M_{t-1} = B_{t-1} + A_{t-1} + P_t (1 - \lambda) \bar{\tau}$$

We will choose the value of  $\bar{\tau}$  so that it coincides with the ‘‘average’’ seignorage in the

economy, as described in [Section 5](#). By doing so we make traders and non-traders completely symmetric, at least in some average long-run sense.<sup>6</sup>

### 3 Approximate aggregation with Segmented Markets

This section studies the determination of inflation in the model with segmentation, i.e. when  $\lambda \in (0, 1)$ . Given the equivalence established by [Proposition 1](#), the argument is developed using the simpler framework without an active bond market. In particular, we allow traders and non-traders to be subject to different arbitrary processes for  $\{\tau_t^N\}_{t=1}^\infty$  and  $\{\tau_t^T\}_{t=1}^\infty$ . We show that, somewhat surprisingly, up to a linear approximation the relation between aggregate inverse velocity  $m_t/y$  and money growth rates  $\{\mu_t\}_{t=1}^\infty$  is independent of the fraction of traders ( $\lambda$ ), and hence that it is the same one obtained in a model with a representative agent where  $\lambda = 1$ .

We assume that both types of agents have the same real income, so that  $y^T = y^N = y$ . We define an approximate equilibrium by replacing the Euler Equations of each agent and the budget constraints by linear approximations around the values that correspond to the steady state of an aggregate model with  $\bar{m}^i = \bar{m}$ ,  $\bar{c}^i = \bar{c} = y$  for  $i = T, N$ , and constant money supply growth  $\bar{\mu} = \bar{\pi}$ , satisfying

$$\bar{\tau} = \bar{m}(\bar{\mu} - 1) \quad , \quad U_1(y, \bar{m}) = \frac{\beta}{\bar{\mu}} [U_1(y, \bar{m}) + U_2(y, \bar{m})] \quad . \quad (15)$$

Let  $\hat{x}_t \equiv x - \bar{x}$  denote the deviation of the variable  $x$  from its steady state value  $\bar{x}$ . The linearization of the Euler equation [\(6\)](#) gives:

$$\bar{U}_{11}\hat{c}_t^i + \bar{U}_{12}\hat{m}_t^i = \frac{\beta}{\bar{\pi}}\mathbb{E}_t [(\bar{U}_{11} + \bar{U}_{21})\hat{c}_{t+1}^i + (\bar{U}_{12} + \bar{U}_{22})\hat{m}_{t+1}^i] - \frac{\beta}{\bar{\pi}^2}\mathbb{E}_t [\bar{U}_1 + \bar{U}_2] \hat{\pi}_{t+1} \quad (16)$$

The linearization of the identity in [\(4\)](#), and the budget constraints in [\(2\)](#) and [\(5\)](#) gives:

$$\hat{\pi}_t = \hat{\mu}_t - \frac{\bar{\mu}}{\bar{m}}(\hat{m}_t - \hat{m}_{t-1}), \quad \hat{m}_t^i = \frac{\hat{n}_{t-1}^i}{\bar{\pi}} - \frac{\bar{n}}{\bar{\pi}^2}\hat{\pi}_t + \hat{\tau}_t^i, \quad \hat{m}_t - \frac{\hat{m}_{t-1}}{\bar{\pi}} + \frac{\bar{m}}{\bar{\pi}^2}\hat{\pi}_t = \lambda\hat{\tau}_t^T + (1-\lambda)\hat{\tau}_t^N, \quad (17)$$

for  $i = N, T$ . We are now ready to define an approximate equilibrium.

**DEFINITION 3.** Given initial conditions  $\{M_0^i\}$  and a fiscal and monetary policy described by  $\{\mu_{t+1}, \tau_{t+1}^i\}_{t=0}^\infty$  an *Approximate Equilibrium* is given by  $\{n_t^i, m_t^i, c_t^i, m_t\}_{t=0}^\infty$  for  $i = T, N$ ,

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<sup>6</sup>We can preserve the symmetry in other ways too. For instance, we can let non-traders have a higher real income, i.e. we can let  $y^N$  be higher than  $y^T$  and assume that non-traders receive zero real lump sum transfer.

and  $P_0, \{\pi_{t+1}\}_{t=0}^\infty$  that satisfy: market clearing (3), the linearized Euler equation (16), and the linearized constraints (17).

Using this definition we state our main result on aggregation:

**PROPOSITION 2.** In an approximate equilibrium the processes for aggregate real balances and inflation  $\{m_t, \pi_{t+1}\}_{t=0}^\infty$  and the initial price level  $P_0$  are the same for any  $\lambda \in [0, 1]$ .

The proof of this result is simple: in the equilibrium without active bond markets traders and non-traders optimal decisions are characterized by the *same* Euler equation, evaluated at different shocks for  $\{\tau_t^i\}$ , and the same inflation process. Linearizing these equations, and using market clearing, one obtains the aggregation result. This result is important for two reasons. First, substantively, it says that the relation between inflation and money growth, to a first order approximation, is independent of the fraction of traders  $\lambda$ . Second, it shows that the equilibrium has a recursive nature. One can determine first the path of aggregate real balances obtaining the process for inflation, as we do in Section 4, and then solve for the decision problem of the non-trader, obtaining the process for the non-trader consumption and real balances. Using feasibility and the process for aggregate real balances, one can finally solve for the traders' real balances and consumption, which in turns gives us the interest rate from equation (11). Since the problem of the non-trader is a key intermediate step to determine the behavior of interest rates, Section 5 analyzes it in detail.

## 4 Velocity and Money Growth

In this section we consider a model with one type of agent, or  $\lambda = 1$ , to obtain a description of inverse velocity and inflation as functions of future expected money growth rates, as in the representative-agent model of Sidrauski, or in Cagan's.<sup>7</sup> Our interest in the setup with  $\lambda = 1$  comes from Proposition 2, which shows that the equilibrium path for aggregate inverse velocity and inflation is the same irrespective of  $\lambda \in [0, 1]$ .

Using market clearing ( $c_t = y$ ) into the f.o.c. for  $m$ , and the inflation identity  $\pi_{t+1} = \mu_{t+1} m_t/m_{t+1}$  we can write

$$U_1(y, m_t) m_t = \mathbb{E}_t \left\{ \frac{\beta}{\mu_{t+1}} [U_1(y, m_{t+1}) + U_2(y, m_{t+1})] m_{t+1} \right\}. \quad (18)$$

Our next task is to analyze the behavior of this system. We first consider the steady state, the case where money supply grows at a constant rate  $\bar{\mu}$  and  $\bar{r}$  is the net interest rate

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<sup>7</sup>Equivalently, we consider a set-up where traders and non-traders are identical, i.e. where  $y^T = y^N = y$ ,  $\tau_t^N = \tau_t^T$  for all  $t \geq 0$  with initial conditions  $M_0^T = M_0^N = M_0$  for any  $\lambda$ .

that corresponds to a constant money growth rate and inflation  $\bar{\mu}$ :

$$\frac{U_2(y, \bar{m})}{U_1(y, \bar{m})} = \bar{r} = \frac{\bar{\mu}}{\beta} - 1. \quad (19)$$

As in Lucas (2000) we interpret the function  $\bar{m}$  of  $\bar{r} = \bar{\mu}/\beta - 1$ , solving equation (19), as the “long run” money demand. For the case where  $U$  is given by equation (12), this money demand has a constant interest rate elasticity  $-1/(1 + \rho)$ .

In what follows we analyze a linearized version of the difference equation (18), expanded around a constant  $\mu = \bar{\mu}$  and  $m = \bar{m}$ . We seek a solution for real balances as a function of the future expected growth of the money supply.

**PROPOSITION 3.** Let  $\hat{m}_t \equiv m_t - \bar{m}$ ,  $\hat{\mu}_t \equiv \mu_t - \bar{\mu}$ . Linearizing (18) around (19) we have:

$$\hat{m}_t = \alpha \mathbb{E}_t [\hat{\mu}_{t+1}] + \phi \mathbb{E}_t [\hat{m}_{t+1}] \quad \text{where} \quad (20)$$

$$\alpha \equiv -\frac{\bar{m}}{\bar{\mu}} \frac{\bar{U}_1}{\bar{U}_1 + \bar{m}\bar{U}_{12}} \quad \text{and} \quad \phi \equiv \frac{\beta}{\bar{\mu}} \left[ 1 + \frac{\bar{U}_2 + \bar{m}\bar{U}_{22}}{\bar{U}_1 + \bar{m}\bar{U}_{12}} \right] \quad (21)$$

and where  $\bar{U}_i$ ,  $\bar{U}_{ij}$  are the derivatives of  $U(\cdot)$  evaluated at  $(y, \bar{m})$ . With  $0 < \phi < 1$  we can express its unique bounded solution as

$$\hat{m}_t = \alpha \sum_{i=1}^{\infty} \phi^{i-1} \mathbb{E}_t [\hat{\mu}_{t+i}] \quad t \geq 0. \quad (22)$$

Thus, if  $0 < \phi < 1$  and  $\alpha < 0$ , future expected money growth reduces current real money balances. We briefly discuss sufficient conditions for this configuration. Equation (21) shows that the condition for  $\alpha < 0$  requires that  $U_1 + mU_{12} > 0$ , which is always the case if  $U_{12} > 0$ . The assumption of  $U_{12} > 0$  has the interpretation that the durable good is a complement to the consumption of non-durables, and will be maintained for the rest of the paper. Notice that  $-\alpha(\bar{\mu}/\bar{m})$  is decreasing in  $U_{12} > 0$ , starting from a value of 1 at  $U_{12} = 0$ .

When  $U$  is given by equation (12) the requirements for  $\phi$  can be written in terms of conditions on:  $\gamma, \rho, \bar{r}$  and  $\bar{m}/y$ . Lemma 1 in Appendix A.4 shows that when  $\rho > -1$ , a sufficient condition for  $\phi < 1$  and  $\alpha < 0$  is

$$\gamma < 2 + \rho + \frac{y}{\bar{r} \cdot \bar{m}} \quad (23)$$

(otherwise if  $\rho = -1$ , then  $\phi = 1$ ). Notice that condition (23) holds for a wide range of

parameters of interest.<sup>8</sup> For instance, with an annual nominal interest rate of 4 percent, annual money-income ratio of 1/4, and an elasticity of substitution between consumption and real balances of 1/2, so that  $\bar{r} = 0.04$ ,  $\bar{m}/y = 1/4$  and  $\rho = 1$ , then  $\gamma$  has to be smaller than 103, a condition that is easily satisfied by any reasonable estimate of the risk aversion parameter  $\gamma$ .

**Lemma 1** also shows that the condition  $0 < \phi$ , which ensures monotone dynamics, is:

$$\gamma < \frac{1}{1 + \bar{r}} \left( 2 + \rho + \frac{y}{\bar{r} \bar{m}} + \bar{r} - \frac{\rho y}{\bar{m}} \right) .$$

This inequality is implied by [equation \(23\)](#) as long as the length of a time period is sufficiently small.<sup>9</sup>

## 4.1 Linear State Space Representation for Velocity and Inflation

We specify a linear time series process for  $\{\hat{\mu}_t\}$  and rewrite the initial conditions exclusively in terms of real variables. We start with a representation for inflation as a function of future money growth rates and the initial aggregate money balances. Using [\(17\)](#) and [\(22\)](#) we obtain

$$\hat{\pi}_t = \hat{\mu}_t + \frac{\bar{\mu}}{\bar{m}} \hat{m}_{t-1} - \frac{\bar{\mu}}{\bar{m}} \alpha \sum_{i=1}^{\infty} \phi^{i-1} \mathbb{E}_t [\hat{\mu}_{t+i}] , \quad t \geq 0 . \quad (24)$$

Notice that [equation \(17\)](#) is defined for  $\pi_0 \equiv P_0/P_{-1}$  and  $\mu_0$ . This representation avoids us to carry a nominal level variable, such as  $M_0$ , as the initial state. Instead, the initial state is the real level of money balances,  $\hat{m}_{-1}$ . We assume that the detrended growth of money supply  $\hat{\mu}_t$  is a linear function of an exogenous state  $z_t$ :

$$\hat{\mu}_{t+1} = \nu z_{t+1} , \quad z_{t+1} = \Theta z_t + \epsilon_{t+1} \quad (25)$$

for  $t \geq 0$ , where  $z_0$  is given,  $\nu$  is a  $k \times 1$  vector,  $\Theta$  is a  $k \times k$  matrix with  $k$  stable eigenvalues, and  $\epsilon_{t+1}$  is a  $k \times 1$  vector of innovations. In this case

$$\hat{m}_t = \alpha \sum_{i=1}^{\infty} \phi^{i-1} \mathbb{E}_t [\hat{\mu}_{t+i}] = \alpha \nu \Theta [I - \phi \Theta]^{-1} z_t , \quad t \geq 0 .$$

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<sup>8</sup>Note that when  $U$  is given by [\(12\)](#) a sufficient condition for  $U_{12} > 0$  is that the intertemporal elasticity of substitution,  $1/\gamma$ , is higher than the intratemporal elasticity of substitution between  $c$  and  $m$ , given by  $1/(1 + \rho)$ .

<sup>9</sup>Note, by contrast, that inequality [\(23\)](#) is independent of the model's time period since the term  $y/(\bar{r} \bar{m})$  contains the ratio of two flows ( $\bar{r}/y$ ), and the rest of the terms are parameters independent of time. We return to this issue in the discussion of [Proposition 5](#).

Replacing (25) into (24) we obtain:

$$\hat{\pi}_t = \nu z_t + \frac{\bar{\mu}}{\bar{m}} \hat{m}_{t-1} - \alpha \frac{\bar{\mu}}{\bar{m}} \nu \Theta [I - \phi \Theta]^{-1} z_t, \quad t \geq 0.$$

For example if  $z_t$  follows the AR(1) process  $z_{t+1} = \theta z_t + \epsilon_{t+1}$  and  $\nu = 1$ , then  $k = 1$ ,  $\hat{\mu}_t = z_t$ , and  $\Theta = \theta$ , so that for  $t \geq 0$ :

$$\hat{m}_t = \alpha \frac{\theta}{1 - \phi \theta} \hat{\mu}_t, \quad \hat{\pi}_t = \frac{\bar{\mu}}{\bar{m}} \hat{m}_{t-1} + \left[ 1 - \alpha \frac{\bar{\mu}}{\bar{m}} \frac{\theta}{1 - \phi \theta} \right] \hat{\mu}_t,$$

given  $\hat{m}_{-1}$ . Recall that  $\alpha < 0$  if (23) holds, so that real balances are decreasing in  $\hat{\mu}_t$  and hence inflation increases more than one for one with  $\hat{\mu}_t$ . This is a well known feature of the standard variable-velocity model.

We summarize the linear equilibrium representation for the aggregate economy as a function of the innovations  $\{\epsilon_t\}_{t=1}^{\infty}$ , the parameters  $\{\bar{\mu}/\bar{m}, \alpha, \phi, \nu, \Theta\}$ , and initial conditions  $z_0$  and  $\hat{m}_{-1}$  as follows:

$$\hat{m}_t = \kappa z_t, \quad \hat{\pi}_t = \left( \frac{\bar{\mu}}{\bar{m}} \right) \hat{m}_{t-1} + \zeta z_t, \quad z_{t+1} = \Theta z_t + \epsilon_{t+1}, \quad (26)$$

for all  $t \geq 0$ , where the vectors  $\zeta$  and  $\kappa$  are given by:

$$\zeta \equiv \nu \left( I - \alpha \frac{\bar{\mu}}{\bar{m}} \Theta [I - \phi \Theta]^{-1} \right), \quad \kappa \equiv \alpha \nu \Theta [I - \phi \Theta]^{-1}. \quad (27)$$

For future reference we also produce a formula for expected inflation:

$$\mathbb{E}_t [\hat{\pi}_{t+j}] = \bar{\Pi} \Theta^{j-1} z_t, \quad \text{where } \bar{\Pi} \equiv \frac{\bar{\mu}}{\bar{m}} \kappa + \zeta \Theta. \quad (28)$$

As a special case notice that if  $\nu = 1$  and  $\Theta = 0$ , so that  $\hat{\mu}_t$  is i.i.d., we have that real money balances are constant ( $\kappa = 0$ ), inflation is equal to money growth ( $\zeta = 1$ ), and hence expected inflation is constant ( $\bar{\Pi} = 0$ ), or:

$$\kappa = \bar{\Pi} = 0, \quad \text{and } \zeta = 1. \quad (29)$$

Finally we examine the behavior of interest rates in the case of  $\lambda = 1$  for the utility function in (12). Denoting  $\hat{r}_t = r_t - \bar{r}$  and linearizing equation (13) we have:  $\hat{r}_t = -(1 + \rho) (\bar{r}/\bar{m}) \hat{m}_t$ . Replacing  $\hat{m}_t$  by (26) we have:

$$\hat{r}_t = -(1 + \rho) \frac{\bar{r}}{\bar{m}} \kappa z_t, \quad t \geq 0. \quad (30)$$

For instance, in the case where  $z_{t+1} = \theta z_t + \epsilon_{t+1}$  and  $\nu = 1$  we have

$$\hat{r}_t = (1 + \rho) \left( -\alpha \frac{\bar{r}}{\bar{m}} \right) \frac{\theta}{1 - \phi\theta} \hat{\mu}_t. \quad (31)$$

Since  $\alpha < 0$  under our maintained assumptions, interest rates move in the same direction than  $\hat{\mu}_t$ , and hence there is *no liquidity effect in the standard model*. Additionally, nominal interest rates inherit the persistence of  $\hat{\mu}_t$ : in the case where  $\theta = 0$ , so that  $\hat{\mu}_t$  is i.i.d., then the nominal interest rate is constant.

## 5 Interest Rates with Segmented Markets

This section analyzes interest rates for the following fiscal-monetary policy. We consider a steady state, i.e a value of  $\bar{m}$  for the aggregate balances that corresponds to a constant money growth rate  $\bar{\mu}$  (the unconditional mean of the process for money growth). We set the fiscal policy as follows:  $\tau_t^N = \bar{\tau}$ , and  $\tau_t^T = 0$ , and endow the traders with an initial bond position that allows them to buy the seignorage not allocated to the non-traders, as outlined in [Section 2.1](#). In the absence of shocks –in a steady state– traders and non-traders are symmetric. Yet when there are shocks, traders must absorb all innovations to the money supply.

While our focus is on interest rates, for simplicity we analyze the equilibrium where there is no active bond markets, which in [Section 2.1](#) was shown to be equivalent to one where all the money injections are carried out through open market operations. In particular, as explained above, the nature of the equilibrium is recursive: we first solve for the process for inflation in the aggregate model, then we solve for the non-trader’s problem –which is done in the next subsection– and finally, using the equivalence of the allocations with and without an active bond market, we solve for interest rates.

We characterize analytically the response of interest rates to money growth shocks with different persistence. [Equation \(36\)](#) gives a closed form solution for the impulse response of interest rates to a monetary shock, [Proposition 8](#) expresses all its coefficients in terms of structural parameters for the case of a once an for all increase in money, and [Section 5.2.2](#) analyzes the cases with persistent increases in money growth. These results allow us to discuss the conditions under which interest rates responses are Fisherian, or display a liquidity effect, and if so how persistent. Furthermore, we show that in the presence of segmentation the short run interest elasticity of the money demand is much smaller than the long run elasticity. [Section 5.2.3](#) defines the short and long term interest rate elasticity of money demand, which is characterized in [Section 5.2.3](#). Both the presence of a persistent liquidity effect after a

monetary shock, and the small interest elasticity of money demand over the short run, have been documented by several authors, see for example section 4.2.2 in [Christiano, Eichenbaum, and Evans \(1999\)](#).

## 5.1 The Non-Trader problem

In this section we consider the problem of a non-trader choosing  $\{n_t\}_{t=0}^{\infty}$ , facing a constant real lump sum transfer  $\tau^N$ , a given process for inflation  $\{\pi_{t+1}\}_{t=0}^{\infty}$  and given initial condition  $m_0 = n_{-1}/\pi_0 + \tau^N$ . We simplify the notation in this section by dropping the superindex  $N$  from  $n$ ,  $c$ , and  $m$ , whenever it is clear from the context.

We start by studying the non-trader problem assuming inflation is constant at  $\bar{\pi} \geq 1$ . In this case the state of the problem is given simply by  $n_{t-1}$ . We solve for the optimal decision rule  $g(\cdot)$ , that gives  $n_t = g(n_{t-1})$ , and find conditions under which it has a unique steady state  $\bar{n} = g(\bar{n})$  that is globally stable. Furthermore we characterize the local dynamics of this problem, i.e. the value of  $g'(\bar{n})$ . In the second part of this section we use these results to characterize the solution of the linearized Euler equation when inflation follows an arbitrary process.

The next proposition uses the Bellman equation for the non-trader problem with  $\bar{\pi} > 1$ :

$$V(n) = \max_{0 \leq \tilde{n} \leq y + \tau^N + n/\bar{\pi}} \{U(y + \tau^N + n/\bar{\pi} - \tilde{n}, n/\bar{\pi} + \tau^N) + \beta V(\tilde{n})\}. \quad (32)$$

to characterize the policy function  $\tilde{n} = g(n)$ , and the uniqueness and stability of the steady state. The proposition also characterizes the value of  $g'(\bar{n})$ , which is important to determine the speed of convergence to the steady state for the non-trader problem.

**PROPOSITION 4.** Assume  $\bar{\pi} > 1$ ,  $\tau^N > 0$ , that  $U$  is strictly concave and bounded above, and  $0 < U_{12} < -U_{11}$ . Then the function  $g(n)$  is strictly increasing, it has a unique interior steady state  $\bar{n} = g(\bar{n})$  that is globally stable, with  $0 < g'(\bar{n}) < 1$ .

Using the decision rule  $g(\cdot)$  and the budget constraint, we can define the optimal consumption rule. For future reference, it turns out to be more convenient to use  $m$  (real cash balances after the transfer) as the state. The budget constraint of the agent is given by  $m = n/\pi + \tau^N$  and  $m + y = c + g(n)$ . Thus  $c(m) \equiv m + y - g([m - \tau^N] \pi)$  and  $c'(m)$  is given by  $1 - \pi g'(n)$ . The elasticity of the ratio  $m/c(m)$  with respect to  $m$  is:

$$\chi(m) \equiv \frac{m}{m/c(m)} \frac{\partial (m/c(m))}{\partial m} = 1 - \frac{m}{c(m)} \frac{\partial c(m)}{\partial m} = 1 - \frac{m}{c(m)} (1 - \pi g'(n)). \quad (33)$$

We are interested in this elasticity because the interest rate response to money shocks depends

on the changes of the  $m/c$  ratio. Equation (33) shows that the eigenvalue  $g'(\bar{n})$ , determining the persistence of the response to monetary shocks, also determines the impact effect of the monetary shock  $\chi$ . Since the nominal interest rate is proportional to the  $m/c$  ratio of traders, see e.g. equation (13), a zero value of  $\chi$  yields no liquidity effect, positive values yields a liquidity effect.

In the next proposition we specialize the utility function to  $U$  CRRA and  $h$  CES, as described in equation (12), and characterize the slope  $g'(\bar{n})$  and the elasticity  $\chi(\bar{m})$ .

**PROPOSITION 5.** Assume  $U$  is given by equation (12),  $0 < U_{12} < -U_{11}$  and  $\bar{\pi} > 1$ . For any values of the triplet  $\rho$ ,  $r = \bar{\pi}/\beta - 1$  and  $\bar{m}/\bar{c} > 1$ , let  $\mathcal{A}$  be such that  $r = U_2/U_1$  evaluated at  $\bar{m}/\bar{c}$ . Then  $g'(\bar{n})$  and  $\chi(\bar{m})$  depend only on  $\bar{m}/\bar{c}$ ,  $\gamma/(1 + \rho)$ ,  $\bar{\pi}$ ,  $\beta$ . Moreover:

$$0 < g'(\bar{n}) < 1 \quad , \quad 0 \leq \chi(\bar{m})$$

and  $g'(\bar{n})$  and  $\chi(\bar{m})$  are increasing in the ratio  $\gamma/(1 + \rho)$ .

Recall that  $1/\gamma$  is the intertemporal elasticity of substitution of the bundle  $h$ , and that  $1/(1 + \rho)$  is the intratemporal elasticity of substitution between cash balances and consumption. Proposition 5 shows that, somewhat surprisingly, the values of  $g'(\bar{n})$  and  $\chi(\bar{m})$  depend on the ratio between the elasticities,  $\gamma/(1 + \rho)$ , as opposed to the values of  $\gamma$  and  $1 + \rho$  separately. The proposition establishes that  $\chi$ , the elasticity of  $m/c$ , is increasing in the ratio:  $\gamma/(1 + \rho)$ . Intuitively, a smaller value of  $\chi$  indicates that  $m/c$  is less responsive to monetary shocks, and hence that the liquidity effect is smaller. For instance,  $\chi = 0$  is zero when  $\gamma = 0$ : in this case the ratio  $m/c$  remains constant, i.e. the adjustment occurs along the inter-temporal margin, which has an infinite substitution elasticity. Analogously  $\chi = 0$  when  $1/(1 + \rho) = 0$ , since in this case the intra-temporal substitution elasticity between  $m$  and  $c$  is nil, and no liquidity effect arises.

This result can be understood by considering the behavior of an agent who starts with cash balances below the steady state  $\bar{m}$ . To reach the steady state the agent must reduce consumption. If the reduction in consumption is large, then convergence to the steady state is fast, and the liquidity effect is short lived. To see why this decision depends on  $\gamma/(1 + \rho)$ , let us consider a case where the convergence to the steady state is fast, i.e.  $g' \approx 0$ . According to Proposition 5 this can happen either because of a small  $\gamma$  or a high  $1 + \rho$ . Let us consider each of these possibilities in turn.

Consider a case where  $\gamma > 0$ , but cash and consumption are poor substitutes, so that  $1/(1 + \rho) \approx 0$  and  $\gamma/(1 + \rho) \approx 0$ . In this case, the agent would like to keep  $m/c$  almost constant, which implies that  $\chi \cong 0$ , and since the initial decrease in consumption is large then  $g'$  must be small. Alternatively, a similar behavior is produced if the intertemporal

substitution elasticity is high. Let  $1/(1 + \rho) > 0$  and assume that the bundles are very close substitutes intertemporally, so that  $1/\gamma \approx \infty$  and  $\gamma/(1 + \rho) \approx 0$ . Since the agent substitutes intertemporally very easily then the speed of convergence is high, or  $g'$  is small, and thus it must be that the initial drop in consumption is large, or  $\chi \cong 0$ .

These examples illustrate why only the ratio matters. To see why  $g'$  (hence  $\chi$ ) is increasing in  $\gamma/(1 + \rho)$  consider now the case where  $\gamma$  is large, so the agent dislikes large variations in the bundle  $h$ . In this case, the initial drop in  $m$  will be partly compensated by not decreasing  $c$  too much, and hence  $\chi > 0$ , and it can be close to one. This also implies that the adjustment to steady state will take more time, or that  $g'$  is high.

We conclude this part with two comments on the role of the steady state value of  $\bar{m}/\bar{c}$  in [Proposition 5](#). First, as a technical comment notice that the condition that  $\bar{m}/\bar{c} > 1$  is only sufficient, but not necessary, for  $0 < g'(\bar{n}) < 1$ .<sup>10</sup> Second, and more importantly, the result on the dependence of the speed of convergence on the ratio  $m/c$  is less standard, but it should be clear in this context. If the stock of money is very small relative to consumption, the effect of starting with a value of this stock below steady state can be quickly corrected: if  $\bar{m}/\bar{c} = 1$  then  $g'(\bar{n}) = 0$ , so the steady state is attained immediately. In other words, if the stock  $m$  is small relative to the flow  $c$ , it must be that the length of the model period is so big that it makes the analysis of convergence uninteresting. While this property was derived for a money-in-the-utility function, we think that inventory theoretical models on the lines of Baumol-Tobin, Miller and Orr, [Alvarez and Lippi \(2009\)](#), must display similar behavior. Indeed, in [Appendix E](#) we analyze the continuous time version of the non-trader problem, that deals more naturally with the stock/flow distinction. In this case the corresponding results of [Proposition 5](#) holds for any value of  $\bar{m}/\bar{c}$ .

So far we have analyzed the problem for a non-trader when inflation is constant. Now we move to the problem of the non-trader facing the (linearized) equilibrium process for inflation. We use the steady state  $\bar{n}$  to define  $\hat{n}_t$ , the deviations of the end-of-period real cash balances  $\hat{n}_t \equiv n_t - \bar{n}$ . Notice that a bounded process  $\{n_t\}_{t=0}^{\infty}$  satisfying the Euler equation [\(6\)](#) is a solution to the non-trader's problem. Now we are ready to state a characterization of the linearized solution to the non-traders' Euler equation. Replacing the budget constraint [\(2\)](#) into the Euler equation [\(6\)](#) for non-traders with constant lump sum transfers  $\tau_t^N = \tau^N$ , and linearizing with respect to  $(c, m, \pi)$  around the values  $(\bar{c}, \bar{m}, \bar{\pi})$  we obtain:

$$\mathbb{E}_t[\hat{n}_{t+1}] = \xi_0 \hat{n}_t - \frac{1}{\beta} \hat{n}_{t-1} + \xi_1 \hat{\pi}_t + \xi_2 \mathbb{E}_t[\hat{\pi}_{t+1}]. \quad (34)$$

---

<sup>10</sup>The following case obtains  $0 < g'(\bar{n}) < 1$ , without requiring  $\bar{m}/\bar{c} > 1$ . Notice that [Proposition 4](#) holds if  $\gamma > 1$ , since in this case the utility is bounded above, and it only requires that in addition  $U_{11} + U_{12} < 0$ , which is weaker than  $\bar{m}/\bar{c} > 1$ .

where the coefficients  $\xi_i$  are functions of the second derivatives of  $U$  evaluated at the steady state as well as  $\beta$  and  $\bar{\pi}$ , given by [equation \(A-10\)](#) in [Appendix A.7](#). The next proposition assumes that inflation is governed by the linearized equilibrium described in [Section 4](#):

**ASSUMPTION 1.** The deviation of inflation and real balances from their steady state value  $\{\hat{\pi}_t, \hat{m}_t\}_{t=0}^\infty$  are given by the linear stochastic difference equation with exogenous driving shocks  $\{z_{t+1}\}_{t=0}^\infty$  described by the matrix and vectors  $\{\Theta, \zeta, \kappa\}$  as detailed in [equation \(26\)](#). They imply that  $\mathbb{E}_t[\hat{\pi}_{t+1}] = \bar{\Pi} z_t$  as given in [equation \(28\)](#).

The state for the dynamic program of the non-trader problem is given by the last period real balances  $\hat{n}_{t-1}$ , and the variables needed to forecast inflation, which given the linear representation in [Section 4](#), are  $(z_t, \hat{m}_{t-1})$ . The next proposition characterizes the solution  $\hat{n}_t = \hat{g}(\hat{n}_{t-1}, z_t, \hat{m}_{t-1})$  for the linearized Euler equation.

**PROPOSITION 6.** Assume that  $U$  is bounded from above, that  $0 < U_{12} < -U_{11}$ , and that the deviation of inflation and real balances from their steady state value  $\{\hat{\pi}_t, \hat{m}_t\}_{t=0}^\infty$  are given by [Assumption 1](#). The unique bounded solution of the linearized Euler equation is given by:

$$\hat{n}_t = \hat{g}(\hat{n}_{t-1}, z_t, \hat{m}_{t-1}) = \varphi_0 \hat{n}_{t-1} + \varphi_1 z_t + \varphi_2 \hat{m}_{t-1} \quad (35)$$

where the  $\varphi_0$  coefficient satisfies:  $0 < \varphi_0 = g'(\bar{n}) < 1$ , and where  $\varphi_i$  are functions of the coefficients  $\xi_i$  of the linearization of the Euler equation, the parameters  $\beta, \bar{\pi}, \bar{m}$ , and the coefficients describing the (linearized) equilibrium law of motion  $\kappa, \Theta, \zeta$  and  $\bar{\Pi}$  are given by [equation \(A-10\)](#) in [Appendix A.7](#).

The result that the slope of the linear optimal policy  $\hat{g}(\cdot)$  that solves the linearized Euler  $\partial \hat{g} / \partial \hat{n} \equiv \varphi_0$  is the same as the slope at the steady state of the optimal decision rule for the non-linear problem  $g'(\bar{n})$ , is a standard one. As it is standard,  $\varphi_0$  is the (stable) solution of a quadratic equation with coefficients defined by  $\xi_0$  and  $\beta$ . Recall that if the growth rate of money  $\hat{\mu}_t$  is i.i.d. then real balances are constant (see [equation \(29\)](#)), inflation is i.i.d., and hence expected inflation is constant or:  $\Theta = \kappa = \bar{\Pi} = 0, \zeta = 1$ . In this case, using our notation for  $\hat{g}(\cdot)$  and the expressions [\(A-10\)](#) for the coefficients  $\varphi_i$  in [Appendix A.7](#) we obtain that

$$\varphi_1 = -\beta \varphi_0 \xi_1 = -\varphi_0 \frac{\bar{n}}{\bar{\pi}}$$

so that for all  $t \geq 0$ , we have  $\hat{m}_t = 0$ , and thus  $\hat{n}_t = \varphi_0 (\hat{n}_{t-1} - \frac{\bar{n}}{\bar{\pi}} \hat{\mu}_t) + \varphi_2 \hat{m}_{t-1}$ . If the economy starts with  $\hat{m}_{-1} = 0$ :

$$\hat{n}_t = g'(\bar{n}) \hat{n}_{t-1} - g'(\bar{n}) \frac{\bar{n}}{\bar{\pi}} \hat{\mu}_t$$

so  $\hat{n}_t$  follows an autoregressive process of order one, with parameter  $g'(\bar{n})$ , and innovations that are proportional to inflation  $\hat{\pi}_t = \hat{\mu}_t$  with a (negative) coefficient given by  $-g'(\bar{n})\bar{n}/\bar{\pi}$ .

As a summary, we describe the evolution of the state of the economy and its dynamics in the next proposition.

**PROPOSITION 7.** The state of the (linearized) equilibrium for economy at time  $t$  can be taken to be  $(\hat{n}_{t-1}, \hat{m}_{t-1}, z_t)$ . Their law of motion is given by equations (26) for  $(\hat{m}_t, z_t)$  and equation (35) for  $\hat{n}_t$ . The eigenvalues of the system are given by those on the exogenous state  $z_t$ , i.e. the eigenvalues of the matrix  $\Theta$ , and by the pair  $(0, \varphi_0)$ .

The zero endogenous eigenvalue comes from the long run money demand, since we can write  $\hat{m}_t = 0\hat{m}_{t-1} + \kappa z_t$ . The other endogenous stable eigenvalue  $\varphi_0 = g'$  comes from the dynamics of the non-trader's problem and depends only on the ratio of the elasticities. Thus the dynamics of any equilibrium variable, including interest rates, depends only on these eigenvalues.

## 5.2 Interest Rates in a Linearized Equilibrium

In this section we use the aggregation result of Section 3, the inflation dynamics of Section 4 and the characterization of the non-trader's dynamic problem of Section 5.1 to solve for the effect of open market operations on interest rates.

We are interested in the following particular monetary-fiscal policy. Non-traders receive a constant real transfer per period, equal to the steady state value of seigniorage. Traders receive the remaining part of the seigniorage. The deviation from the steady state growth of money supply evolves according to  $\hat{\mu}_t = \nu z_t$ , for an exogenous process  $z_t$  as described in (25). Equivalently, as shown in Section 2.1, we can regard this equilibrium as one in which non-traders receive a constant real tax rebate  $\tau_t^N$  and traders receive no lump sum rebate but participate in open market operations. Returning to the equilibrium without an active bond market, the values of  $\tau_t^N$  and  $\tau_t^T$  are given as follows. Let  $\bar{m}(\bar{\mu} - 1)/\bar{\mu}$  be the average seigniorage,

$$\tau_t^N = \frac{\bar{m}(\bar{\mu} - 1)}{\bar{\mu}} \quad \text{and} \quad \tau_t^T = \frac{1}{\lambda} \left[ \frac{M_t - M_{t-1}}{P_t} \right] - \frac{(1 - \lambda)\bar{m}(\bar{\mu} - 1)}{\lambda \bar{\mu}}.$$

where  $\bar{m}$  solves  $\frac{U_2(y, \bar{m})}{U_1(y, \bar{m})} = \bar{r} \equiv \frac{\bar{\mu}}{\beta} - 1$ . In steady state (i.e. when  $\mu_t = \bar{\mu}$  all  $t$ ), the value of  $\tau_t^T$  is also constant, and hence  $\tau_t^N = \tau_t^T = \bar{m}(\bar{\mu} - 1)/\bar{\mu}$ . It is straightforward to verify that these choices satisfy the government budget constraint (5). Also it is easy to verify that with these choices for  $\tau_t^N$  and  $\tau_t^T$  if  $M_t$  grows at a constant rate  $\bar{\mu}$ , then traders and non-traders will have the same consumption and money holdings,  $\bar{m}^T = \bar{m}^N = \bar{m}$  and  $\bar{c}^T = \bar{c}^N = y$ .

Now we turn to the determination of the path of interest rates. To do so, let's use a first order approximation around  $\mu_t = \pi_t = \bar{\mu}$  and  $m_t = m_t^T = m_t^N = \bar{m}$ ,  $c_t^T = c_t^N = y$  where  $\hat{r}_t = r_t - \bar{r}$ . Linearizing the f.o.c. of the traders (11) with respect to  $c^T$  and  $m^T$ , replacing  $\hat{c}_t^T$ ,  $\hat{m}_t^T$  using market clearing for goods and money (3) to write the resulting expression in terms of  $\hat{c}_t^N$ ,  $\hat{m}_t^N$ , and using that the elasticity of substitution between  $m$  and  $c$  is given by  $1/(1 + \rho)$ , we obtain

$$\frac{\hat{r}_t}{\bar{r}} = -(1 + \rho) \left( \frac{\hat{m}_t^T}{\bar{m}} - \frac{\hat{c}_t^T}{\bar{c}} \right) = -\frac{(1 + \rho)}{\lambda} \left( \frac{\hat{m}_t}{\bar{m}} - (1 - \lambda) \left( \frac{\hat{m}_t^N}{\bar{m}} - \frac{\hat{c}_t^N}{\bar{c}} \right) \right)$$

Finally we use this equation to solve for interest rates as follows. The term  $\hat{m}_t$  is determined by the equilibrium in the aggregate economy, i.e.  $\hat{m}_t = \kappa z_t$ . The terms  $\hat{m}_t^N$ ,  $\hat{c}_t^N$  are determined by the solution of the non-trader problem. Using the budget constraint of the non-trader we have  $m_t^N = n_{t-1}^N/\pi_t + \tau^N$  and  $c_t^N = y + n_{t-1}^N/\pi_t + \tau^N - n_t^N$ . Linearizing these expressions gives

$$\frac{\hat{m}_t^N}{\bar{m}} - \frac{\hat{c}_t^N}{\bar{c}} = \hat{n}_{t-1} \frac{1}{\bar{\pi}} \left( \frac{1}{\bar{m}} - \frac{1}{y} \right) + \frac{\bar{n}}{\bar{\pi}^2} \left( \frac{1}{y} - \frac{1}{\bar{m}} \right) \hat{\pi}_t - \frac{1}{y} \hat{n}_t .$$

Using the decision rule of non-traders:  $\hat{n}_t = \varphi_0 \hat{n}_{t-1} + \varphi_1 z_t + \varphi_2 \hat{m}_{t-1}$ , to replace  $\hat{n}_t$ , and that inflation dynamics are given by  $\hat{\pi}_t = \left(\frac{\bar{\mu}}{\bar{m}}\right) \hat{m}_{t-1} + \zeta z_t$  we obtain:

$$\begin{aligned} \frac{\hat{r}_t}{\bar{r}} &= \frac{(1 + \rho)(1 - \lambda)}{\bar{\pi} \lambda} \left[ \frac{1}{\bar{m}} - \frac{1}{y} + \frac{\bar{\pi} \varphi_0}{y} \right] \hat{n}_{t-1} \\ &+ \frac{(1 + \rho)}{\lambda} \left( \frac{(1 - \lambda)}{\bar{\pi}} \left[ \frac{\bar{n}}{\bar{\pi}} \left( \frac{1}{y} - \frac{1}{\bar{m}} \right) \zeta + \frac{\bar{\pi} \varphi_1}{y} \right] - \frac{\kappa}{\bar{m}} \right) z_t \\ &- (1 + \rho) \frac{(1 - \lambda)}{\lambda} \left[ \frac{\bar{n}}{\bar{\pi}^2} \left( \frac{1}{y} - \frac{1}{\bar{m}} \right) \left( \frac{\bar{\mu}}{\bar{m}} \right) + \frac{\varphi_2}{y} \right] \hat{m}_{t-1} . \end{aligned} \quad (36)$$

The interest rate is a function of  $(\hat{n}_{t-1}, z_t, \hat{m}_{t-1})$ . The variables  $\hat{m}_{t-1}$  and  $\hat{n}_{t-1}$  encode the distributional effects between traders and non-traders. Indeed, as  $\lambda \rightarrow 1$ , the expression for the interest rate converges to [equation \(30\)](#) and, as already noticed, there is no liquidity effect.

### 5.2.1 Unexpected once-and-for-all increase in the money supply

This section studies the impulse-response of nominal interest rates when the growth rate of money supply follows an iid process. Equivalently, we analyze the effect of starting the system at the steady state corresponding to  $\bar{\mu}$  and then shock it with an unexpected transitory one time increase in the growth rate of the money supply at  $t$ , i.e.,  $\mu_t > \bar{\mu}$  and  $\mu_{t+s} = \bar{\mu}$  for all

$s \geq 1$ , i.e. a once and for all permanent increase in the level of the money supply.

Let the initial conditions  $\hat{m}_{t-1} = \hat{n}_{t-1}^N = 0$ , if  $\hat{\mu}_t$  is iid we have that, as shown in [equation \(29\)](#),  $\kappa = \Theta = 0$ ,  $\nu = \zeta = 1$ . This gives

$$\hat{\pi}_t = \hat{\mu}_t > 0, \quad \hat{\pi}_{t+s} = \hat{\mu}_{t+s} = 0 \quad \text{all } s \geq 1, \quad \text{and } \hat{m}_{t+s} = 0, \quad s \geq 0.$$

For the non-traders we have that for all  $s \geq 0$ :

$$\hat{n}_{t+s} = \varphi_0 \hat{n}_{t+s-1} + \varphi_1 \hat{z}_{t+s} + \varphi_2 \hat{m}_{t+s-1} = \varphi_0 \hat{n}_{t+s-1} = [\varphi_0]^s \hat{n}_t.$$

Using  $\bar{\pi} = \bar{\mu}$  and that for iid shocks  $\varphi_1 = -\varphi_0 \frac{\bar{n}}{\bar{\pi}}$  we derive the following proposition, which assumes that  $m/c > 1$  (a condition related to the choice of time units discussed in the comment to [Proposition 5](#)).

**PROPOSITION 8.** The effect of an unexpected once and for all increase in the money supply at time  $t$  of size  $(\mu_t - \bar{\mu}) / \bar{\mu}$  is to decrease interest rates on impact, and gradually return to the steady state value  $\bar{r}$ , according to

$$\frac{\hat{r}_{t+s}}{\bar{r}} = - \frac{(1-\lambda)(1+\rho)\bar{n}}{\lambda\bar{m}\bar{\pi}} \chi(\bar{m}) [\varphi_0]^s \left( \frac{\mu_t - \bar{\mu}}{\bar{\mu}} \right) \quad (37)$$

for all  $s = 0, 1, 2, \dots$ , where  $\chi(\bar{m}) = 1 - \frac{\bar{m}}{y}(1 - \bar{\pi}\varphi_0)$ .

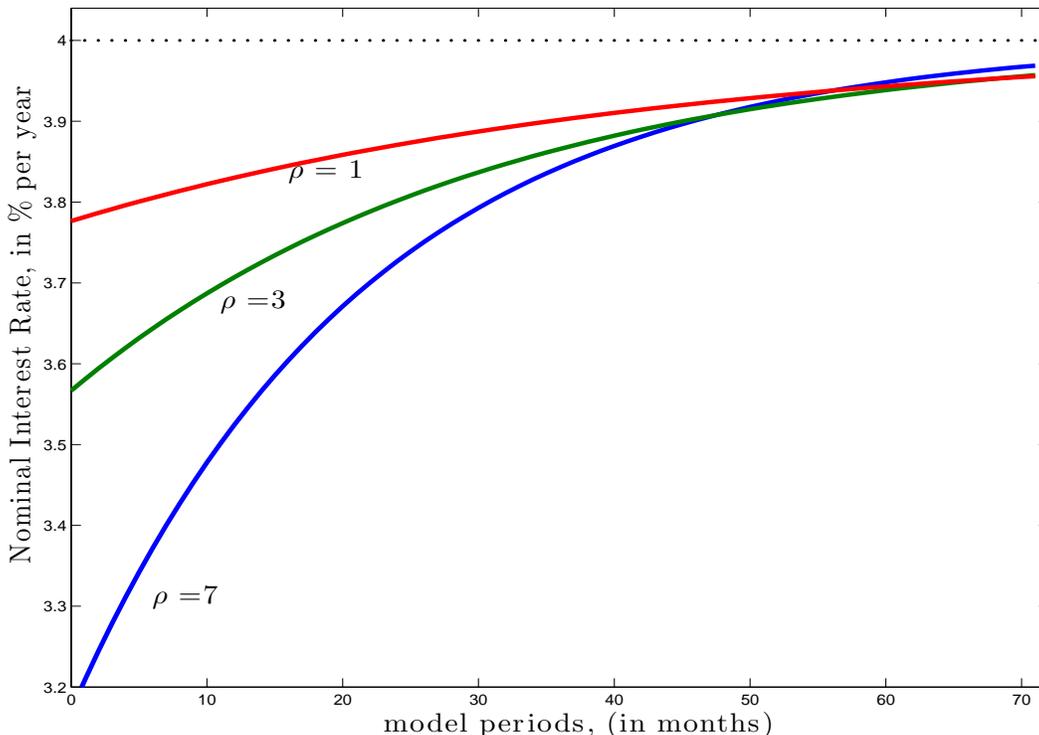
[Equation \(37\)](#) shows that the sign and persistence of the liquidity effect depend on the magnitude of  $\varphi_0$ . The impact effect is negative, i.e. the nominal interest rate decreases when money increases, if  $\chi(\bar{m}) > 0$  a condition established in [Proposition 5](#).

To understand the mechanics of the liquidity effect, note that the effect of an iid shock to money supply to the non-trader is to increase the price level, thus decreasing the post-transfer real money balances  $m$  of the non-trader. If the consumption elasticity is smaller than one then the ratio of money to consumption for the non-traders decreases, i.e.  $\chi(\bar{m}) > 0$ . Since with an iid shock aggregate real balances remain the same, this implies that the ratio of money to consumption must increase for traders. In turn a higher  $m/c$  ratio for traders implies, by [equation \(11\)](#), that the nominal interest rate must decrease.

The size of the impact decrease in interest rate after a once and for all increase in money increases with  $\chi(\bar{m}) > 0$ . Recall that [Proposition 5](#) establishes that  $\chi$  is an increasing function of  $\gamma / (1 + \rho)$ . The persistence of the liquidity effect also depends on the magnitude of  $\varphi_0 = g'(\bar{n})$ , which is also increasing in  $\gamma / (1 + \rho)$ . The closer the value of  $\varphi_0$  is to one, the more persistent the liquidity effect is. Finally, note that the more segmented markets are (the smaller is  $\lambda$ ), the larger the (absolute value of the) liquidity effect at all horizons.

To illustrate this proposition we compute the impulse response to a once and for all shock to the money supply for different parameter values. In all the impulse responses we use annual inflation and real rate of 2 percent,  $U$  given by (12) and choose the value of the parameter  $\mathcal{A}$  to obtain a steady-state value of  $\bar{m}/\bar{c}$  equal to 0.25 at annual frequency. We let the model period to be a month. In Figure 1 we plot the impulse response for a once and for all shock to the money supply which implies a 1 percent increase in the price level on impact.<sup>11</sup> In the figure we use  $\lambda = 0.25$ ,  $\gamma = 4$  and we vary the value of  $\rho$ . As it is clear from Proposition 8, different values of  $\lambda$  scale the distance to the steady state by the same proportion at all horizons.

Figure 1: Response to a once-and-for-all money supply shock



The shock is an unanticipated increase of money causing a 1% increase of the price level. The other parameters are:  $\gamma = 4$ ,  $\lambda = 0.25$ .

Figure 1 shows that for larger values of  $\rho$ , and hence lower elasticity of the long-run money demand, there are larger liquidity effects at impact, with shorter life-times. The value of  $\rho$  has two opposite effects on the impulse response of interest rates, as can be seen from equation (37). The first is a direct effect of the preferences: a large value of  $1 + \rho$

<sup>11</sup>In the case considered in this section, where the money supply follows an iid process, this requires shocking the money supply by the same amount, namely  $(\mu_0 - \bar{\mu})/\bar{\mu} = 0.01$ , see Section 4.1 for details.

raises the sensitivity of the marginal rate of substitution of traders, increasing the effect of a given change in the money-consumption ratio  $m/c$  onto the interest rate. The second effect operates through the equilibrium determination of the elasticity  $\chi$ , that was discussed in [Proposition 5](#). For a fixed value of  $\gamma$  larger values of  $\rho$  decrease  $\chi$  and hence imply a smaller decrease at impact on the ratio  $m/c$ . The impulse responses in [Figure 1](#) show that the first effect almost completely dominates the second one, since the vertical distance between the impulse responses at  $t = 0$ , is almost proportional to the change in the value of  $1 + \rho$ . Additionally, different values of  $\rho$  correspond to different persistence of the liquidity effect, through changes in  $g'(\bar{n})$ . Higher values of  $\rho$ , as shown in [Proposition 5](#), imply faster convergence, as the figure shows.

[Table 1](#) complements [Figure 1](#) by computing two of the determinants of the impulses response of interest rates after a once and for all change in the money supply for different combinations of intertemporal elasticity of substitution  $1/\gamma$ , and intratemporal elasticity of substitution  $1/(1 + \rho)$ . One of the determinants, the half-life of the shock, is a simple transformation of  $\varphi_0$ , and is given by  $\tau \equiv [\log(1/2) / \log(\varphi_0)]$ , expressed in years. The other determinant, denoted by  $\chi$ , is the impact elasticity of  $m/c$  with respect to a change in  $m$ , which is also a simple function of  $\varphi_0$ .

Table 1: Once and for all shock to  $\mu$ : half-life ( $\tau$ , in years), and elasticity of  $m/c$  ( $\chi$ )

$\gamma$	$\rho$	1	3	7
2		$\chi = 0.91$	$\chi = 0.87$	$\chi = 0.82$
		$\tau = 1.77$	$\tau = 1.25$	$\tau = 0.89$
4		$\chi = 0.94$	$\chi = 0.91$	$\chi = 0.87$
		$\tau = 2.53$	$\tau = 1.77$	$\tau = 1.25$
8		$\chi = 0.96$	$\chi = 0.94$	$\chi = 0.91$
		$\tau = 3.63$	$\tau = 2.53$	$\tau = 1.77$

$\tau = \log(1/2) / \log(\varphi_0)$  half-life of the interest shock,  $\chi(\bar{m}) = 1 - \frac{\bar{m}}{c} \frac{dc}{dm}$  elasticity of  $\frac{m}{c}$  w.r.t  $m$  on impact.

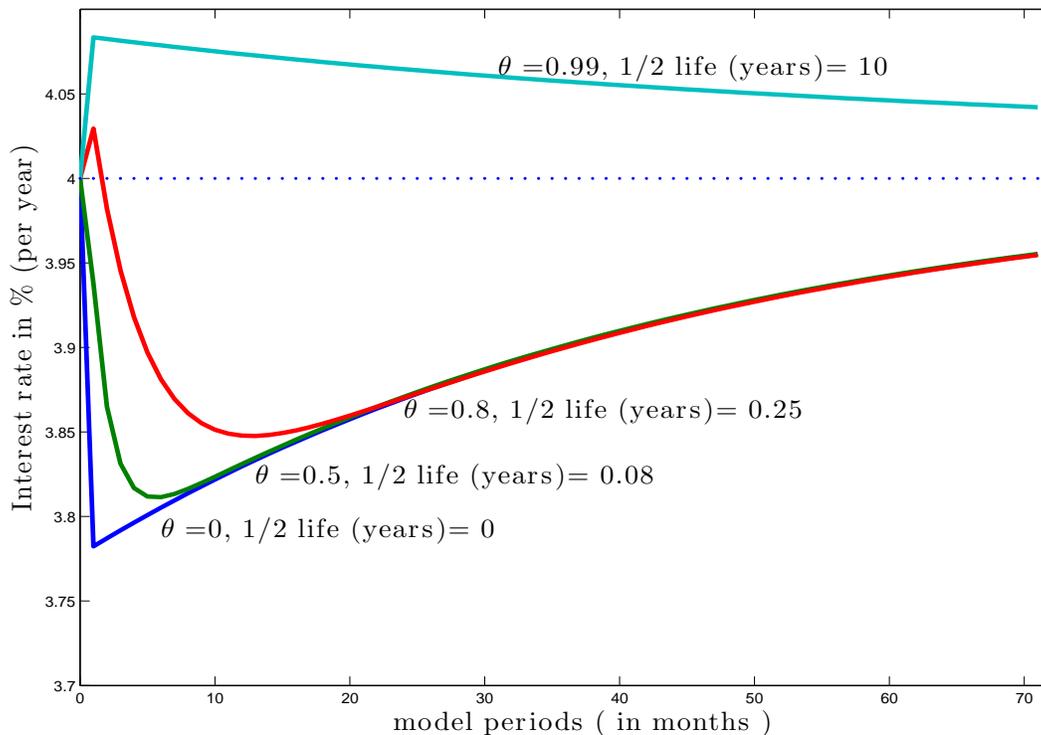
The values for  $\rho$  and  $\gamma$  for [Table 1](#) are chosen so that the ratio  $\gamma / (1 + \rho)$  is constant on the diagonal. The values of  $\tau$  and  $\chi$  across the diagonal of [abreftab-iid-mu-shocks](#) are the same, which follows from [Proposition 5](#), where it is shown that the decision rules  $c(\cdot)$  and  $g(\cdot)$  are functions of the ratio of the elasticities  $\gamma / (1 + \rho)$ . The value of  $\chi$  has the interpretation of the elasticity of the ratio  $m/c$  with respect to an unanticipated once and for all shock to the price level (hence to  $m$ ). The values for this elasticity varies between 0.82 and 0.95 across the values of  $\gamma / (1 + \rho)$  reported in the table. The range of half-lives across the values of

$\gamma/(1 + \rho)$  reported in Table 1 is between a bit less than a year, to more than 3 and a half years.

### 5.2.2 Persistent increase in the growth rate of the money supply

This section analyzes the effect of a persistent increase in the growth rate of the money supply on interest rates. We use the general expression for  $r_t$  in (36), the evolution of the state  $m_t$  given by (26), and the evolution of  $n_t$  given by (35).

Figure 2: Response to a persistent money supply shock



The shock is an unanticipated increase of money causing a 1% increase of the price level. The other parameters are:  $\gamma = 4$ ,  $\lambda = 0.25$ .

Figure 2 plots the impulse responses of interest rates to money shocks under the assumption that the growth rate of the money supply is an AR(1) with autocorrelation  $\theta$ , as opposed to iid. Otherwise the parameters are the ones used in Figure 1 in the case where  $\rho = 1$  (so the long run elasticity of the money demand is 1/2). We plot the impulse response for four values of  $\theta$ , corresponding to a half life of zero months, one month, three months, and ten years. The zero half life coincides with the iid case of Figure 1, and is included to help in the comparisons. The size of the initial shock to money is chosen so that the effect on the price level on impact is an increase of 1 percent, as in the case of iid money growth.

As can be seen from the impulse responses in [Figure 2](#), the monetary shocks with a shorter half-life produce a liquidity effect. If the monetary shock is very persistent, instead, the Fisherian aspects of the model take over, expected inflation rises considerably on impact, and there is no liquidity effect. Notice that for intermediate values of  $\theta$  the impulse response has an inverted hump shape, attaining the minimum some periods after the impact effect. The hump shape of the impulse response is due to the fact that the dynamic system has two eigenvalues:  $\theta$ , governing aggregate real balances and inflation, and  $\varphi_0$ , governing the non-traders adjustment of their real balances. Note that for the three smallest values of  $\theta$ , the impulse response converge to the same line, since the short run behavior is dominated by  $\theta$  and the long run by  $\varphi_0$ . Instead, for the case where  $\theta$  is much larger, the fisherian effect dominates and the impulse response is almost identical to the one where markets are not segmented.

To identify the effect of segmented markets on interest rates, [Figure 3](#) displays the impulse response (to the same shocks) for the model with  $\lambda = 1$ , which has no liquidity effects.<sup>12</sup> When shocks are short lived, so that there are no movements in the expected growth rate of money, interest rates remain almost constant at the steady state level. Comparing [Figure 2](#) and [3](#) shows that when monetary shocks are very persistent the behavior of interest rates in the model with segmented markets ( $\lambda = 0.25$ ) is similar to the one in model with homogenous agents ( $\lambda = 1$ ).

### 5.2.3 Short vs Long Run Money Demand Elasticities and the Liquidity Effect

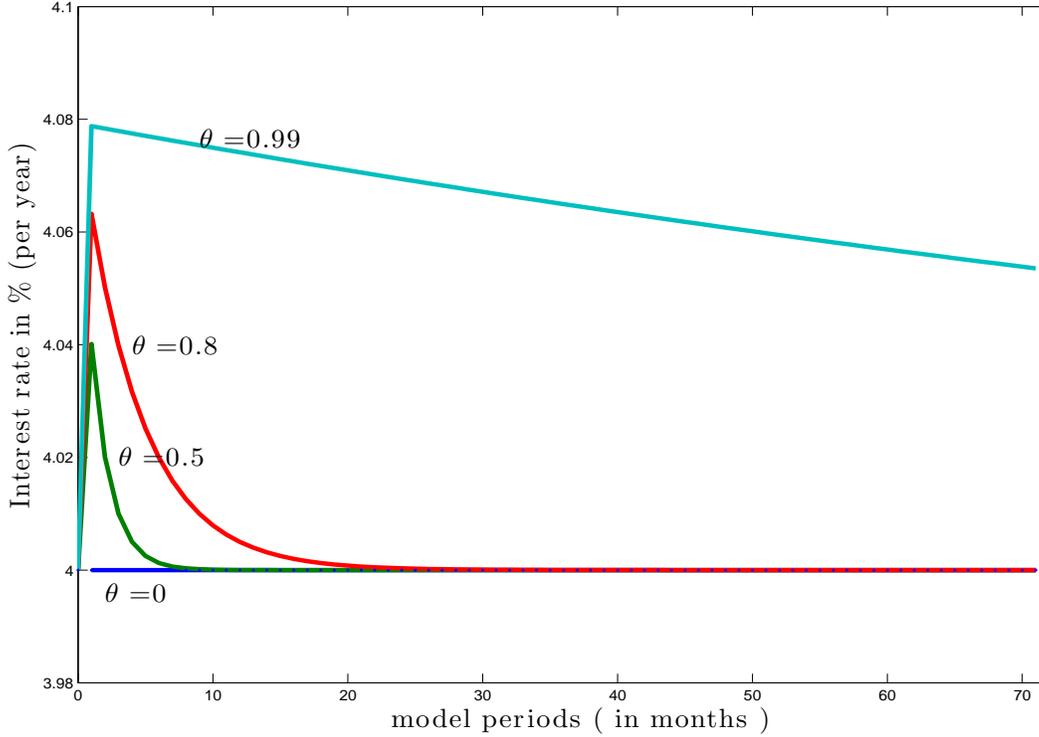
We conclude the section with a comment on the relation between the liquidity effect and the interest elasticity of money demand. The thought experiment that reveals a liquidity effect on interest rate is an open market operation, i.e. an increase of the money supply. Instead, the slope of the money demand is a relationship between real money balances –or velocity– and interest rates. As explained, in this model the “long run” interest-elasticity of the money demand is  $-1/(1 + \rho)$ . The liquidity effect of an increase in the (nominal) money supply, too, depends on  $\rho$ , among other parameters.

As done in the literature –see e.g. [Christiano, Eichenbaum, and Evans \(1999\)](#)–, we define as the “short-run money demand elasticity” the ratio of the impact effect on real balances relative to the impact effect on interest rates following a monetary shock. We argue that there is no “constant” short-run elasticity of the money demand in the model. We emphasize that this is consistent with the unstable estimates of the interest-elasticity of money demand equations that are obtained using high-frequency data.

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<sup>12</sup>As shown in [Section 4](#), in the model with  $\lambda = 1$  interest rates move in the same direction than expected changes in money supply.

Figure 3: Response with no segmentation ( $\lambda = 1$ )



The shock is an unanticipated increase of money causing a 1% increase of the price level. The other parameters are:  $\gamma = 4$ ,  $\lambda = 1.0$  (i.e. no segmentation).

To fix ideas consider the case where the growth rate of the money supply follows an AR(1) process with parameter  $\theta$ . From our previous analysis we have that the decrease on impact of aggregate real balances after a shock to the money supply is given by [equation \(26\)](#):

$$\left. \frac{1}{m} \frac{dm}{d\hat{\mu}} \right|_{m=\bar{m}, \hat{\mu}=0} = \frac{\kappa}{\bar{m}} = \frac{\alpha}{\bar{m}} \frac{\theta}{1 - \phi\theta},$$

where  $\alpha < 0$  and  $0 < \phi < 1$ . Hence real balances decrease after a money growth shock, the more so the more persistent is the shock. From our analysis of the impact on interest rates of a monetary shock, [equation \(36\)](#), we have that:

$$\left. \frac{1}{r} \frac{dr}{d\hat{\mu}} \right|_{m=\bar{m}, \hat{\mu}=0, n=\bar{n}} = \frac{(1 + \rho)}{\lambda} \left( \frac{(1 - \lambda)}{\bar{\pi}} \left[ \frac{\bar{n}}{\bar{\pi}} \left( \frac{1}{y} - \frac{1}{\bar{m}} \right) \zeta + \frac{\bar{\pi}\varphi_1}{y} \right] - \frac{\kappa}{\bar{m}} \right)$$

where  $\zeta$  is given in [\(27\)](#) and the expression for  $\varphi_1$  is given in [Appendix A.7](#) by [\(A-10\)](#). Thus

we define the short run elasticity of the money demand as the ratio:

$$\eta \equiv \frac{r}{m} \frac{dm}{dr} = \frac{\frac{1}{m} \frac{dm}{d\hat{\mu}}}{\frac{1}{r} \frac{dr}{d\hat{\mu}}} \Bigg|_{m=\bar{m}, \hat{\mu}=0, n=\bar{n}} .$$

To sign this expression notice that  $(1/m)(dm/d\mu) < 0$ , so the sign of the elasticity  $\eta$  depends on whether there is a liquidity effect or not. If there is a liquidity effect then the elasticity is positive on impact, otherwise it is negative. We consider two interesting special cases:

$$\eta \equiv \frac{r}{m} \frac{dm}{dr} = 0 \text{ if } \theta = 0 \text{ , } \eta \equiv \frac{r}{m} \frac{dm}{dr} = -\frac{1}{1+\rho} \text{ if } \lambda = 1,$$

In the case of a once-and-for-all increase in the money supply ( $\theta = 0$ ) expected inflation is constant and thus aggregate real balances remain constant ( $\kappa = 0$ , and  $\hat{m}_t = 0$ ). Thus the impulse response of a purely transitory shock on the growth rate of the money supply  $\mu_t$  will display a short-run interest elasticity of the money demand equal to zero. Instead, in the case where  $\lambda = 1$ , i.e. when markets are not segmented, the short-run and long-run elasticities are the same since the standard interest elastic money demand equation holds at all frequencies. To illustrate the effect of other parameters' values on this elasticity, [Table 2](#) reports the value of  $\eta$  using three values of  $\theta$  used in [Figure 2](#), denoted by the corresponding half-life of the shocks  $\tau(\theta)$ , two values of  $\rho$ , determining the long-run elasticity  $-1/(1+\rho)$ , and two values of  $\lambda$ , determining the degree of market segmentation. The table shows that when the monetary shock is persistent the short-run elasticity is closer to the long-run elasticity  $-1/(1+\rho)$ , although the limit as  $\theta \rightarrow 1$  depends on  $\lambda$  and does not converge exactly to  $-1/(1+\rho)$ . Instead, when the monetary shock is short-lived, the liquidity effect makes the sign of the short-run elasticity the opposite of the one on the long-run elasticity.

Table 2: Interest elasticity of money demand at various horizons

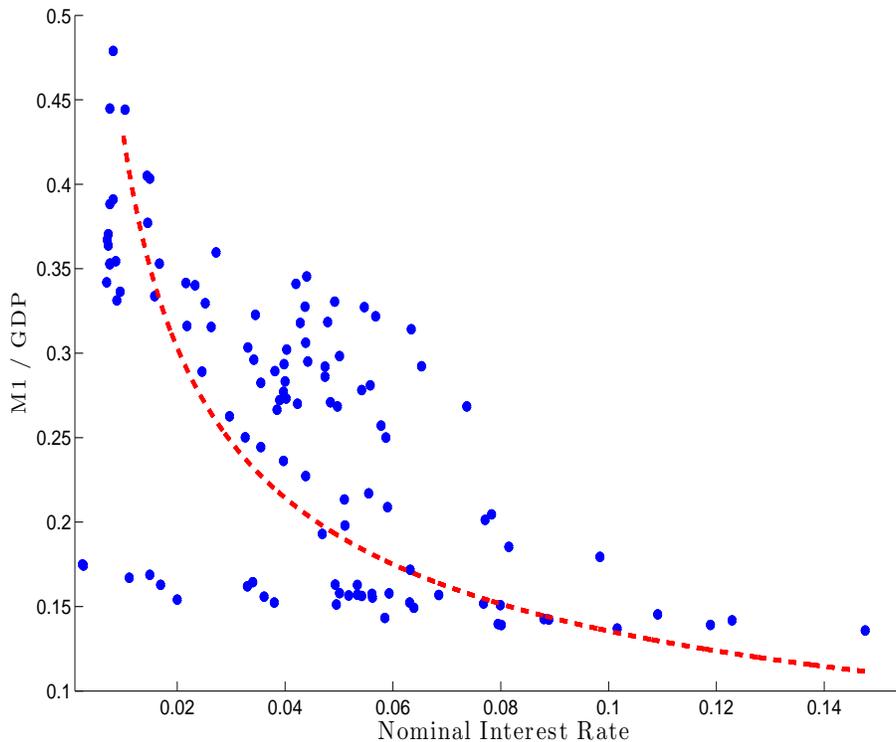
shock half-life: $\tau(\theta)$	$\rho = 1$ , $\lambda = 0.2$	$\rho = 1$ , $\lambda = 0.1$	$\rho = 3$ , $\lambda = 0.1$
1-month	+ 0.34	+ 0.08	+ 0.19
4-months	+ 0.93	+ 1.25	- 0.40
10-years	- 0.47	- 0.43	-0.23

The shock is an unanticipated increase of money causing a 1% increase of the price level. The other parameters are:  $\gamma = 4$ .

## 6 A calibration of the model on the US data

In this section we present a calibration of the model on the US data. The objective is to compare the model predictions with the historical evidence on money demand. In particular, we estimate a stochastic process for the growth rate of M1 that is the sum of two independent AR(1) processes, one with high-frequency (a half life of a few months) the other with low-frequency (a half life of 10 years). We then feed this process to the linearized equilibrium and compare the model predictions to the US data on velocity and interest rates in the 20-th century at different frequencies. Next we describe the data, the calibration of the model, and compare the two.

Figure 4: US Money Demand 1900-2010



Note: Annual data denoted by (blue) dots, the source is [Ireland \(2009\)](#), updated to 2010 data. The (red) dashed line plots a log-log money demand with a  $1/2$  interest elasticity, which pass through the mean of both variables.

The scatter plot in [Figure 4](#) shows the annual US data (blue dots) on the money/income ratio and the nominal interest rate over the period 1900-2006 discussed, among others, by [Lucas \(2000\)](#) and [Ireland \(2009\)](#). With these data in mind, we calibrate a monthly model featuring a money to income ratio equal to 0.25 at annual frequencies, consistent with the

data in the figure, this implies a value of  $\mathcal{A} = 33.33$  for the monthly model. We assume  $U(c, m)$  is CRRA and  $h$  CES as in [equation \(12\)](#) and assume a long-run interest elasticities of money equal to  $1/2$ , which corresponds to a value of  $\rho = 1$ . This yields the long-run money demand schedule in [equation \(13\)](#), i.e. the constant-elasticity dashed-line plotted in the figure (in red). This interest elasticity is the one used by [Lucas \(2000\)](#) in his analysis of the welfare cost of inflation and is consistent with the micro-evidence in [Alvarez and Lippi \(2009\)](#). We set the intertemporal substitution elasticity equal to  $1/4$ , or  $\gamma = 4$ , as is common in the literature and within the range of estimates by [Ogaki and Reinhart \(1998\)](#) and others. The values of  $\beta$  and  $\bar{\mu}$  are chosen so that the (unconditional mean) annual real interest rates and inflation rates are two percent (the values for our monthly model are  $\beta = 0.9983$ , and  $\bar{\mu} = 1.0017$ ). We set the fraction of traders to  $\lambda = 0.5$  (more on this parameter choice below).

What is left to be specified is a process for the (M1) money supply. Recall that a main result from the theoretical analysis of [Section 5.2](#) is that permanent shocks to the money supply will generate data that trace a classic interest-elastic money demand. Hence, if shocks are permanent (or, very long lasting) the model generated data will produce the dashed-line that is depicted in the [Figure 4](#). A regression on these data (in logs) will then uncover an elasticity of  $-1/2$ . If, on the contrary, the money supply shocks are iid then the money/income ratio does not respond to the shocks and, provided that financial markets are segmented, the liquidity shocks will produce data that display a zero elasticity between real balances and the interest rate -whose variance will be increasing in the degree of segmentation. A money supply process that combines independent temporary and permanent shocks will then produce an estimated interest elasticity that is in between 0 and  $1/2$ , depending on the relative importance (variance) of the two shocks, as well as on the degree of segmentation.

We assume the stochastic process for M1 growth,  $\hat{\mu}_t/\bar{\mu}$ , is the sum of two independent AR(1) processes. The calibration below uses an estimate, discussed in [Appendix B](#), where one component is highly persistent, with a monthly autocorrelation equal to 0.995, which corresponds to a half-life of about 12 years. The other component is less persistent, with a monthly autocorrelation 0.5, i.e. a half-life of a month. The (annualized) unconditional standard deviation of these processes is 0.006 and 0.065, respectively, which shows that the largest part of the year to year variation in money growth is explained by the high-frequency innovations.

To characterize the time-series behavior produced by the calibration we compute 500 simulations, each one 100-years long of our monthly model, on real balances  $m$  and the nominal interest rate  $r$  and run the linear regression (on annual simulated data):  $\log(m_t) = \hat{b}_0 + \hat{b}_1 \log(r_t) + \epsilon_t$ . The objective is to compare these simulated data with the annual US data on inverse velocity ( $M1/GDP$ ) and the short term interest rate of [Figure 4](#). The first

Table 3: Actual vs simulated money demand regressions

	US Data		Model	
	[1] Raw Data	[2] HP filtered	[3] Raw Data	[4] HP filtered
$\hat{b}_1$	-0.31	-0.46	-0.25	-0.36
$R^2$	0.45	0.49	0.47	0.69
$DW$	0.09	0.01	0.29	0.01
$N.obs$	107	107	107×500	107×500

Note: All regressions include a constant. The dependent variable is the inverse velocity:  $M1/GDP$  in the data,  $m/c$  in the model. US annual data on M1 and the federal funds rate over 1900-2009. The  $R^2$  is the squared errors of fit statistic,  $DW$  is the Durbin-Watson test. The model statistic are the mean value obtained from 500 simulations. The regressions in columns [2] and [4] use HP filtered data with parameter  $\lambda_{HP} = 10^4$ . The model is solved and simulated at monthly frequency, and the data is annualized, i.e. we use non-overlapping averages during 12 months for each variable.

column of [Table 3](#) reports the results of a regression over the annual US data from the 20th century. The estimated interest elasticity coefficient of the money demand is  $-0.3$ .<sup>13</sup> The low value of the Durbin-Watson statistics points to a large positive autocorrelation of the residuals. We report the Durbin-Watson statistics as a simple way to measure the persistence of the deviations from fitted money demand, which in our model will be related with the persistence of the liquidity effects. The third column of the table reproduces this regression using the model-generated data. We run 500 simulations, each 100 year long, and report in the table the mean value of the estimated regression coefficient (the median values yield similar results),  $R^2$  and Durbin-Watson statistics.

Recall that in the case of no segmentation,  $\lambda = 1$ , there is no liquidity effect, so that the model-generated data obey the “long run” money demand at each point in time and the regression fit is perfect (i.e.  $R^2 = 1$ ). The segmentation of  $\lambda = 0.5$  was thus chosen to align the magnitude of the regression  $R^2$  produced by the model to the one obtained from the data (0.47 vs. 0.45, respectively). The table shows that the interest elasticity detected on the model simulations is not very different from the one that is obtained from the data:  $-0.25$  vs  $-0.31$ , respectively. Moreover, the model data generating process gives rise to serially correlated residuals which, as in the data, yields a very low Durbin-Watson statistics. Notice that the positive serial correlation of shocks is the consequence, in the theoretical model, of the gradual propagation of the liquidity effect.

Another consequence of the liquidity effect (i.e. of the high frequency shocks) is to

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<sup>13</sup> As discussed in [Appendix D.1](#), the estimated interest elasticity ranges between 1/4 and 1/2 in the literature.

produce a downward bias in the estimated long-run interest elasticity. The regression on the model data detect an elasticity of  $-0.25$ , instead of the “true” value of  $-0.5$ , i.e. the value of the parameter  $1/(1 + \rho)$ . As mentioned above, this is due to the presence of the liquidity shocks. The estimated regression coefficient is a weighted average of the conditional correlation (of size  $-0.5$ ) produced by the nearly permanent shocks, and the (near-zero) correlation produced by the transitory shocks. To capture the low frequency correlation implicit in the data the regressions in columns [2] and [4] in [Table 3](#) we use HP filtered data with a weight parameter of 10,000. The resulting trend series, for money and interest rates, are then used in the regression. Consistently with the model predictions the interest elasticity of the money demand becomes larger, both in the US data and in the model simulations.

In this section we characterized the patterns for velocity and interest rates that the model produces by studying the implied correlations revealed by the coefficient of a classic money demand regression. Compared to the quantitative analysis in [Hodrick, Kocherlakota, and Lucas \(1991\)](#), where the correlation between velocity and the nominal rate is essentially zero because of an (almost) always binding cash-in-advance constraint, our model does generate a positive covariance between velocity and interest rates, at least at low frequencies. As in their model, however, the model produces too little variation for the *absolute* variance of interest rates and velocity compared to the data. More shocks are needed, possibly to money demand, to improve the model performance in this dimension.

## 7 Concluding remarks

This paper presented a theoretical mechanism that gives rise to a persistent liquidity effect, and characterized its relationship with a long-run interest-elastic money demand. One key feature of our model is that the presence and the persistence of the liquidity effect is determined by a simple combination of the intertemporal elasticity of substitution and of the intra-temporal elasticity of substitution of cash vs consumption. It was shown that the long-run (i.e. low frequency) properties of money demand determine strength and persistence of the short run liquidity effects. This provided a unified theory for the low and high frequency movements of interest rates. The simplicity of the logic of the argument for the presence of a liquidity effect can be seen by noticing that a once and for all increase of money supply must necessarily imply a liquidity effect.

Our interest in monetary models that feature liquidity effects based on segmented asset markets is to provide a framework for studying policy questions on the effects of monetary shocks, such as those analyzed in e.g. [Lahiri, Singh, and Vegh \(2007\)](#), [Nakajima \(2006\)](#), [Lama and Medina \(2007\)](#), [Khan and Thomas \(2007\)](#), [King and Thomas \(2008\)](#), [Bilbiie \(2008\)](#),

Curdia and Woodford (2008), and Zervou (2008). Comparing with this literature, we have deliberately kept the model as simple as possible. The simplicity allowed us to give a sharp characterization of the theoretical results, such as the relationship between money and prices in the presence of segmentation, and the impact effect and persistence of the liquidity effect as a function of simple elasticities. The disadvantage of this simplicity is that this version of the model lacks some features that may be interesting for some policy questions. For instance, the model has exogenous endowment, flexible prices, and constant exogenous participation rates in the bond markets. In this regard we view the result of the model as applying to the aggregate nominal demand. We think that it is feasible and interesting to extend the model in several dimensions: to have variable inputs, so that the model can be used to study the “real” effects of monetary shocks, and/or to introduce nominal rigidities, so that we can use it to study whether liquidity effects are substitutes or complements of them, or to introduce differential shocks to participants and non-participants in bond markets, to study the welfare effect of interest rate smoothing.

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Online Appendices (NOT FOR PUBLICATION)

Persistent Liquidity Effect and  
Long Run Money Demand

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November 2, 2011

# A Proofs

## A.1 Proof of Proposition 1.

The tilde allocation satisfies market clearing for cash balances and consumption, since they are satisfied in the original equilibrium. Market clearing of bonds is satisfied trivially by construction. The budget constraint of the government is satisfied by construction too, given that the budget constraint for the original equilibrium is satisfied. The tilde allocation solve the problem of the non-traders since their budget constraint is identical. Finally, for traders, at the tilde equilibrium prices and interest rates and chosen initial conditions, the allocation  $\{c_t^T, m_t^T\}_{t=0}^\infty$  is budget feasible, and the f.o.c. for the bond holdings is satisfied. Hence,  $\{c_t^T, m_t^T\}_{t=0}^\infty$  solve the trader's problem. *Q.E.D.*

## A.2 Proof of Proposition 2.

By adding the Euler equation of the traders times  $\lambda$  to the Euler equation of the non-traders times  $(1 - \lambda)$  we obtain the following Euler equation for the aggregate real balances:

$$\bar{U}_{11}\hat{c}_t + \bar{U}_{12}\hat{m}_t = \mathbb{E}_t \frac{\beta}{\bar{\pi}} [(\bar{U}_{11} + \bar{U}_{21}) \hat{c}_{t+1} + (\bar{U}_{12} + \bar{U}_{22}) \hat{m}_{t+1}] - \mathbb{E}_t \frac{\beta}{\bar{\pi}^2} [\bar{U}_1 + \bar{U}_2] \hat{\pi}_{t+1}.$$

Feasibility implies that  $\hat{c}_t = 0$ , hence the Euler equation becomes

$$\bar{U}_{12}\hat{m}_t = \mathbb{E}_t \frac{\beta}{\bar{\pi}} [(\bar{U}_{12} + \bar{U}_{22}) \hat{m}_{t+1}] - \mathbb{E}_t \frac{\beta}{\bar{\pi}^2} [\bar{U}_1 + \bar{U}_2] \hat{\pi}_{t+1}$$

Using the linearized money growth identity in [equation \(17\)](#) to replace  $\hat{\pi}_{t+1}$  into the Euler equation we get

$$\bar{U}_{12}\hat{m}_t = \mathbb{E}_t \frac{\beta}{\bar{\pi}} [(\bar{U}_{12} + \bar{U}_{22}) \hat{m}_{t+1}] - \mathbb{E}_t \frac{\beta}{\bar{\pi}^2} [\bar{U}_1 + \bar{U}_2] \left( \hat{\mu}_{t+1} + \frac{\bar{\mu}}{\bar{m}} \hat{m}_t - \frac{\bar{\mu}}{\bar{m}} \hat{m}_{t+1} \right)$$

or

$$\bar{U}_{12}\hat{m}_t + \frac{\beta}{\bar{\pi}^2} [\bar{U}_1 + \bar{U}_2] \frac{\bar{\mu}}{\bar{m}} \hat{m}_t = \mathbb{E}_t \frac{\beta}{\bar{\pi}} [(\bar{U}_{12} + \bar{U}_{22}) \hat{m}_{t+1}] - \mathbb{E}_t \frac{\beta}{\bar{\pi}^2} [\bar{U}_1 + \bar{U}_2] \left( \hat{\mu}_{t+1} - \frac{\bar{\mu}}{\bar{m}} \hat{m}_{t+1} \right)$$

multiplying by  $\bar{m}$  and using that in the steady state  $\bar{\pi} = \bar{\mu}$  and  $\bar{U}_1 = \beta/\bar{\mu} (\bar{U}_1 + \bar{U}_2)$ :

$$[\bar{U}_{12}\bar{m} + \bar{U}_1] \hat{m}_t = -\mathbb{E}_t \frac{\beta}{\bar{\mu}^2} [\bar{U}_1 + \bar{U}_2] \bar{m} \hat{\mu}_{t+1} + \mathbb{E}_t \frac{\beta}{\bar{\pi}} [\bar{m} (\bar{U}_{12} + \bar{U}_{22}) + \bar{U}_1 + \bar{U}_2] \hat{m}_{t+1} \quad (\text{A-1})$$

which is identical to the Euler Equation for the aggregate model that is obtained by linearizing [\(18\)](#) around the steady state. *Q.E.D.*

## A.3 Proof of Proposition 3.

Linearizing [\(18\)](#) around the steady state gives [equation \(A-1\)](#). Using the definitions for  $\alpha$  and  $\phi$  in [equation \(21\)](#) gives [\(20\)](#). *Q.E.D.*

## A.4 Sufficient conditions for $0 < \phi < 1$ and $\alpha < 0$

LEMMA 1. Let the utility be given by equation (12). Then  $\phi < 1$  and  $\alpha < 0$  if

$$\gamma < 2 + \rho + \frac{1}{r(m/c)} \quad \text{and} \quad \rho > -1$$

If  $\rho = -1$  then  $\phi = 1$ . Finally  $\phi > 0$  if

$$\gamma < \frac{1}{1+r} \left( 2 + \rho + \frac{1}{r(m/c)} + r - \frac{\rho}{(m/c)} \right) .$$

**Proof.** Simple algebra using equation (12) gives

$$\begin{aligned} U_2 &= \left[ c^{-\rho} + \frac{1}{\mathcal{A}} m^{-\rho} \right]^{-\frac{1-\gamma}{\rho}-1} \frac{1}{\mathcal{A}} m^{-\rho-1} \\ mU_{22} &= \left[ c^{-\rho} + \frac{1}{\mathcal{A}} m^{-\rho} \right]^{-\frac{1-\gamma}{\rho}-1} \frac{1}{\mathcal{A}} m^{-\rho-1} (1 - \gamma + \rho) \frac{\frac{1}{\mathcal{A}} m^{-\rho}}{\left[ c^{-\rho} + \frac{1}{\mathcal{A}} m^{-\rho} \right]} \\ &\quad - \left[ c^{-\rho} + \frac{1}{\mathcal{A}} m^{-\rho} \right]^{-\frac{1-\gamma}{\rho}-1} \frac{1}{\mathcal{A}} m^{-\rho-1} (\rho + 1) \end{aligned}$$

and

$$\begin{aligned} U_1 &= \left[ c^{-\rho} + \frac{1}{\mathcal{A}} m^{-\rho} \right]^{-\frac{1-\gamma}{\rho}-1} c^{-\rho-1} \\ mU_{12} &= \left[ c^{-\rho} + \frac{1}{\mathcal{A}} m^{-\rho} \right]^{-\frac{1-\gamma}{\rho}-1} c^{-\rho-1} (1 - \gamma + \rho) \frac{\frac{1}{\mathcal{A}} m^{-\rho}}{\left[ c^{-\rho} + \frac{1}{\mathcal{A}} m^{-\rho} \right]} \end{aligned}$$

thus

$$\phi \equiv (\beta/\bar{\mu}) \left[ 1 + \frac{\bar{U}_2 + \bar{m}\bar{U}_{22}}{\bar{U}_1 + \bar{m}\bar{U}_{12}} \right] = \frac{1}{1+r} \left[ 1 + r \frac{(1-\gamma) r(m/c) - \rho}{(2-\gamma+\rho) r(m/c) + 1} \right] .$$

Sufficient conditions for  $0 < \phi < 1$  are the following. The condition for  $\phi < 1$  requires:

$$\frac{(1-\gamma) r(m/c) - \rho}{(2-\gamma+\rho) r(m/c) + 1} < 1$$

If  $(2-\gamma+\rho) r(m/c) + 1 > 0$ , this inequality is:

$$0 < (1+\rho) (1+rm/c)$$

which holds if  $\rho > -1$ . If  $\rho = -1$ , then  $\phi = 1$ . This establishes the condition for  $\phi < 1$ :

$$\gamma < 2 + \rho + \frac{1}{r(m/c)} \quad \text{and } \rho > -1 \quad .$$

The condition for  $\phi > 0$  requires

$$-1 < r \frac{(1 - \gamma) r(m/c) - \rho}{(2 - \gamma + \rho) r(m/c) + 1}$$

Since the denominator is positive (whenever  $\phi < 1$ ), this inequality implies

$$-(2 - \gamma + \rho) r(m/c) - 1 < r [(1 - \gamma) r(m/c) - \rho]$$

which, after rearranging terms, gives the second inequality in [Lemma 1](#) .

Finally the condition for  $\alpha < 0$  if and only if  $\bar{U}_1 + \bar{m}\bar{U}_{12} > 0$ . Using [equation \(13\)](#) and the expressions computed above for the utility function [equation \(12\)](#) we have that

$$\frac{U_1 + mU_{12}}{\left[ c^{-\rho} + \frac{1}{\mathcal{A}} m^{-\rho} \right]^{-\frac{1-\gamma}{\rho}-1} c^{-\rho-1}} = \frac{(2 - \gamma + \rho) r(m/c) + 1}{1 + r(m/c)}$$

so  $U_1 + mU_{12} > 0$  holds if  $(2 - \gamma + \rho) r(m/c) + 1 > 0$  or, as is immediate to verify, the condition for  $\phi < 1$  holds.

Using the steady state equilibrium condition in [equation \(13\)](#) and the definition for the nominal interest rate in [equation \(19\)](#) we can rewrite the condition for  $\alpha < 0$  and  $\phi < 1$  in terms of exogenous parameters as

$$\gamma < 2 + \rho + \mathcal{A}^{\frac{1}{1+\rho}} \left( \frac{\bar{\mu}}{\beta} - 1 \right)^{-\frac{\rho}{1+\rho}} \quad \text{and } \rho > -1 \quad .$$

*Q.E.D.*

## A.5 Proof of [Proposition 4](#).

Under the stated conditions the value function  $V$  is strictly concave and differentiable. The f.o.c. for this problem is

$$U_1 \left( y + \tau^N + \frac{n}{\bar{\pi}} - g(n), \frac{n}{\bar{\pi}} + \tau^N \right) = \beta V' (g(n)) \quad (\text{A-2})$$

for all  $n \geq 0$ . Since  $V$  is concave,  $V'$  is decreasing and since  $U_{11} + U_{12} < 0$  the LHS of the f.o.c. [\(A-2\)](#), for a fixed value  $n' = g(n)$ , is decreasing in  $n$ , and hence  $g(\cdot)$  is increasing. The envelope gives:

$$V'(n) = \frac{1}{\bar{\pi}} U_1 \left( y + \tau^N + \frac{n}{\bar{\pi}} - g(n), \frac{n}{\bar{\pi}} + \tau^N \right) + \frac{1}{\bar{\pi}} U_2 \left( y + \tau^N + \frac{n}{\bar{\pi}} - g(n), \frac{n}{\bar{\pi}} + \tau^N \right).$$

Using the f.o.c. and the envelope evaluated at steady state we obtain:

$$U_1 \left( y + \tau^N - \bar{n} \left( 1 - \frac{1}{\bar{\pi}} \right), \frac{\bar{n}}{\bar{\pi}} + \tau^N \right) \left( 1 - \frac{\beta}{\bar{\pi}} \right) = \frac{\beta}{\bar{\pi}} U_2 \left( y + \tau^N - \bar{n} \left( 1 - \frac{1}{\bar{\pi}} \right), \frac{\bar{n}}{\bar{\pi}} + \tau^N \right)$$

Under the assumption that  $U_{12} \geq 0$  and  $\bar{\pi} > 1$  it is easy to see that there is a unique steady state  $\bar{n}$  satisfying this equation.

We now show that this steady state is globally stable. Suppose not, i.e. that  $g'(n) > 1$ , and assume that  $n_0 > \bar{n}$ , then  $\lim n_t = \infty$ . But notice that  $V$  is bounded below, since  $V(n) \geq V(0) \geq U(y + \tau^N, \tau^N) / (1 - \beta)$ . Additionally we assume that  $U$  is bounded above. In this case, since  $V$  is concave,

$$V(n_t) \geq V(0) + V'(n_t) n_t$$

and hence as  $n_t \rightarrow \infty$  it must be that

$$V'(n_t) = U_1(n_t/\bar{\pi} + y + \tau^N - n_{t+1}, n_t/\bar{\pi} + \tau^N) + U_2(n_t/\bar{\pi} + y + \tau^N - n_{t+1}, n_t/\bar{\pi} + \tau^N) \rightarrow 0.$$

and since  $U_2 \geq 0$ :

$$\lim_{t \rightarrow \infty} U_1(n_t/\bar{\pi} + y + \tau^N - n_{t+1}, n_t/\bar{\pi} + \tau^N) = 0$$

Since  $U_{12} \geq 0$  then

$$U_1(n_t/\bar{\pi} + y + \tau^N - n_{t+1}, n_t/\bar{\pi} + \tau^N) \geq U_1(n_t/\bar{\pi} + y + \tau^N - n_{t+1}, \tau^N)$$

but if  $n_{t+1} > n_t$ , for  $\bar{\pi} > 1$

$$U_1(n_t/\bar{\pi} + y + \tau^N - n_{t+1}, \tau^N) \geq U_1(y + \tau^N, \tau^N) > 0$$

a contradiction. Hence  $\lim_t n_t$  must be bounded, and thus  $g'(\bar{n}) < 1$ . *Q.E.D.*

## A.6 Proof of Proposition 5.

-Part I. Using  $\hat{n}_t = g' \cdot \hat{n}_{t-1}$  for the policy rule gives  $\hat{m}_t = \hat{n}_{t-1}/\bar{\pi}$ ,  $\hat{c}_t = (1/\bar{\pi} - g') \hat{n}_{t-1}$  and  $\hat{m}_{t+1} = g' \hat{n}_{t-1}/\bar{\pi}$ ,  $\hat{c}_{t+1} = (1/\bar{\pi} - g') g' \hat{n}_{t-1}$ . Totally differentiating the Euler equation (6) and using the above policy functions gives a second order ODE with characteristic equation

$$0 = \beta (\varphi_0)^2 + b\varphi_0 + 1,$$

that has  $\varphi_0 \equiv g'(\bar{n})$  as its smallest root, where the coefficient  $b$  is:

$$-b \equiv \frac{\bar{\pi} + \frac{\beta}{\bar{\pi}} \left( 1 + 2 \frac{\bar{U}_{12}}{\bar{U}_{11}} + \frac{\bar{U}_{22}}{\bar{U}_{11}} \right)}{1 + \frac{\bar{U}_{12}}{\bar{U}_{11}}} > 0. \quad (\text{A-3})$$

Note that  $-b > 0$  under the assumption  $U_{11} + U_{12} < 0$  and  $\pi > 1$ . As an intermediate step for the proof, the next lemma gives the properties of  $\varphi_0$  as a function of  $-b$ .

LEMMA 2. The expression for the root that is smaller in absolute value is:

$$\varphi_0 = \frac{-b - \sqrt{b^2 - 4\beta}}{2\beta} \quad (\text{A-4})$$

with  $0 < \varphi_0 \leq 1$ , provided that  $-b \geq 1 + \beta$ . Moreover,  $\varphi_0$  is decreasing in  $-b$ .

**Proof of Lemma 2.** A real solution requires  $b^2 - 4\beta \geq 0$ . If  $-b \geq 1 + \beta$ , then  $b^2 \geq (1 + \beta)^2 = 1 + \beta^2 + 2\beta$ , and  $b^2 - 4\beta \geq (1 - \beta)^2 > 0$ . If  $-b = 1 + \beta$  then  $\varphi_0 = 1$ . To see that  $\varphi_0$  is decreasing in  $-b$ :

$$\frac{\partial \varphi_0}{\partial (-b)} = \frac{\partial}{\partial (-b)} \left( \frac{-b - \sqrt{(-b)^2 - 4\beta}}{2\beta} \right) = \frac{1}{2\beta} \left( 1 - \frac{1}{\sqrt{b^2 - 4\beta}} \right) \leq \frac{1}{2\beta} \left( 1 - \frac{1}{1 - \beta} \right) < 0.$$

*Q.E.D.*

- Part II. We show that the coefficients of the equation that defines  $\varphi_0 \equiv g'(\bar{n})$  are a function of  $\gamma/(1 + \rho)$ ,  $\beta$ ,  $\pi$  and  $m/c$ . Using  $U(c, m) = (h(c, m)^{1-\gamma} - 1)/(1 - \gamma)$  gives:

$$\frac{U_{22}}{U_{11}} = \frac{h_{22}/h_{11} + \gamma r^2 (h_1 h_1) / (-h h_{11})}{1 + \gamma (h_1 h_1) / (-h h_{11})}, \quad \frac{U_{12}}{U_{11}} = \frac{h_{12}/h_{11} + \gamma r (h_1 h_1) / (-h h_{11})}{1 + \gamma (h_1 h_1) / (-h h_{11})}.$$

Using that  $h$  is CES we have

$$\begin{aligned} h_{11} &= (1 + \rho) \left[ c^{-\rho} + \frac{1}{\mathcal{A}} m^{-\rho} \right]^{-1/\rho-1} c^{-\rho-2} \left\{ \frac{c^{-\rho}}{[c^{-\rho} + \frac{1}{\mathcal{A}} m^{-\rho}]} - 1 \right\}, \\ h_{22} &= (1 + \rho) \left[ c^{-\rho} + \frac{1}{\mathcal{A}} m^{-\rho} \right]^{-1/\rho-1} \frac{1}{\mathcal{A}} m^{-\rho-2} \left\{ \frac{\frac{1}{\mathcal{A}} m^{-\rho}}{[c^{-\rho} + \frac{1}{\mathcal{A}} m^{-\rho}]} - 1 \right\}, \\ h_{12} &= (1 + \rho) \left[ c^{-\rho} + \frac{1}{\mathcal{A}} m^{-\rho} \right]^{-1/\rho-2} c^{-\rho-1} m^{-\rho-1} / \mathcal{A}. \end{aligned}$$

Thus

$$\frac{h_{22}}{h_{11}} = \left( \frac{c}{m} \right)^2 \quad \text{and} \quad \frac{h_{12}}{h_{11}} = - \left( \frac{c}{m} \right).$$

And using that, as from [equation \(13\)](#),  $(\frac{m}{c})^{-\rho} / \mathcal{A} = r m/c$

$$\frac{h_1 h_1}{-h h_{11}} = \frac{1}{(1 + \rho) r (m/c)}.$$

Plugging these expressions into the ones for  $U_{22}/U_{11}$  and  $U_{12}/U_{11}$  gives

$$\frac{U_{22}}{U_{11}} = \frac{\left( \frac{c}{m} \right)^2 + r \left( \frac{c}{m} \right) \left( \frac{\gamma}{1+\rho} \right)}{1 + \frac{1}{r} \left( \frac{c}{m} \right) \left( \frac{\gamma}{1+\rho} \right)}, \quad \frac{U_{12}}{U_{11}} = \frac{- \left( \frac{c}{m} \right) + \left( \frac{c}{m} \right) \left( \frac{\gamma}{1+\rho} \right)}{1 + \frac{1}{r} \left( \frac{c}{m} \right) \left( \frac{\gamma}{1+\rho} \right)} \quad (\text{A-5})$$

which are functions of the ratio  $\gamma/(1+\rho)$ , the number  $r$ , and the ratio  $m/c$ .

Replacing (A-5) into (A-3) gives:

$$-b = \frac{\bar{\pi} + \frac{\beta}{\bar{\pi}}(1-x)^2 + x\frac{\tilde{\gamma}}{r}(\bar{\pi} + \frac{\beta}{\bar{\pi}}(1+r)^2)}{1-x + x\frac{\tilde{\gamma}}{r}(1+r)} \quad (\text{A-6})$$

where  $x \equiv c/m \in (0, 1)$  and  $\tilde{\gamma} \equiv \gamma/(1+\rho)$ .

The partial derivative of (A-6) with respect to  $\tilde{\gamma}$  gives

$$\frac{\partial(-b)}{\partial\tilde{\gamma}} = \frac{-\frac{x}{r}(x+r)}{(1-x + x\frac{\tilde{\gamma}}{r}(1+r))^2} \left( \bar{\pi} - \frac{\beta}{\bar{\pi}}(1+r)(1-x) \right) \quad (\text{A-7})$$

Noting that at the steady state  $\frac{\beta}{\bar{\pi}}(1+r) = 1$  (by equation (13)) establishes that  $-b$  is decreasing in  $\gamma/(1+\rho)$ . This, by Lemma 2, implies that the root  $\varphi_0$  is increasing in  $\gamma/(1+\rho)$ .

-Part III. We conclude the proof by showing that  $0 < \chi(\bar{m})$  for  $\bar{\pi} > 1$  and  $m/c > 1$ . The proof is in two parts. We first analyze the case of  $\gamma = 0$ . Then we extend the results for the case of  $\gamma > 0$ .

Assume  $\gamma = 0$ . We show that the elasticity of  $c(m)$  with respect to  $m$ , evaluated at  $m = \bar{m}$ , is smaller than one, which implies that  $\chi(\bar{m}) > 0$ . Note by equation (33) that  $\frac{m}{c} \frac{\partial c}{\partial m} < 1$  requires  $(1 - \frac{c}{m}) \frac{1}{\pi} \leq \varphi_0$ . Using equation (A-4) this inequality becomes

$$-b - 2\beta \left(1 - \frac{c}{m}\right) \frac{1}{\pi} - \sqrt{b^2 - 4\beta} \geq 0 \quad (\text{A-8})$$

Using equation (A-6) for  $\gamma/(1+\rho) = 0$  gives

$$-b = \frac{\pi}{1-x} + \frac{\beta}{\pi}(1-x) \quad (\text{A-9})$$

where  $x \equiv c/m$ . We want to show that inequality (A-8) holds for  $x \in (0, 1)$ , i.e. that

$$a(x) \equiv -b(x) - 2\beta(1-x)\frac{1}{\pi} - \sqrt{b(x)^2 - 4\beta} > 0 .$$

This follows because:  $a(0) > 0$  and  $a'(x) > 0$  for  $x \in (0, 1)$ . These two inequalities follow from:

$$a(0) = \pi - \frac{\beta}{\pi} - \sqrt{\pi^2 + \left(\frac{\beta}{\pi}\right)^2 + \frac{2\beta}{\pi} - 4\beta}$$

Using that  $\pi > 1$ , we have

$$a(0) > \pi - \frac{\beta}{\pi} - \sqrt{\pi^2 + \left(\frac{\beta}{\pi}\right)^2 + \frac{2\beta}{\pi} - 4\frac{\beta}{\pi}} = \pi - \frac{\beta}{\pi} - \sqrt{\left(\pi - \frac{\beta}{\pi}\right)^2} = 0 .$$

Finally for  $a'(x) > 0$  we have

$$a' = -b' \left[ 1 + \frac{b}{\sqrt{b(x)^2 - 4\beta}} \right] + 2\frac{\beta}{\pi} = \left( \frac{\pi}{(1-x)^2} - \frac{\beta}{\pi} \right) \left[ 1 + \frac{b}{\sqrt{b^2 - 4\beta}} \right] + 2\frac{\beta}{\pi} > 0 .$$

We conclude the proof for the  $\gamma = 0$  case by showing that  $-b > 1 + \beta$  (an assumption in [Lemma 2](#)). From [equation \(A-6\)](#) with  $\tilde{\gamma} = 0$ , simple algebra shows that the inequality  $-b > 1 + \beta$  holds if  $\pi > 1$  and  $m/c > 1$ .

These results extend to the case where  $\gamma > 0$ . As above, the inequality  $0 < \chi$  requires  $(1 - \frac{c}{m}) \frac{1}{\pi} \leq \varphi_0$ . This inequality was shown to hold for  $\gamma = 0$ . Since  $\varphi_0$  is increasing in  $\tilde{\gamma}$ , then it holds *a fortiori* for  $\gamma > 0$ . The inequality  $-b \geq 1 + \beta$  holds, since  $-b$  is decreasing in  $\tilde{\gamma}$  and  $\lim_{\tilde{\gamma} \rightarrow \infty} (-b) = 1 + \beta$ .

## A.7 Expressions for the linearized Non-Trader problem

Coefficients for the linearization of the Euler equation of the Non-Trader's problem.

$$\begin{aligned} \xi_0 &= \frac{[\bar{\pi}\bar{U}_{11} + \frac{\beta}{\pi}(\bar{U}_{11} + \bar{U}_{12} + \bar{U}_{21} + \bar{U}_{22})]}{\beta[\bar{U}_{11} + \bar{U}_{21}]} , & \xi_1 &= \frac{\bar{n}}{\beta\bar{\pi}}, \\ \xi_2 &= - \left[ \frac{\frac{\beta}{\bar{n}}(\bar{U}_1 + \bar{U}_2) + \frac{\beta}{\pi}(\bar{U}_{11} + \bar{U}_{12} + \bar{U}_{21} + \bar{U}_{22})}{[\bar{U}_{11} + \bar{U}_{21}]} \right] \frac{\bar{n}}{\beta\bar{\pi}}. \end{aligned}$$

Coefficients for the equilibrium solution of the Non-Trader's problem:

$$\begin{aligned} (\varphi_0 - \xi_0) \varphi_0 + \frac{1}{\beta} &= 0, \quad , \quad (\varphi_0 - \xi_0) \varphi_2 = \xi_1 \left( \frac{\bar{\mu}}{\bar{m}} \right), \\ (\varphi_0 - \xi_0) \varphi_1 + \varphi_1 \Theta + \varphi_2 \kappa &= \xi_1 \zeta + \xi_2 \bar{\Pi} \end{aligned}$$

## A.8 Proof of [Proposition 6](#).

We try a solution of the form

$$n_t = \varphi_0 \hat{n}_{t-1} + \varphi_1 z_t + \varphi_2 \hat{m}_{t-1}$$

with coefficients  $\varphi_0$ ,  $\varphi_1$  and  $\varphi_2$  to be determined. Replacing the hypothesis for inflation and expected inflation the Euler equation:

$$\mathbb{E}_t[\hat{n}_{t+1}] = \xi_0 \hat{n}_t - \frac{1}{\beta} \hat{n}_{t-1} + \xi_1 \left[ \left( \frac{\bar{\mu}}{\bar{m}} \right) \hat{m}_{t-1} + \zeta z_t \right] + \xi_2 \bar{\Pi} z_t .$$

Taking expected values on the guess for the solution of the policy:

$$\mathbb{E}_t[\hat{n}_{t+1}] = \varphi_0 \hat{n}_t + \varphi_1 \mathbb{E}_t[z_{t+1}] + \varphi_2 \hat{m}_t = \varphi_0 \hat{n}_t + \varphi_1 \Theta z_t + \varphi_2 \kappa z_t$$

Equating the two terms:

$$\varphi_0 \hat{n}_t + \varphi_1 \Theta z_t + \varphi_2 \kappa z_t = \xi_0 \hat{n}_t - \frac{1}{\beta} \hat{n}_{t-1} + \xi_1 \left[ \left( \frac{\bar{\mu}}{\bar{m}} \right) \hat{m}_{t-1} + \zeta z_t \right] + \xi_2 \bar{\Pi} z_t$$

rearranging

$$(\varphi_0 - \xi_0) \hat{n}_t + \varphi_1 \Theta z_t + \varphi_2 \kappa z_t = -\frac{1}{\beta} \hat{n}_{t-1} + \xi_1 \left[ \left( \frac{\bar{\mu}}{\bar{m}} \right) \hat{m}_{t-1} + \zeta z_t \right] + \xi_2 \bar{\Pi} z_t$$

Replacing again the guess for the optimal policy in  $\hat{n}_t$  :

$$(\varphi_0 - \xi_0) [\varphi_0 \hat{n}_{t-1} + \varphi_1 z_t + \varphi_2 \hat{m}_{t-1}] + \varphi_1 \Theta z_t + \varphi_2 \kappa z_t = -\frac{1}{\beta} \hat{n}_{t-1} + \xi_1 \left[ \left( \frac{\bar{\mu}}{\bar{m}} \right) \hat{m}_{t-1} + \zeta z_t \right] + \xi_2 \bar{\Pi} z_t$$

rearranging:

$$\begin{aligned} & (\varphi_0 - \xi_0) \varphi_0 \hat{n}_{t-1} + [(\varphi_0 - \xi_0) \varphi_1 + \varphi_1 \Theta + \varphi_2 \kappa] z_t + (\varphi_0 - \xi_0) \varphi_2 \hat{m}_{t-1} \\ &= -\frac{1}{\beta} \hat{n}_{t-1} + (\xi_1 \zeta + \xi_2 \bar{\Pi}) z_t + \xi_1 \left( \frac{\bar{\mu}}{\bar{m}} \right) \hat{m}_{t-1} \end{aligned}$$

matching coefficients:

$$(\varphi_0 - \xi_0) \varphi_0 = -\frac{1}{\beta} \quad , \quad (\varphi_0 - \xi_0) \varphi_2 = \xi_1 \left( \frac{\bar{\mu}}{\bar{m}} \right) \quad , \quad (\varphi_0 - \xi_0) \varphi_1 + \varphi_1 \Theta + \varphi_2 \kappa = \xi_1 \zeta + \xi_2 \bar{\Pi}$$

That  $\varphi_0 = g'(\bar{n})$  can be verified immediately by linearizing the f.o.c. of the problem for the non-trader with constant inflation  $\bar{\pi}$ . Then, using [Proposition 4](#), we then have  $0 < \varphi_0 = g'(\bar{n}) < 1$ . *Q.E.D.*

## B Estimating a two-component process for $\mu_t$

In this appendix we setup and estimate a two-component process for money growth using monthly US data on M1 for the period 1959.1-2009.9. Let  $\{\mu_t\}_{t=1, \dots, T}$  be the sample of observed money growth rates, i.e. the log change in the levels of the nominal money stock. We assume that it is the sum of two unobserved, independent processes, each of them of the form:

$$\mu_{i,t+1} - \bar{\mu} = \theta_i (\mu_{i,t} - \bar{\mu}) + \epsilon_{i,t+1}$$

where  $i = 1, 2$ , the innovations  $\{\epsilon_{i,t}\}$  are i.i.d., through time and independent of each other, normally distributed with mean zero and variance  $\sigma_i^2$ . We have then  $\mu_t = \mu_{1,t} + \mu_{2,t}$  for  $t = 1, \dots, T$ . Assuming the process was in place for a long time before the beginning of the sample implies that the initial observation is drawn from the invariant distribution, which is normal with mean  $\mu$  and variance  $\sum_{i=1,2} \sigma_i^2 / (1 - \theta_i^2)$ .

The theoretical autocovariance function for  $\mu$  is given by:

$$\hat{\Omega}(k) = \sigma_1^2 \frac{\theta_1^k}{1 - \theta_1^2} + \sigma_2^2 \frac{\theta_2^k}{1 - \theta_2^2}$$

for  $k = 0, 1, 2, 3, \dots$

To estimate the 4 parameters  $(\theta_1, \theta_2, \sigma_1, \sigma_2)$ , we use a GMM algorithm that minimizes the distance between  $J$  selected elements from the ACF function of M1 and this theoretical counterpart, where identification obviously requires that  $J \geq 4$ , giving an equal weight to each of these moments. Let  $\Omega(k)$  be the sample measure of the autocovariance function. Then

$$\theta_1^*, \theta_2^*, \sigma_1^*, \sigma_2^* = \underset{\theta_1, \theta_2, \sigma_1, \sigma_2}{\operatorname{argmin}} \sum_{j \in J} \left( \Omega(j-1) - \hat{\Omega}(j-1) \right)^2$$

The table below reports one set of estimates based on monthly data, for a choice of  $J$  that matches very high frequencies (1-three months) ( $J = 1 - 3$ ), business cycle ( $J = 57 - 60$ ), and 10 year frequencies ( $J = 117 - 120$ )

Table A: Estimates for money growth (M1) process and implied half-life of shocks ( $\tau$ )

	$\theta_1^*$	$\sigma_1^*$	$\tau_1$	$\theta_2^*$	$\sigma_2^*$	$\tau_2$
$J = [1 : 3], [57 : 60], [117 : 120]$	0.995	0.00005	154	0.466	0.0048	0.9

Note: Based on M1 monthly data (not seasonally adjusted) over the sample period 1959:1-2009:9. Data source: Board of Governors of the Federal Reserve System. The half life is  $\tau_i = \log(1/2)/\log(\theta_i)$

## C Relation to sticky-price models (with no capital)

There are two properties of agents' preferences that are required to make the model consistent with both low and high frequency data: (1) a unitary income elasticity of money demand and (2) the stylized facts along a BGP. The latter means that the model must be consistent with a BGP with constant real interest rates, constant leisure, and the following growing at the same rate: real wages, real balances and consumption, and this must be so for arbitrary nominal wages and initial real wages. These points are important for us because we want to connect the low-frequency and high-frequency features of the data within the same model. The set of preferences with these properties are given by

$$U(c, m, l) = \begin{cases} \log h(c, m) + v(l) \\ \frac{h(c, m)^{1-\gamma}}{1-\gamma} \cdot v(l) \end{cases} \quad \text{where } h(c, m) > 0 \text{ is h.o.d. 1 in } c \text{ and } m \quad (\text{A-10})$$

These preferences correspond to the first and second case (equations 10 and 11) in [Andres, David Lopez-Salido, and Valles \(2002\)](#). Note that the preference satisfy, for any fixed  $l$ , the BGP requirement that  $U_c/U_l$  is homogenous of degree 1 in  $(c, m)$  and that  $U_m/U_c$  is homogenous of degree 0 in  $(c, m)$ .<sup>14</sup>

Some parametric conditions must be satisfied to have a liquidity effect. Typically these imply a low intertemporal substitution elasticity ( high  $\sigma$  in [Gali \(2002\)](#) ) in a model without

<sup>14</sup>The first condition ensures consistency with a unit income elasticity, the second that the ratio  $m/c$  is a function of the interest rate (see the discussion in Section 6 of [Chari, Kehoe, and McGrattan \(1996\)](#))

capital. To see this note that from the money demand condition  $M/(PY) = R^{-\eta}$  the presence of the liquidity effect means that  $R$  fall if  $M$  increases. To have this effect it must be that the denominator  $PY$  does not increase much as  $M$  increases. Now if  $P$  is "fixed" because of sticky prices, this requires that  $Y$  does not increase much; this condition obtains if the intertemporal substitution elasticity is low.

Imposing requirements (1) and (2) in the NK models (without capital) implies that preferences must either be of the first type (log-log), but then this gives no liquidity effect given the low substitution elasticity of the log case ( $\sigma = 1$ , see [Gali \(2002\)](#) ). One must then use the non-separable specification. Of course non-separability implies a different set of equations than the ones used in NK models, where the money balances do not affect the determination of real variables. [Andres, David Lopez-Salido, and Valles \(2002\)](#) discuss the case of non-separable preferences (their utility function 11, section 4.2) and find that no parametrization allows them to produce a liquidity effect on impact (see their figure 2) .

## D Literature Review

### D.1 Interest elasticity, liquidity effects and the price puzzle

Here we give a brief account of estimates of parameters that quantify two key experiments for the model: the long run interest rate elasticity of the money demand, and the size and persistence of the liquidity effect.

There are many estimates of the interest rate elasticity of money demand. In our model, the parameter  $1/(1 + \rho)$  correspond to the long run elasticity of the money demand, so we refer to the studies that better much this concept. [Lucas \(2000\)](#) and [Stock and Watson \(1993\)](#) identify this elasticity using the long-run behavior of interest rates and velocity using almost a century of annual data. Lucas preferred estimate is 0.5 for the elasticity of M1 velocity with respect to interest rates, or equivalently a semi-elasticity of velocity with respect to interest rates of 0.8, while Stock and Watson's estimate of the M1 semi-elasticity is 1.0. [Hoffman, Rasche, and Tieslau \(1995\)](#) find long run elasticities similar to the ones estimated by Lucas for five industrialized countries.<sup>15</sup> There is a widespread view that the short run money demand elasticity is substantially smaller than the long run elasticity, see, for instance, [Goodfriend \(1991\)](#). [Lucas \(2000\)](#) and [Stock and Watson \(1993\)](#) interpret the lack of stability as evidence that it takes a very long-span on data to have enough low frequency variation to uncover the elasticities.

Papers that incorporate more recent data, such as [Ball \(2001\)](#) and [Ireland \(2009\)](#), find evidence of instability on the long-run money demand using M1, and also evidence of smaller elasticities. [Ireland \(2009\)](#) finds evidence of a downward shift on money demand after 1980. Ireland estimates elasticities of M1 velocity with respect to interest rates of about 0.05 and semi-elasticities about 1.5 using data after 1980. Ireland, as well as others, interpret the shift on money demand as due to financial innovation and deregulation. One approach suggested

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<sup>15</sup>The difference between the interest rate elasticities around 0.5 found by [Lucas \(2000\)](#) and [Hoffman, Rasche, and Tieslau \(1995\)](#), and the one in [Stock and Watson \(1993\)](#) around 0.1 is due to the treatment of the income elasticity of money demand. In [Lucas \(2000\)](#) and [Hoffman, Rasche, and Tieslau \(1995\)](#) a unitary elasticity is imposed, while in [Stock and Watson \(1993\)](#) income elasticities are estimated.

by this finding is to use a different monetary aggregate constructed to better reflect the changes in regulatory environment, such as MZM, or “money of zero maturity”, or M1S (M1 plus balances “swept”, see [Dutkowsky, Cynamon, and Jones \(2006\)](#) for details). Studies using MZM, such as [Teles and Zhou \(2005\)](#) and [Carlson et al. \(2000\)](#), find a more stable money demand. [Teles and Zhou \(2005\)](#) report elasticities of the MZM velocity with respect to interest rates of about 0.25, and [Carlson and Keen \(1996\)](#) find a interest rate semi-elasticity of around 4.0 (See table 10).

Summarizing, estimates of long run interest rate elasticities for money demand range between 0.25 and 0.5, depending on the time periods and monetary aggregates. These estimates are based on low frequency, decade to decade, changes on velocity and interest rates.

While the liquidity effect is present in most description of how monetary shocks affect interest rates, innovations on interest rates and monetary growth do not display the negative correlation suggested by the liquidity effect –see, for instance, [Leeper and Gordon \(1992\)](#). The most common interpretation of this feature of the data is that there are lots of changes in the money supply that are accommodating high frequency changes on money demand. Following this interpretation the literature has used a variety of identifying assumptions to isolate the changes in money supply that accommodate high frequency money demand shocks.

There is a large literature on the effect of “monetary policy shocks” using VARs. The estimates in this literature differ according to the assumptions used to identify the monetary policy shock, the time period and the variables included in the VARs. Yet, almost all of the estimates are based on the assumption that there is a liquidity effect. While the focus of this literature is on the effect of monetary shocks on output, the estimated impulse responses can be used to quantify the size of the impact effect, as well as the persistence of interest rates and monetary aggregates after a monetary shock.

Estimates using VARs typically display persistent effects on interest rates of an innovation on monetary policy, or a monetary policy shock. See for instance the survey of [Christiano, Eichenbaum, and Evans \(1999\)](#) where the impulse response of interest rates seem to be well described as an AR process, with a half life of about half a year. The VARs also estimate the path of monetary aggregates that will follow a monetary shock. For the purpose of the theory developed in this paper, the persistence of the liquidity effect, as well as its magnitude depends on the persistence of the path of the relevant monetary aggregate that follows a monetary shock. Depending on the assumption on identification of a “monetary shock”, and the choices of monetary aggregate, different behavior of monetary aggregates is found.

For instance, one of the identification schemes used in the literature focuses on the behavior of Non-borrowed reserves (or the ratio of borrowed to non-borrowed reserves), as opposed to a broader monetary aggregate such as M1 or M2. This assumption is meant to reflect the operating procedure of the Fed, which accommodates high frequency shocks to reserves. While this assumption may be a good description of the Fed operating procedures, it is not part of the theoretical model, which complicates the mapping between the estimated impulse response and the theoretical path described in the models. In the survey by [Christiano, Eichenbaum, and Evans \(1999\)](#) they compare different identification schemes, and in particular the behavior of Non-Borrowed reserves and broader monetary aggregates. Indeed the patterns of adjustment seem to differ across monetary aggregates. Non-borrowed reserves,

M0, and M1 seem to display “u” shaped patterns after a contractionary shock, while M2 seem to display a persistent decrease.

Another statistic of interest that appears in the literature is the difference between the long and short run elasticities of the money demand. For instance [Christiano, Eichenbaum, and Evans \(1999\)](#) compute the short run money demand elasticity, by the ratio of the impact effects on interest rates and money on the impulse response of the estimated VAR. They find that the semi-elasticity of the money demand is smaller than what other estimate based on long run data, say -1.0, to be compared with values between -4.0 and -8.0 discussed above.

Finally, we comment briefly on the behavior of the price level in the model and in the estimated impulse responses. A common specification in the literature is that, by construction, the price level cannot jump contemporaneously with the monetary contraction. The estimated subsequent behavior of the price level after a monetary contraction, depends on the variables included in the system, and to a lesser extent, the period considered. Indeed, the simpler system tend to find an increase in the price level after a contractionary shock, a feature that is called “the price puzzle”, since it is against the hypothesis of nominal rigidities.<sup>16</sup> Specifications where the price puzzle is not observed, which typically consist on including variables such as the price of commodities, display a slow decrease in the price level after a contractionary shock. This pattern is consistent with model with nominal rigidities.

The model in this paper has no rigidities on the price adjustment, so the price level can jump on impact after a monetary shock. The importance of nominal rigidities is not a central topic of this paper, indeed one can consider the same model with sticky prices. In the current version with flexible prices, the reaction of inflation diminishes the strength of the liquidity effect. Recall that in the case of i.i.d. shocks to the growth rate of money  $\mu_t$ , interest rates decrease after an open market operation because the price level increase proportionally with the amount of money, but traders receive a more than proportional amount. Instead, in a sticky price version of the model, the increase on traders’ real balances will be even higher.

[Christiano, Eichenbaum, and Evans \(1999\)](#) use M2 as a monetary aggregate, and in a quarterly model find that after a monetary expansion, interest rates decrease in a persistent fashion. The point estimates indicate that the response of interest rates is a bit more persistent than the response of M2, which can be well approximated by an AR(2) with autocorrelation 0.5 and hence a half-life of 3 months. Their estimates display the “price puzzle”, i.e. inflation has a small protracted initial decrease. The estimates in [Christiano, Eichenbaum, and Evans \(2005\)](#) also show similar patterns, where the point estimates of the interest rate impulse response is more persistent than the one for money growth. [Altig et al. \(2005\)](#) estimate the effect of a monetary policy shock, using MZM as a measure of money. The result are roughly similar to those in [Christiano, Eichenbaum and Evans \(2001, 2005\)](#) but they also feature velocity. They estimate that the impulse response of velocity and interest rates comoves, as if were generated by a money demand with a small interest rate semi-elasticity.

In summary, the literature present plenty of estimates of the liquidity effect after a monetary shock, but the emphasis and interest of the papers is not on measuring monetary aggregates or the relative persistence. As a consequence, future version of this paper will include estimated VARs that are tailored to estimate the relative persistence of interest rates

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<sup>16</sup>Notice that an alternative explanation of this puzzle is that the shock identify the economy behaves according to the Fisherian fundamentals, i.e. higher interest rates are associated with higher inflation.

and velocity of monetary aggregates. In particular the monetary aggregates will be chosen to be consistent with the ones that produce stable long run money demands.

## D.2 Monetary Segmented Asset Models

In [Lucas \(1990\)](#) asset and good markets are separated within the period, but at the end of the period all agents pool their resources. Similar assumptions are made in the base-line models of [Fuerst \(1992\)](#) and [Christiano and Eichenbaum \(1992\)](#), who use set-ups closely related to the one in [Lucas \(1990\)](#) to study the effects on output of a monetary shock. In [Alvarez and Atkeson \(1997\)](#) agents visit the asset market at exogenously randomly distributed times, at which they effectively –via the complete market assumption– pool resources. In [Alvarez, Atkeson, and Kehoe \(2002\)](#) agents face a fixed cost of accessing the asset market, making participation endogenous and time-varying. Yet the parameters considered and the nature of the equilibrium is such that, at the end of the period the distribution of asset is degenerate. Finally, the closest environments to the one consider in this paper are the ones in [Alvarez, Lucas, and Weber \(2001\)](#) and in [Occhino \(2004\)](#). In these set ups a fraction  $\lambda$  of the agents have permanent access to asset markets, the remaining fraction is excluded from the asset markets permanently. Apart from these differences in the way that agents are segmented in their participation in asset markets, these models have a binding cash-in-advance with exogenous velocity.

ADDITIONAL MATERIAL  
(NOT FOR PUBLICATION)

Persistent Liquidity Effect and  
Long Run Money Demand

*Fernando Alvarez and Francesco Lippi*

November 2, 2011

## E The Non-Trader problem in continuous time

Since in our discrete time analysis, the speed of convergence and the elasticity of  $m/c$  depend on the value of a stock/flow, namely the steady state value of  $m/c$ , we analyze the continuous time version of the non-trader problem. The continuous time version has two advantages. One is that it deals more naturally with the stock/flow distinction. The other one is that it simplifies some expression, which are key for the impact of money on interest rate shocks.

Consider the problem of a non-trader maximizing

$$\max \int_0^{\infty} e^{-\delta t} U(c(t), m(t)) dt$$

subject to

$$\dot{m}(t) + c(t) = y + \tau - \pi m(t)$$

and  $m(0) > 0$  given. Here  $\pi$  is the continuously compounded inflation rate and  $\delta$  is the discount rate. The steady state of this problem is the same as the discrete time version, given by

$$\frac{U_m(\bar{c}, \bar{m})}{U_c(\bar{c}, \bar{m})} = \delta + \pi \equiv r.$$

In the case where  $U$  is given by (12), recall that  $\bar{c}$  and  $\bar{m}$  are independent of  $\gamma$  and depend only on the properties of  $h$ , in particular we obtain  $\bar{m}/\bar{c} = (1/\mathcal{A}) r^{-1/(1+\rho)}$ . We let  $c = \psi(m)$  the optimal decision rule for consumption. From the budget constraint the dynamics of  $m(t)$  for values close to the steady state  $\bar{m}$ , is governed by  $\partial \dot{m}(t) / \partial m = -\pi - \psi'(\bar{m})$ . In the next proposition we keep fixed the value of  $\delta$ ,  $\pi$ , and regard  $\gamma$ ,  $\rho$  and the steady state value of  $\bar{m}/\bar{c}$  as parameters. The interpretation is that the value of the parameter  $\mathcal{A}$  is changed as a function of  $\rho$  and  $\bar{m}/\bar{c}$ .

**PROPOSITION 9.** Assume that  $\pi \geq 0$ , and that  $U$  is given by (12). Keeping the steady state value  $\bar{m}/\bar{c}$  fixed, the slope of the optimal consumption function evaluated at steady state is a function of  $\gamma/(1+\rho)$  and satisfy

$$\frac{c}{m} \frac{\partial c(m)}{\partial m} \Big|_{m=\bar{m}} \equiv \frac{\bar{m}}{\bar{c}} \psi'(\bar{m}) = \begin{cases} 1 & \text{for } \gamma/(1+\rho) = 0 \\ < 1 & \text{for } \frac{\gamma}{1+\rho} > 0 \end{cases}$$

and  $\frac{\bar{m}}{\bar{c}} \psi'(\bar{m})$  is decreasing in  $\frac{\gamma}{1+\rho}$ .

As anticipated in the continuous time limit the particular threshold value of  $m/c = 1$  plays no role.

**Proof of Proposition 9.** Using  $\lambda$  for the co-state, the Hamiltonian  $H$  is

$$H(c, m) = U(c, m) + \lambda[-m\pi + y + \tau - c]$$

The f.o.c. are:

$$\dot{\lambda} = -\rho\lambda - H_m : \dot{\lambda} = \lambda(\delta + \pi) - U_m(c, m), \text{ and } H_c = 0 : U_c(c, m) = \lambda.$$

The steady state is then:

$$\frac{U_m}{U_c}(\bar{c}, \bar{m}) = \delta + \pi \quad , \quad \pi \bar{m} + \bar{c} = y + \tau.$$

From  $U_c(c(t), m(t)) = \lambda(t)$  we obtain:

$$\dot{c}U_{cc} + \dot{m}U_{cm} = \dot{\lambda},$$

replacing into the  $\dot{\lambda}$  expression:

$$\dot{c}U_{cc}(c, m) + \dot{m}U_{cm}(c, m) = U_c(c, m)(\pi + \delta) - U_m$$

and using the budget constraint for  $\dot{m}$ :

$$\dot{c}U_{cc}(c, m) = U_c(c, m)(\pi + \delta) - [y + \tau - m\pi - c]U_{cm}(c, m) - U_m(c, m)$$

Linearizing this ODE around  $(\dot{c}, \dot{m}, c, m) = (0, 0, \bar{c}, \bar{m})$  we obtain

$$\begin{aligned} \dot{c} &= (\pi + \delta) \frac{U_{cc}}{U_{cc}}(c - \bar{c}) + (\pi + \delta) \frac{U_{cm}}{U_{cc}}(m - \bar{m}) + \frac{U_{cm}}{U_{cc}}(c - \bar{c}) \\ &\quad + \frac{\pi U_{cm}}{U_{cc}}(m - \bar{m}) - \frac{U_{mm}}{U_{cc}}(m - \bar{m}) - \frac{U_{mc}}{U_{cc}}(c - \bar{c}) \end{aligned}$$

where all the second derivatives are evaluated at the steady state values. Summarizing we have the linear system:

$$\begin{aligned} \dot{c} &= a(c, m) \equiv [\pi + \delta](c - \bar{c}) + \left[ \frac{U_{cm}(2\pi + \delta) - U_{mm}}{U_{cc}} \right] (m - \bar{m}) \\ \dot{m} &= b(c, m) \equiv y + \tau - m\pi - c \end{aligned}$$

We are looking for a solution of the form

$$c = \psi(m) = \bar{c} + \psi'(\bar{m})(m - \bar{m}).$$

We use the method of undetermined coefficients:

$$\frac{\partial \psi(\bar{m})}{\partial m} \equiv \psi'(\bar{m}) = \frac{dc/dt}{dm/dt} = \frac{a(\psi(\bar{m}), \bar{m})}{b(\psi(\bar{m}), \bar{m})} = \frac{0}{0} = \frac{(\pi + \delta)\psi'(\bar{m}) + \left[ \frac{U_{cm}(2\pi + \delta) - U_{mm}}{U_{cc}} \right]}{-\pi - \psi'(\bar{m})},$$

where the last equality uses L'Hopital rule. The quadratic equation for  $\psi'$  is then

$$-\psi'(\pi + \psi') + (\pi + \delta)\psi' + \left[ \frac{U_{cm}(2\pi + \delta) - U_{mm}}{-U_{cc}} \right] = 0.$$

The stable solution is given by

$$\psi' = \frac{-[r + \pi] + \sqrt{[r + \pi]^2 + 4 \frac{U_{cm}(r+\pi) - U_{mm}}{(-U_{cc})}}}{2}.$$

Now we specialize the utility function  $U$  to (12). First we show that if  $\gamma = 0$ , then  $\psi'(\bar{m}) = m/c$ . Using that  $h$  is a CES we obtain that

$$\frac{h_{22}}{h_{11}} = \frac{1}{(m/c)^2}, \quad h_{12}/h_{11} = -\frac{1}{m/c}$$

we have

$$\psi' = \frac{-[r + \pi] + \sqrt{[r + \pi]^2 + 4 \frac{h_{12}(r+\pi) - h_{22}}{-h_{11}}}}{2}$$

so

$$\frac{h_{12}(r + \pi) - h_{22}}{-h_{11}} = \frac{(\pi + r)}{m/c} + \left(\frac{1}{m/c}\right)^2$$

Thus

$$[r + \pi]^2 + 4 \frac{h_{12}(r + \pi) - h_{22}}{-h_{11}} = [r + \pi]^2 + 2(\pi + r) \left(\frac{2}{m/c}\right) + \left(\frac{2}{m/c}\right)^2 = \left([r + \pi] + \left(\frac{2}{m/c}\right)\right)^2$$

and hence

$$\psi' = \frac{-[r + \pi] + \sqrt{\left([r + \pi] + \left(\frac{2}{m/c}\right)\right)^2}}{2} = \frac{-[r + \pi] + [r + \pi] + \left(\frac{2}{m/c}\right)}{2} = \frac{1}{m/c}.$$

In the case of  $\gamma > 0$  we have  $U = f(h)$  :

$$U_{11} = f'h_{11} + f''h_1h_1 = f' \left\{ h_{11} + \frac{f''}{f'} h_1 h_1 \right\}, \quad U_{22} = f' \left\{ h_{22} + \frac{f''}{f'} h_2 h_2 \right\}$$

$$U_{12} = f'h_{12} + f''h_1h_2 = f' \left\{ h_{12} + \frac{f''}{f'} h_1 h_2 \right\}$$

so

$$\frac{U_{22}}{U_{11}} = \frac{h_{22} + \frac{f''}{f'} h_2 h_2}{h_{11} + \frac{f''}{f'} h_1 h_1} = \frac{h_{22}/h_{11} + \frac{f''}{f'} h_2 h_2 / h_{11}}{1 + \frac{f''}{f'} h_1 h_1 / h_{11}} = \frac{h_{22}/h_{11} + \gamma r^2 (h_1 h_1) / (-h h_{11})}{1 + \gamma (h_1 h_1) / (-h h_{11})}$$

or, since

$$h_1 = \left[ c^{-\rho} + \frac{1}{\mathcal{A}} m^{-\rho} \right]^{-1/\rho-1} c^{-\rho-1}, \quad h_{11} = (1 + \rho) \left[ c^{-\rho} + \frac{1}{\mathcal{A}} m^{-\rho} \right]^{-1/\rho-1} c^{-\rho-2} \left\{ \frac{c^{-\rho}}{\left[ c^{-\rho} + \frac{1}{\mathcal{A}} m^{-\rho} \right]} - 1 \right\}$$

$$\frac{h_1 h_1}{-h h_{11}} = - \frac{[c^{-\rho} + \frac{1}{\mathcal{A}} m^{-\rho}]^{-2/\rho-2} c^{2(-\rho-1)}}{(1+\rho) [c^{-\rho} + \frac{1}{\mathcal{A}} m^{-\rho}]^{-2/\rho-1} c^{-\rho-2} \left\{ \frac{c^{-\rho}}{[c^{-\rho} + \frac{1}{\mathcal{A}} m^{-\rho}]} - 1 \right\}} = \frac{1}{(1+\rho) r (m/c)}$$

Thus

$$\frac{U_{22}}{U_{11}} = \frac{h_{22}/h_{11} + \gamma r^2 (h_1 h_1) / (-h h_{11})}{1 + \gamma (h_1 h_1) / (-h h_{11})} = \frac{\left(\frac{1}{m/c}\right)^2 + \frac{\gamma r^2}{(1+\rho)r(m/c)}}{1 + \frac{\gamma}{(1+\rho)r(m/c)}}$$

and

$$\frac{U_{12}}{U_{11}} = \frac{h_{12}/h_{11} + \gamma r (h_1 h_1) / (-h h_{11})}{1 + \gamma (h_1 h_1) / (-h h_{11})}$$

so

$$\begin{aligned} \Delta(\gamma) &\equiv \frac{U_{cm}(2\pi + \delta) - U_{mm}}{(-U_{cc})} = \frac{\left(\frac{1}{m/c} - \frac{\gamma}{(1+\rho)(m/c)}\right) [\pi + r] + \left(\frac{1}{m/c}\right)^2 + \frac{\gamma r}{(1+\rho)(m/c)}}{1 + \frac{\gamma}{(1+\rho)r(m/c)}} \\ &= \frac{\frac{1}{m/c} [\pi + r] - \frac{\gamma}{(1+\rho)(m/c)} \pi + \left(\frac{1}{m/c}\right)^2}{1 + \frac{\gamma}{(1+\rho)r(m/c)}} = \frac{\frac{1}{m/c} [\pi + r] + \left(\frac{1}{m/c}\right)^2}{1 + \frac{\gamma}{1+\rho} \frac{1}{r(m/c)}} - \frac{\frac{\pi}{(m/c)}}{\frac{(1+\rho)}{\gamma} + \frac{1}{r(m/c)}} \end{aligned}$$

Thus  $\Delta(\gamma/(1+\rho))$  is decreasing in  $\gamma$  provided that  $\pi > 0$ . Since

$$\psi'(\bar{m}, \gamma) = \frac{-[r + \pi] + \sqrt{[r + \pi]^2 + 4\Delta(\gamma)}}{2}$$

then  $\psi'(\gamma)$  is decreasing in  $\gamma$ . *Q.E.D.*

## F Linearizing the model for a generic $\mu_t$ process

Uhlig (1998) develops a convenient matlab code to solve linear models. A linear model is defined by the following equations,

$$0 = A x_t + B x_{t-1} + C y_t + D z_t \quad (\text{A-11})$$

$$0 = E_t [F x_{t+1} + G x_t + H x_{t-1} + J y_{t+1} + K y_t + L z_{t+1} + M z_t] \quad (\text{A-12})$$

$$z_{t+1} = N z_t + \varepsilon_{t+1} \quad \text{with } \mathbb{E}_t(\varepsilon_{t+1}) = 0 \quad (\text{A-13})$$

where  $A, B, C, D, F, G, H, J, K, L, M, N$  are matrices and  $x_t, y_t$  and  $z_t$  are vectors. One should think of  $x_t$  as endogenous state-type variables,  $y_t$  as control-type variables and  $z_t$  as exogenous state-type variables. The timing of the model is that the beginning of a period  $x_{t-1}$ , and  $z_t$  are given by history and  $x_t$  and  $y_t$  should be determined so that the equations above hold. Harald's code is a sophisticated version of the method of undetermined

coefficients, i.e. it gives a linear solution as

$$\begin{aligned} x_t &= P x_{t-1} + Q z_t \\ y_t &= R x_{t-1} + S z_t \end{aligned}$$

i.e. the code gives the matrices  $P, Q, R$  and  $S$ .

In our case the vector  $x, y$  and  $z$  are

$$x_t = (\hat{m}_t, \hat{n}_t^N), \quad z_t = (\hat{\mu}_{1,t}, \hat{\mu}_{2,t}) \quad \text{and} \quad y_t = (\hat{c}_t^T, \hat{c}_t^N, \hat{m}_t^T, \hat{m}_t^N, \hat{\pi}_t, \hat{r}_t)$$

where  $\mu_{1,t}$  and  $\mu_{2,t}$  are two AR(1) process that compose money growth according to  $\mu_t = \mu_{1,t} + \mu_{2,t}$ . The equations in (A-11) are the market clearing for consumption, market clearing for money, definition of  $\hat{n}_t^N$  in terms of  $\hat{m}_t^N$ , the definition of  $\hat{\pi}_t$  in terms of  $\hat{\mu}_t, \hat{m}_t$  and  $\hat{m}_{t-1}$ , the definition of the interest rate, and the budget constraint of the non-traders:

$$\begin{aligned} \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} &= \begin{bmatrix} 0 & 0 \\ -1 & 0 \\ 0 & 0 \\ -\bar{\mu}/\bar{m} & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \hat{m}_t \\ \hat{n}_t^N \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 1/\bar{\pi} \\ \bar{\mu}/\bar{m} & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \hat{m}_{t-1} \\ \hat{n}_{t-1}^N \end{bmatrix} \\ &+ \begin{bmatrix} \lambda & 1-\lambda & 0 & 0 & 0 & 0 \\ 0 & 0 & \lambda & 1-\lambda & 0 & 0 \\ 0 & 0 & 0 & -1 & -\bar{n}/\bar{\pi}^2 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ (1+\rho)/\bar{c} & 0 & -(1+\rho)/\bar{m} & 0 & 0 & -1/\bar{r} \\ 0 & 1 & 0 & -1 & 0 & 0 \end{bmatrix} \begin{bmatrix} \hat{c}_t^T \\ \hat{c}_t^N \\ \hat{m}_t^T \\ \hat{m}_t^N \\ \hat{\pi}_t \\ \hat{r}_t \end{bmatrix} \\ &+ \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 1 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \hat{\mu}_{1,t} \\ \hat{\mu}_{2,t} \end{bmatrix} \end{aligned}$$

the equations in (A-12) are the aggregate money demand, and the Euler equation for the

non-traders

$$\begin{aligned}
\begin{bmatrix} 0 \\ 0 \end{bmatrix} &= \begin{bmatrix} \phi & 0 \\ 0 & -\beta [\bar{U}_{11} + \bar{U}_{21}] / \bar{\pi} \end{bmatrix} \begin{bmatrix} \hat{m}_{t+1} \\ \hat{n}_{t+1}^N \end{bmatrix} \\
&+ \begin{bmatrix} -1 & 0 \\ 0 & \bar{U}_{11} + \frac{\beta}{\bar{\pi}^2} (\bar{U}_{11} + \bar{U}_{12} + \bar{U}_{21} + \bar{U}_{22}) \end{bmatrix} \begin{bmatrix} \hat{m}_t \\ \hat{n}_t^N \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & -(\bar{U}_{11} + \bar{U}_{12}) / \bar{\pi} \end{bmatrix} \begin{bmatrix} \hat{m}_{t-1} \\ \hat{n}_{t-1}^N \end{bmatrix} \\
&+ \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -[\frac{\beta}{\bar{\pi}^2} (\bar{U}_1 + \bar{U}_2) + \frac{\beta \bar{n}}{\bar{\pi}^3} (\bar{U}_{11} + U_{12} + U_{21} + U_{22})] & 0 \end{bmatrix} \begin{bmatrix} \hat{c}_{t+1}^T \\ \hat{c}_{t+1}^N \\ \hat{m}_{t+1}^T \\ \hat{m}_{t+1}^N \\ \hat{\pi}_{t+1} \\ \hat{r}_{t+1} \end{bmatrix} \\
&+ \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & [\bar{U}_{11} + \bar{U}_{12}] \frac{\bar{n}}{\bar{\pi}^2} & 0 \end{bmatrix} \begin{bmatrix} \hat{c}_t^T \\ \hat{c}_t^N \\ \hat{m}_t^T \\ \hat{m}_t^N \\ \hat{\pi}_t \\ \hat{r}_t \end{bmatrix} + \begin{bmatrix} \alpha & \alpha \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \hat{\mu}_{1,t+1} \\ \hat{\mu}_{2,t+1} \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \hat{\mu}_{1,t} \\ \hat{\mu}_{2,t} \end{bmatrix}
\end{aligned}$$

and the equation in (A-13) is the law of motion for the growth rate of the money supply.

$$\begin{bmatrix} \hat{\mu}_{1,t+1} \\ \hat{\mu}_{2,t+1} \end{bmatrix} = \begin{bmatrix} \theta_1 & 0 \\ 0 & \theta_2 \end{bmatrix} \begin{bmatrix} \hat{\mu}_{1,t} \\ \hat{\mu}_{2,t} \end{bmatrix} + \begin{bmatrix} \varepsilon_{1,t+1} \\ \varepsilon_{2,t+1} \end{bmatrix}.$$