Heterogeneous beliefs and recursive preferences

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Introduction

- Market selection
  - *Alchian (1950)*, *Friedman (1953)*
  - question of extinction of agents with incorrect beliefs as a supporting argument for the rational expectations hypothesis
  - many results for setups with time-separable preferences (*Sandroni (2000)*, *Blume and Easley (2006)*, *Kogan et al. (2009)*)

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  - many results for setups with time-separable preferences (*Sandroni (2000)*, *Blume and Easley (2006)*, *Kogan et al. (2009)*)
- How does the analysis change when we introduce recursive, time-nonseparable utility?
  - survival of agents with distorted beliefs is a generic result, occurring for a wide range of empirically plausible parametrizations.

Outline

- General approach in the market selection literature
- Recursive utility in continuous time
  - introducing belief distortions
  - planner's problem and decentralization
  - numerically tractable framework
- Quantitative results
  - survival
  - price impact
- Learning

Market selection — general approach

- Consider the Euler equation for a (marginal) agent

$$P_t = E_t^Q \left[\frac{MC_{t+s}}{MC_t} Z_{t+s} \right] = E_t \left[\frac{M_{t+s}}{M_t} \frac{MC_{t+s}}{MC_t} Z_{t+s} \right]$$

  - Q is agent's subjective probability measure, MC marginal utility of consumption, Z asset payoff, P asset price, and M the Radon-Nikodym derivative martingale (belief ratio)

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- How to avoid extinction with diverging beliefs?
  - unbounded endowment *and* unbounded relative risk aversion (*Kogan et al. (2009)*)
  - incomplete markets (*Blume and Easley (2006)*)
  - market frictions to distort the Euler equation or make some agents nonmarginal
  - recursive utility? $\Longrightarrow$ **this paper**

Belief distortions

- Belief distortions modeled by a martingale M representing the Radon-Nikodym derivative dQ/dP

$$M_t = \exp\left(\int_0^t \left(-\frac{1}{2}\left|u_s\right|^2 ds + u_s dW_s\right)\right)$$

  - Why? This allows to incorporate a specific type of optimism/pessimism. Consider aggregate endowment with growth rate

$$d \log e_t = \mu_e dt + \sigma_e dW_t$$

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  - With learning, u is endogenously determined.

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  - Also a result of specific *robust utility models* (*Hansen, Sargent et al.*)
  - With learning, u is endogenously determined.
- Agent's beliefs are common knowledge, so there is no strategic behavior.

Literature

- *Kreps and Porteus (1978)*, *Epstein and Zin (1989)* — discrete time recursive preferences
- *Duffie and Epstein (1992)* — continuous time
- *Geoffard (1996)*, *El Karoui et al. (1997)* — nonrecursive representation
- *Dumas, Uppal and Wang (2000)* — Pareto optimal allocations using an approach analogous to the Pareto problem from the time-separable case

Stochastic differential utility — Duffie, Epstein (1992)

- Study a class of preferences that can be characterized by the stochastic differential utility

$$V_t = E_t \left[\int_t^T f\left(c_s, V_s\right) ds \right] \qquad V_T = 0$$

  - recursive formulation, time separable preferences as a special case
  - marginal utility MC_t now contains a contribution from the continuation value V_t, not only the current consumption $C_t \implies$ *this is a way how to potentially generate survival.*

- Examples will focus on the Duffie-Epstein-Zin specification

$$f\left(c, u\right) = \frac{\beta}{\rho} \left[c^{\rho} \left(\gamma u\right)^{1-\frac{\rho}{\gamma}} - \gamma u \right]$$

  - generalization of power utility (RA $1-\gamma$, IES $\frac{1}{1-\rho}$)

Nonrecursive representation

We rewrite the recursive problem

$$V_t = E_t\left[\int_t^T f\left(c_s, V_s\right) ds\right]$$

in a nonrecursive form. Defining the *felicity function*

$$F\left(c,\nu\right) \equiv \inf_u \left[f\left(c,u\right) + u\nu\right]$$

it can be shown that V_t is equivalently part of the solution to the minimization problem

$$\begin{aligned} \lambda_t V_t &= \inf_\nu E_t\left\{\int_t^T \lambda_s F\left(c_s,\nu_s\right) ds\right\} \\ \frac{d\lambda_s}{\lambda_s} &= -\nu_s ds. \end{aligned}$$

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- ν is the endogenously determined *discount rate process*, and λ is the *discount factor process*.
  - for time-separable preferences, $\nu_t = \beta$, and $\lambda_t = \lambda_0 e^{-\beta t}$.

Incorporating belief distortions

- Use the same concept under distorted beliefs

$$\lambda_t V_t = \inf_{\nu} E_t^Q \left\{ \int_t^T \lambda_s F(c_s, \nu_s)\, ds \right\}$$

$$M_t \lambda_t V_t = \inf_{\nu} E_t \left\{ \int_t^T M_s \lambda_s F(c_s, \nu_s)\, ds \right\}$$

and define the *augmented discount factor* as

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- Take $\bar{\lambda}$ and ν as the solution to the inf problem. Itō's lemma implies

$$\frac{d\bar{\lambda}_t}{\bar{\lambda}_t} = -\nu_t dt + u_t dW_t$$

  - belief distortion introduces a diffusion term to the evolution of the discount factor — additional contribution to risk prices

Pareto problem in an endowment economy

- Assume an N agent economy with an aggregate endowment process e. The social planner solves, subject to feasibility, the objective

$$J_t = \sup_{(c)} \sum_{n=1}^{N} \alpha^n V_0^n(c^n) = \sup_{(c)} \inf_{(\nu)} \sum_{n=1}^{N} E_t\left(\int_0^T \bar{\lambda}_s^n F^n(c_s^n, \nu_s^n)\, ds\right)$$

subject to

$$\begin{aligned} \frac{d\bar{\lambda}_t^n}{\bar{\lambda}_t^n} &= -\nu_t^n dt + u_t^n dW_t \\ \bar{\lambda}_0^n &= \alpha^n \qquad\qquad n = 1, \ldots, N \end{aligned}$$

and the feasibility constraint $\sum_{n=1}^{N} c^n \leq e$.

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and the feasibility constraint $\sum_{n=1}^{N} c^n \leq e$.

- In Markov environment, this leads to a HJB equation and numerical solutions.
  - $\bar{\lambda}$ has the interpretation of *time-varying Pareto weights*
  - importantly, introduction of fixed belief distortions does not lead to any additional state variables

Survival

- From now on, consider a parsimonious endowment economy
  - 2 agents, agent 2 rational ($u^2 = 0$, without loss of generality)
  - iid growth of aggregate endowment

$$d \log e_t = \mu_e dt + \sigma_e dW_t$$

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- Rather than looking at the Pareto weights

$$\frac{d\bar{\lambda}_t^n}{\bar{\lambda}_t^n} = -\nu_t^n dt + u_t^n dW_t$$

  we look at the Pareto share of agent 1 (scaling of weights is innocuous)

$$\theta_t^1 = \frac{\bar{\lambda}_t^1}{\bar{\lambda}_t^1 + \bar{\lambda}_t^2}$$

  θ^1 is our measure of *relative survival*.

- A sufficient condition for survival is the existence of a nondegenerate stationary distribution for θ^1

Survival

Proposition

Given the dynamics of the Pareto share

$$d\theta_t^1 = \theta_t^1 \left(1 - \theta_t^1\right) \left(\nu_t^2 - \nu_t^1 - \theta_t^1 \left(u^1\right)^2\right) dt + \theta_t^1 \left(1 - \theta_t^1\right) u^1 dW_t,$$

it is sufficient for the existence of a stationary density for θ^1 *if there exist thresholds* $\underline{\theta} \leq \bar{\theta}$ *such that*

i) $\nu^2\left(\theta^1\right) - \nu^1\left(\theta^1\right) > \frac{1}{2}\left(u^1\right)^2$ *on* $(0, \underline{\theta})$

ii) $\nu^2\left(\theta^1\right) - \nu^1\left(\theta^1\right) < \left(u^1\right)^2$ *on* $\left(\bar{\theta}, 1\right)$

iii) $\nu^2\left(\theta^1\right) - \nu^1\left(\theta^1\right)$ *is bounded on* $\left(\underline{\theta}, \bar{\theta}\right)$.

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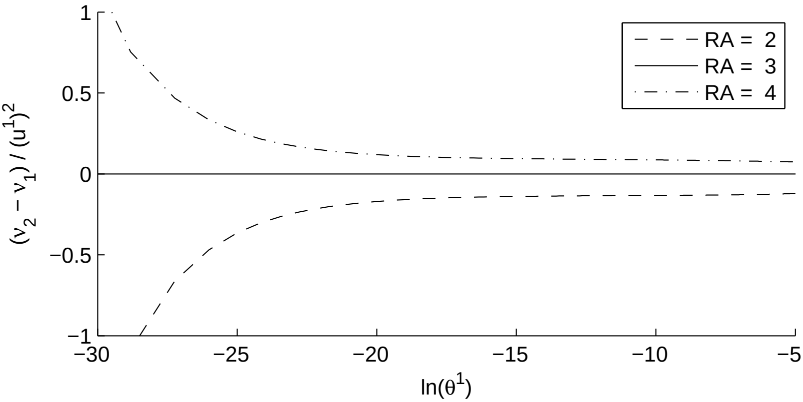
Corollary

Under conditions in Proposition 1, the agent with distorted beliefs survives with probability one.

Example — behavior of discount rates, IES $= 1/3$



Survival result for high risk aversion

Proposition

Under time-separable preferences (when $\gamma = \rho$), the consumption share of the irrational agent with $u^1 \neq 0$ goes to zero with probability one, and the irrational agent thus faces extinction.

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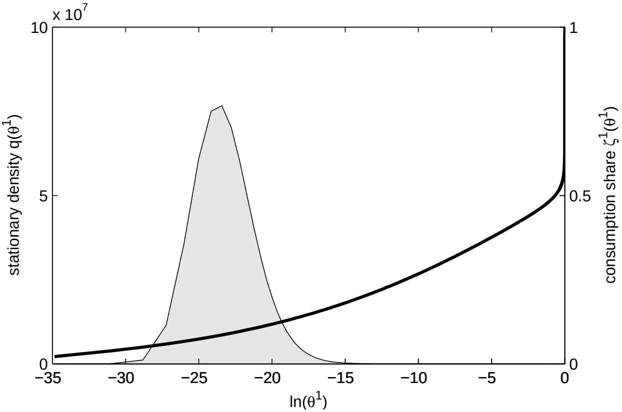
Claim

Given a belief distortion u^1 and intertemporal elasticity of substitution ρ, agent with distorted beliefs will survive in the long run when the risk aversion in the economy is sufficiently high (i.e. γ sufficiently low).

Example — stationary distribution, IES = 1/3, RA = 10



Asset pricing and risk prices

- Having solved for allocations and continuation values in the planner's problem, we can use agents' implied stochastic discount factors

$$S_{t,s} = \exp\left(\int_0^s f_v\left(C_{t+u}, V_{t+u}\right) du\right) \frac{f_c\left(C_{t+s}, V_{t+s}\right)}{f_c\left(C_t, V_t\right)}$$

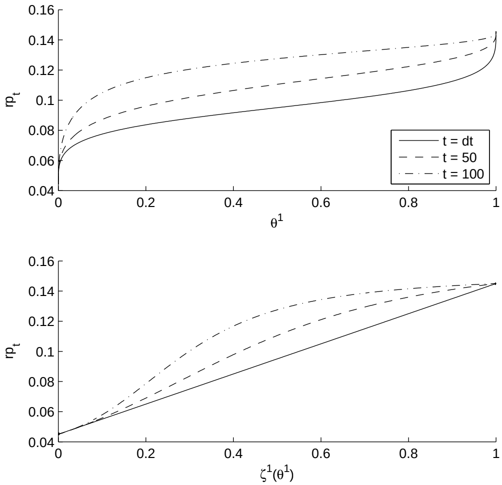
- In line with *Hansen (2008)*, we are interested in the term structure of risk prices
  - Consider a martingale payoff

$$\frac{D_{t+s}}{D_t} = \exp\left(-\frac{1}{2}\sigma_D^2 s + \sigma_D \int_0^s dW_{t+u}\right)$$

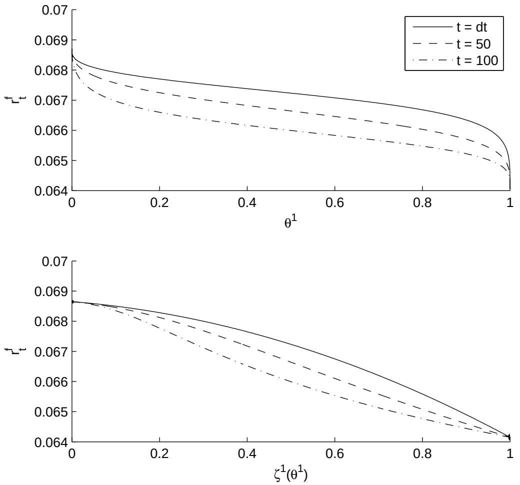
    and the price $P_{t,s}^D$ of a strip paid at time $t+s$
  - The *risk price* is the increase in the expected per-period return resulting from a marginal increase in the risk exposure σ_D

$$rp_{t,s}^D = \frac{d}{d\sigma_D}\left[-\frac{1}{s}\log\frac{P_{t,s}^D}{D_t}\right]$$

Example — risk prices, pessimistic agent



Example — risk-free rate, pessimistic agent



Learning

- We now show how learning about the true model can be incorporated.
- Let us take a model with iid growth of aggregate endowment

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- Let us take a model with iid growth of aggregate endowment

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  - the learning agent has beliefs π_t across a vector κ of possible drift terms (each specifies one model)
  - true evolution can be written as

$$d \log e_t = \kappa \pi_t dt + \sigma_e \underbrace{\left[\frac{\mu_e - \kappa \pi_t}{\sigma_e} dt + dW_t \right]}_{d\tilde{W}_t}$$

    where $\tilde{W}$ is a martingale in the agent's information set.
  - thus

$$u_t = -\frac{\mu_e - \kappa \pi_t}{\sigma_e}$$

Learning

- The evolution equation for the beliefs can be shown to be

$$d\pi_t = \Delta(\pi_t)\,\sigma d\tilde{W}_t$$

where $\Delta(\pi_t)$ is a regression coefficient

- Beliefs π_t form a new state variable vector

Summary

- Recursive preference framework that allows the introduction belief heterogeneity
  - a planner's problem with time-varying Pareto weights to sidestep the direct computation of decentralized equilibria
  - numerically tractable
  - learning is easy to incorporate
  - fits specific robust utility models

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  - a planner's problem with time-varying Pareto weights to sidestep the direct computation of decentralized equilibria
  - numerically tractable
  - learning is easy to incorporate
  - fits specific robust utility models
- Survival results qualitatively different from the time-separable case
  - holding IES constant, a sufficiently high RA guarantees survival
- Price impact
  - stronger impact of belief heterogeneity on the prices of long-horizon payoffs.