Ideologues or Pragmatists?*

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Abstract

We study a model of electoral control where the politician is a policy expert, but the voter is not. First, we focus on the case of an "ideologue," i.e., a politician who always wants the same policy implemented regardless of the state. We show that the voter's lack of policy expertise comes at no cost to him, but may come at an electoral cost to the politician. Next, we focus on the case of a "pragmatist," i.e., a politician whose preferences are state contingent. We show that the voter's lack of policy expertise does come at a cost to him. As a consequence, the voter may fare better with an ideologue than with a pragmatist. This can occur even if the pragmatist's preferences are arbitrarily close to perfectly correlated with the voter's.

What type of politicians do voters prefer? Often, the argument is made that voters prefer politicians who serve their interest on some important dimension. For instance, voters may prefer congruent to non-congruent politicians (Canes-Wrone, Herron and Shotts, 2001; Maskin and Tirole, 2004; Besley, 2006; Fox, 2007), honest to dishonest politicians (Callander and Wilkie, 2007), non-corrupt to corrupt politicians (Myerson, 1993), and high- to low-ability politicians (Persson and Tabellini, 2000; Ashworth, 2005; Ashworth and Bueno de Mesquita, 2006).

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Here, we focus on a distinction between what we will call ideologues and pragmatists. We view an ideologue as a politician whose preferences about some particular policy do not depend on the data. Put differently, an ideologue is a politician whose preferences are not state contingent. For instance, a politician who supports or opposes stem cell research, regardless of the scientific evidence, is an ideologue (on that policy dimension).

Contrast this with a pragmatist. In our account, a pragmatist is a politician whose preferences about some particular policy depend on the data. Put differently, a pragmatist is a politician whose preferences over a particular policy are state contingent. For instance, a politician whose position on stem cell research depends on scientific facts (i.e., relating to the likely impact of such research or the likely success of alternative approaches) is a pragmatist (on that policy dimension). Such a politician's preferences may change with new information.

The distinction we focus on is in line with Canes-Wrone and Shotts's (2007) notion of ideological rigidity.[1] This idea revolves around whether or not a politician's preferences are state contingent. It is important to note that ideologues do not (necessarily) care more about policy than pragmatists. Likewise, ideologues may be willing to compromise their principles, just as pragmatists are. In this paper, we draw only one distinction—whether preferences are sensitive to new information or not. Indeed, we consider the case where politicians—be they ideologues or pragmatists—care both about policy and about electoral benefits. Both types of politicians may be prepared to compromise on their policy preferences if electoral incentives are sufficiently strong.

Intuition might suggest that a pragmatic voter will always prefer a pragmatic politician, so long as the politician's response to new information is similar to the voter's. Indeed, when the voter and politician have access to the same information, the voter will prefer such a pragmatist. The rationale is as follows. Because the voter has access to the same information as the politician, the voter can offer electoral incentives that depend on both the voter's (and politician's) information and on the policy the politician chose.

We will see that this need not be the case when the politician is a policy expert—i.e., when the politician has access to information (about the state) that is not available to the voter. In this case, the voter does not know the true state, and so does not know the pragmatist's actual preferences over policy. This uncertainty can make it difficult to induce the politician to choose a policy that reflects the voter's interests.

Consider the case of an ideologue. In the absence of electoral incentives, ideologues are not inclined to respond to the state of the world. This is bad from the pragmatic voter's

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[1] Within their model, an ideologue (in our sense of the term) is a politician with β equal to zero or one.

perspective. That said, the voter knows exactly where the ideologue stands on the issues. We show that the voter can use this knowledge to gain electoral control.

This tradeoff raises a question: Does a pragmatic voter prefer an ideologue or a pragmatist? To answer this question, we consider a model of electoral control in which the incumbent is a policy expert relative to the voter. We are interested in the level of compliance the voter can achieve (under a Bayesian equilibrium analysis). That is, in what situations can the voter induce the politician to choose his ideal point?

We begin with a politician who is an ideologue, and show three surprising results. First, the voter can achieve the same level of compliance that he could if he had full information. Thus, the absence of information comes at no cost to the voter. But to achieve this level of compliance, the voter uses a particular voting rule, and this voting rule may differ from the one used when he (i.e., the voter) is a policy expert. This leads to the second result. Because the voter uses a different voting rule, his lack of information may come at an electoral cost to the politician. The third result follows from the particular voting rule used. We show that an uninformed voter, faced with an incumbent ideologue, should bias her reelection decisions to reward incumbents who choose "extreme" policies—i.e., policies far from the incumbent's own ideal point.[2]

We next turn to a politician who is a pragmatist. We consider two (particular) formalizations of a pragmatic politician. In each formalization, the voter's and pragmatist's preferences are positively correlated. Indeed, the voter's and pragmatist's ideal points are arbitrarily close to perfectly correlated. We show another surprising result: the voter may be better off with an ideologue than with a pragmatist. How can this happen?

The voter's means for inducing the politician to choose his (i.e., the voter's) ideal point is by offering electoral incentives—that is, by choosing different probabilities of reelection for different policy choices. Because the voter is not a policy expert, these electoral incentives cannot depend on the state (or, put differently, the information). They only depend on the policy the politician actually chooses.

In the case of an ideologue, the voter can infer the politician's cost of choosing any given policy over her (i.e., the politician's) own ideal point. The voter can use this information to design electoral incentives, so that the politician chooses the voter's ideal point in a wide variety of scenarios.

In the case of a pragmatist, however, the voter cannot infer the politician's cost of choosing any given policy over her (i.e., the politician's) own ideal point. (This is the case, even though the voter can infer the politician's cost of choosing the voter's ideal point over

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[2] We thank an anonymous referee for bringing this third point to our attention.

her own ideal point.) As such, it is difficult for the voter to design electoral incentives so that the politician chooses the voter's ideal point in a wide variety of scenarios. In particular, unlike the case of an ideologue, here the voter faces a tradeoff: By offering electoral incentives to choose the voter's ideal point in certain scenarios, the voter must give up on the politician choosing his (i.e., the voter's) ideal point in other scenarios.

To sum up, we will see that the voter may fare better with an ideologue than with a pragmatist—provided that the politician is a policy expert and the voter is not. We view the case of policy expertise as "natural," and indeed it has been studied by a number of papers in the literature. See, for instance, Gilligan and Krehbiel (1987, 1989, 1990), Schultz (1996), and Callander (2008, 2009).

The paper proceeds as follows. Section 1 lays out the game. Section 2 defines the notion of compliance. In Section 3, we study the case of an ideologue. We show that, in a wide variety of scenarios, the voter can achieve the same level of compliance as he could if he had full information. In Section 4, we turn to the case of a pragmatist. We show that there are several limits on the voter's ability to achieve compliance, and we make steps toward characterizing the best equilibrium from the voter's perspective. Section 5 shows that these limitations on the voter's ability to achieve compliance are important. Indeed, the voter may prefer an ideologue over a pragmatist, even when the pragmatist's preferences are arbitrarily close to perfectly positively correlated with the voter's. Finally, Section 6 discusses the case where the voter cares about the future. We verbally argue that our results also cover this case.

1 The Game

Throughout we adopt the following conventions: Endow a metrizable space X with its Borel sigma-algebra. If $Y \subseteq X$, endow Y with the relative topology, so that it is again metrizable. Also, endow the product of metrizable spaces with the product topology. Write $\mathcal{M}(X)$ for the set of probability measures on X, and endow $\mathcal{M}(X)$ with the topology of weak convergence.

There are two players, the Politician and the Voter. We refer to the Politician as "she" and the Voter as "he." The order of play is as follows: Nature chooses a state, viz. ω, from a metrizable space Ω. The state determines the players' policy preferences. The Politician observes the true state and then chooses a policy, viz. $p \in \mathbb{R}$. The Voter observes the Politician's policy choice. Finally, the Voter decides whether or not to reelect the Politician. That is, the Voter chooses $r \in \{0, 1\}$, where $r = 1$ represents the decision to reelect the Politician.

The Voter does not know the true state chosen by Nature, viz. $\omega \in \Omega$. Instead, he has a prior μ on Ω, which is "transparent" to the players.

Let $x_P : \Omega \to \mathbb{R}$ (resp. $x_V : \Omega \to \mathbb{R}$) be an integrable real-valued random variable. The interpretation is that $x_P(\omega)$ (resp. $x_V(\omega)$) is the Politician's (resp. Voter's) ideal point when the true state is ω. When the Politician is an Ideologue, her ideal point does not vary with the state. In this case, we will take $x_P(\Omega) = \{0\}$. When the Politician is a Pragmatist there are states $\omega, \omega' \in \Omega$, so that $x_P(\omega) \neq x_P(\omega')$. The Voter is pragmatic. Moreover, the Voter's preferences are rich—for any given policy p, there is some state ω, so that the Voter's ideal point at ω is p. That is, x_V is surjective. Notice, since the Voter is uncertain about the state, he faces uncertainty about how policies map into outcomes.

The Politician has quadratic preferences over policy and seeks reelection. Given a policy p and an ideal point of $x_P(\omega)$, the Politician gains a payoff $-(p - x_P(\omega))^2$. If reelected, the Politician receives a payoff of $B > 0$. Taken together, these imply that the Politician's payoffs, at a state ω, from a policy p and reelection decision r are

$$u_P(\omega, p, r) = -(p - x_P(\omega))^2 + rB.$$

The Voter also has quadratic preferences over policy. Formally, at a state ω, the Voter's payoffs from a policy p are

$$u_V(\omega, p) = -(p - x_V(\omega))^2.$$

We will focus on equilibria in behavioral strategies: A (behavioral) strategy for the Politician maps each state into a probability measure on the policy space, viz. $s_P : \Omega \to \mathcal{M}(\mathbb{R})$. A (behavioral) strategy for the Voter maps each policy choice into a measure on reelection decisions, i.e., is a mapping $p \mapsto \mathcal{M}(\{0,1\})$. Note, we can view, instead, a strategy for the Voter as a mapping $s_V : \mathbb{R} \to [0,1]$, where $s_V(p)$ is the probability that the Voter reelects the Politician after seeing policy p.[3] We loosely refer to $s_V(p)$ as a measure on $\{0,1\}$. No confusion should result.

Given a strategy, viz. s_V, write $\mathbb{E}u_P(\omega, p, s_V)$ for the Politician's expected payoffs from choosing policy p when the Voter chooses the reelection strategy s_V, i.e.,

$$\mathbb{E}u_P(\omega, p, s_V) = -(p - x_P(\omega))^2 + s_V(p)B.$$

Consider the case when the Politician chooses a mixture, viz. $\sigma_P \in \mathcal{M}(\mathbb{R})$. If s_V is a

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[3] Here, we implicitly use Lemma A2 in Appendix A.

measurable strategy, we can think of the image measure of σ_V under s_V, viz. $s_V[\sigma_P]$. So, $s_V[\sigma_P]$ gives the probability that the Politician is reelected if she chooses a mixture σ_P. That is, $s_V[\sigma_P]$ is a measure on $\{0,1\}$ where, for each event E in $\mathcal{M}(\{0,1\})$, $s_V[\sigma_P](E) = \sigma_P((s_V)^{-1}(E))$.

If the true state is ω and the Politician chooses a mixture of policies, viz. σ_P, then her expected payoffs under the Voter's strategy s_V are

$$\mathbb{E}u_P(\omega, \sigma_P, s_V) = -\textstyle\int_{\mathbb{R}} (p - x_P(\omega))^2 \, d\sigma_P + s_V[\sigma_P] B.$$

We will abuse notation and write $\mathbb{E}u_P(\omega, s_P, s_V)$ for $\mathbb{E}u_P(\omega, s_P(\omega), s_V)$, i.e., for the expected utility of the Politician, at the state ω, under the strategy profile (s_P, s_V).

Likewise, if the true state is ω and the Politician chooses a mixture, viz. σ_P, the Voter's expected payoffs are

$$\mathbb{E}u_V(\omega, \sigma_P) = -\int_{p\in\mathbb{R}} (p - x_V(\omega))^2 \, d\sigma_P.$$

Again, we abuse notation and write $\mathbb{E}u_V(\omega, s_P)$ for $\mathbb{E}u_V(\omega, s_P(\omega))$.

This framework is quite general, and so allows us to capture a number of interesting cases. For instance, some examples of metrizable spaces, viz. Ω, include the real line, a compact subset of the real line, or even multidimensional versions thereof. Take the case where $\Omega = \mathbb{R}$. Here, x_V could simply be the identity map, i.e., saying that, if the state is $\omega \in \Omega = \mathbb{R}$, then the Voter's ideal point is exactly ω. Or, alternatively, the Voter's ideal point could be some particular policy plus a shock, i.e., x_V can simply be $x^* + \omega$ for some policy $x^* \in \mathbb{R}$ and some $\omega \in \Omega = \mathbb{R}$. But, our framework allows for other scenarios—e.g., it allows for the case where small changes to the state cause big changes for the Voter's (or Politician's) most preferred policy.

In light of the discussion above, it should be clear that the current framework captures a number of assumptions made in the literature. As an example, in sender-receiver games (e.g., Crawford and Sobel (1982) and Gilligan and Krehbiel (1990)), it is often assumed that the outcome is the policy chosen plus a random shock. Specifically, in such games, $\Omega = \mathbb{R}$ and both the sender's and receiver's ideal points take the form $x^* + \omega$ (where x^* is a different constant for the sender and receiver).

On the other hand, the game itself has some limitations. First, in this game, the Voter cannot become a policy expert. Section 3.4 moves away from this assumption. Second, and more importantly, in this game, the Voter does not care about the future when making reelection decisions. We discuss an extension to this case in Section 6. Finally, we consider the case where both the Voter and Politician have quadratic preferences. We do so for analytic convenience.

2 Compliance

When can the Voter induce the Politician to choose his ideal point? Or, put differently, when can the Voter achieve compliance?

Of course, we want to ask this question under a Bayesian equilibrium analysis of the game. So, we begin there. We then introduce the idea of compliance, which is defined relative to a particular equilibrium. Specifically, the Voter achieves compliance over a set of policies if the Voter can induce the Politician to choose his ideal point whenever it is contained in that set.

Definition 2.1 *A strategy profile* (s_P^*, s_V^*) *is a* ***Bayesian equilibrium*** *if*

(i) s_P^* *and* s_V^* *are measurable; and*

(ii) for each $\omega \in \Omega$, $\mathbb{E}u_P(\omega, s_P^*, s_V^*) \geq \mathbb{E}u_P(\omega, \sigma_P, b_V^*)$ *for all* $\sigma_P \in \mathcal{M}(\mathbb{R})$.

First, we require that the Politician's and Voter's strategies must both be measurable. We require that s_P^* be measurable, so that the Voter can compute his expected payoffs. (Note, we need $\mathbb{E}u_V(\cdot -, s_P^*) : \Omega \rightarrow \mathbb{R}$ to be measurable, so that it is indeed integrable.) Likewise, we require that s_V^* is measurable so that the Politician can do the same. (Note, $s_V^*\left[s_p^*(\omega)\right]$ is only defined if s_V^* is measurable.) The final requirement is that, at each state, the Politician must choose a policy that maximizes her expected payoffs, given the Voter's actual "reelection rule," i.e., given the Voter's actual strategy s_V^*. There is no analogous requirement for the Voter. The reason is that he makes his reelection decision at the end of the game, so his choice does not directly affect his payoffs. As such, an explicit optimization requirement for the Voter can be omitted.[4]

Because the Voter's reelection decision does not directly affect his payoffs, the game has many equilibria. As such, we focus on the question of which equilibrium is "best" from the Voter's perspective. That is, what is the maximum level of control the Voter can hope to obtain? This question is of substantive interest because it provides a normative benchmark—i.e., it identifies the most the Voter can hope to achieve with his vote.

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[4] For the same reason, we omit a requirement on updating beliefs. Imposing the natural requirement yields an equivalent definition.

Toward this end, we now turn to a notion of compliance. A starting point is: Can the Voter induce the Politician to choose his ideal point when the true state is ω? In fact, we will ask for more. Consider some Bayesian equilibrium, viz. (s_P^*, s_V^*). Can the Voter induce the Politician to choose his ideal point whenever it is the policy p? Informally, we will say that the Voter achieves C-compliance if the Voter can induce the Politician to choose his ideal point whenever it is contained in some set C.

Definition 2.2 *Fix a set $C \subseteq \mathbb{R}$. Call a Bayesian equilibrium, viz. (s_P^*, s_V^*),* ***C-compliant*** *if, for each state $\omega \in (x_V)^{-1}(C)$, $s_P^*(\omega)$ assigns probability one to $x_V(\omega)$.*

Fix an equilibrium, viz. (s_P^*, s_V^*). By definition, there is some set C so that (s_P^*, s_V^*) is C-compliant. (Of course, here, we allow C to be empty.) Also, if (s_P^*, s_V^*) is C-compliant and $D \subseteq C$, then (s_P^*, s_V^*) is also D-compliant. If (s_P^*, s_V^*) is C-compliant and it is not D-compliant, for each D that strictly contains C, say C is the **maximal compliance induced by** (s_P^*, s_V^*).

The idea of maximal compliance will be useful in our analysis. But it is important to note that it does not characterize the Voter's welfare under different equilibria. In particular, it only considers whether the Voter's actual ideal point is chosen. However, the Voter may also care about what policy is chosen in those states where his ideal point is not chosen. Thus, we may have two equilibria with the same level of maximal compliance, but the Voter may prefer one over the other. We could even have one equilibrium preferred to a second, despite the fact that it has a lower level of maximal compliance.

3 The Ideologue

In this section, we focus on the Politician who is an Ideologue. We will see that the Voter can achieve a high level of compliance—namely, the same level he could achieve if he too were a policy expert (i.e., if the Voter learned the true state before voting). We first analyze this latter situation. It should be viewed as a benchmark case.

3.1 A Benchmark

Consider a different game—one in which the Voter learns the true state before making his reelection decision. What is the best that the Voter can achieve in this game?

First, notice that if (s_P^*, s_V^*) is a C-compliant equilibrium then $C \subseteq [-\sqrt{B}, \sqrt{B}]$. Put differently, the Voter can never induce the Ideologue to choose his ideal point when it is further than $\sqrt{B}$ from the Ideologue's ideal point. The electoral incentives for choosing such a policy are necessarily insufficient. (Recall, electoral incentives are bounded from above by B.) See Lemma B1 in Appendix B.

But there does exist a $[-\sqrt{B}, \sqrt{B}]$-compliant equilibrium. Consider a strategy of the Voter where, at each state ω, he reelects the Ideologue if and only if she actually chose $x_V(\omega)$. That is, the Voter reelects the Ideologue if and only if the Ideologue chose the Voter's actual ideal point. Now, it is a best response for the Ideologue to choose the Voter's (resp. her own) ideal point, whenever it is closer (resp. further) than $\sqrt{B}$ from her own ideal point.

The Voter can improve on this. Suppose, instead, the Voter uses the following rule. At each state, viz. ω, with $x_V(\omega) \in [-\sqrt{B}, \sqrt{B}]$, reelect the Ideologue if and only if she actually chose $x_V(\omega)$. At each state, viz. ω, with $x_V(\omega) < -\sqrt{B}$ (resp. $x_V(\omega) > \sqrt{B}$), reelect the Ideologue if and only if she actually chose $-\sqrt{B}$ (resp. $\sqrt{B}$) Now, there is an associated equilibrium where the Ideologue chooses the Voter's ideal point whenever it lies within $\sqrt{B}$ of zero and otherwise chooses $\pm\sqrt{B}$, as per the Voter's preference. Moreover, this equilibrium is "best" from the Voter's perspective. (See Proposition 3.2 to come.)

3.2 Compliance with Pure Strategies

Now turn to the game described in Section 1, where the Voter is not a policy expert. Here, the Voter does not know the true state at the point where he makes his reelection decision. As such, the reelection rule cannot be state contingent.

Does a $[-\sqrt{B}, \sqrt{B}]$-compliant equilibrium exist? At first blush, the answer may appear to be no. To see why, consider the case where the Voter is restricted to the use of pure strategies. Fix an equilibrium where, at some state, the Ideologue chooses a policy $p > 0$. Then the Voter must offer electoral incentives for choosing policy p. If not, the Ideologue would strictly prefer to choose her own ideal point at the given state. Moreover, the Voter must offer no electoral incentives for choosing policies closer to the Ideologues's ideal point (i.e., for choosing policies in $(-p, p)$). If not—i.e., if the Voter also reelects the Ideologue when she chooses some $q \in (-p, p)$—then the Ideologue would strictly prefer to choose this policy, viz. q, over the policy p. So, if there is a pure-strategy equilibrium where the Voter can induce the Ideologue to choose two distinct policies $p, q \neq 0$, then these policies must be equidistant from the Ideologue's ideal point, i.e., $p = -q$.

This discussion suggests that, in this case, "the best" the Voter can hope for is one of the following two scenarios:

Moderate Two-Sided Voting Rule There is some policy $p^* \in (0, \sqrt{B})$ so that the Voter reelects the Ideologue if and only if she chooses either $-p^*$ or p^*. In any state, the Ideologue chooses either $-p^*$ or p^*, depending on which is closest to the Voter's ideal point.

Extreme Two-Sided Voting Rule The Voter reelects the Ideologue if and only if she chooses either $-\sqrt{B}$ or $\sqrt{B}$. The Ideologue chooses either $0, -\sqrt{B}$, or $\sqrt{B}$, depending on which is closest to the Voter's ideal point.

Proposition B1 in Appendix B states that this is indeed the case. (See, also, the Online Appendix.)

Both the Moderate and Extreme Two-Sided Voting rules exhibit some—but very little—compliance. The set $\{-p, p\}$ is the maximal compliance induced by the Moderate Two-Sided Voting Rule. The set $\{-\sqrt{B}, 0, \sqrt{B}\}$ is the maximal compliance induced by the Extreme Two-Sided Voting Rule. In either case, the Ideologue only chooses the Voter's ideal point on a "small" set of policies.

3.3 Compliance with Behavioral Strategies

In the analysis above, the Voter was only able to achieve a small level of compliance. However, we also only allowed the Voter to use a weak method of maintaining electoral control. Specifically, for any given policy the Ideologue chose, the Voter was restricted to an up-or-down reelection decision. That is, we restricted the Voter to use pure strategies.

Now, consider the case where the Voter can respond probabilistically to policy choices by the Ideologue. We will see that the Voter can achieve a greater level of compliance. Why is this the case?

Begin with the case where the Voter must make an up-or-down electoral decision. Here, the Voter cannot offer electoral incentives to choose distinct policies p and q, where q lies closer to the Ideologue's ideal point than does p. The electoral incentives for choosing q conflict with the electoral incentives for choosing p. However, when the Voter can respond probabilistically, such a conflict need not arise. The Voter can now offer different levels of electoral incentives for choosing different policies, and thereby eliminate the conflict.

Proposition 3.1 *There exists a Bayesian equilibrium in behavioral strategies, viz.* (s_P^*, s_V^*), *so that:*

(i) if $x_V(\omega) \in [-\sqrt{B}, \sqrt{B}]$, *then* $s_P^*(\omega)$ *assigns probability one to* $x_V(\omega)$;

(ii) if $x_V(\omega) < -\sqrt{B}$, *then* $s_P^*(\omega)$ *assigns probability one to* $-\sqrt{B}$; *and*

(iii) if $x_V(\omega) > \sqrt{B}$, then $s_P^*(\omega)$ assigns probability one to $\sqrt{B}$.

Proof. Let s_P^* be a strategy as in the statement of the result. Construct s_V^* so that

$$s_V^*(p) = \begin{cases} \frac{p^2}{B} & \text{if } p \in [-\sqrt{B}, \sqrt{B}] \\ 1 & \text{if } p \in \mathbb{R} \backslash [-\sqrt{B}, \sqrt{B}]. \end{cases}$$

We will show that (s_P^*, s_V^*) is a Bayesian equilibrium. Part (i) of Definition 2.1 is shown as Lemmata A3-A4 in Appendix A. As such, we focus on part (ii) of Definition 2.1.

Fix some $\omega \in \Omega$. It suffices to show that

$$\mathbb{E}u_P(\omega, s_P^*, s_V^*) = 0 \geq \mathbb{E}u_P(\omega, p, s_V^*)$$

for each p with $s_P^*(\omega)(p) = 0$. If so, then, certainly, for any $\sigma_P \in \mathcal{M}(\mathbb{R})$,

$$\mathbb{E}u_P(\omega, s_P^*, s_V^*) \geq \mathbb{E}u_P(\omega, \sigma_P, s_V^*),$$

as required.

Begin by showing that $\mathbb{E}u_P(\omega, s_P^*, s_V^*) = 0$. First, suppose that $B \geq [x_V(\omega)]^2$. Here,

$$\mathbb{E}u_P(\omega, s_P^*, s_V^*) = -[x_V(\omega)]^2 + \frac{[x_V(\omega)]^2}{B} B = 0,$$

as required. Next, suppose that $[x_V(\omega)]^2 > B$. Here too,

$$\mathbb{E}u_P(\omega, s_P^*, s_V^*) = -B + B = 0,$$

as required.

Now fix some p with $s_P^*(\omega)(p) = 0$. If $B \geq p^2$, then

$$\mathbb{E}u_P(\omega, p, s_V^*) = -p^2 + \frac{p^2}{B} B = 0,$$

as required. If $p^2 > B$, then

$$\mathbb{E}u_P(\omega, p, s_V^*) = -p^2 + B < 0,$$

as required. ■

We have the following immediate consequence of Proposition 3.1.

Corollary 3.1 *There exists a $[-\sqrt{B}, \sqrt{B}]$-compliant Bayesian equilibrium in behavioral strategies.*

There are three somewhat surprising implications from Proposition 3.1. First, the Voter's lack of information comes at no cost to him. Specifically, the Voter can achieve the same level of electoral control whether he is a policy expert (as in Section 3.1) or not (as in Proposition 3.1).

Second, the Voter's lack of information may make the Ideologue worse off. To see this, consider the strategy s_P^* defined in Proposition 3.1. This strategy is associated with a Bayesian equilibrium, both when the Voter has full information and when the Voter has no information. (Of course, from the Voter's perspective, the equilibria associated with this strategy are the "best" in their respective games.) When the Voter knows the true state, there is an equilibrium in which the Ideologue chooses s_P^* and is reelected with certainty. When the Voter does not know the true state, there is also an equilibrium in which the Ideologue chooses s_P^*. In any such equilibrium, the Voter reelects the Ideologue with probability strictly less than one whenever the Voter's ideal point lies in $(-\sqrt{B}, \sqrt{B})$. Thus, the Voter's lack of information—while imposing no cost on him—does come at an electoral cost to the Ideologue.

Third, notice, the equilibrium constructed has the property that the Voter reelects the incumbent Ideologue with probability $\frac{p^2}{B}$ for $p \in [-\sqrt{B}, \sqrt{B}]$. (Indeed, any $[-\sqrt{B}, \sqrt{B}]$-compliant equilibrium must have this feature). So, the Voter is more likely to reelect the incumbent only if she chooses sufficiently "extreme" policies. That is, an uninformed Voter, faced with an incumbent Ideologue, should bias his vote toward rewarding policy choices that are far from the incumbent's ideal point.

3.4 Robustness

We began by restricting the Voter to use a deterministic voting rule and saw that he could achieve little compliance. Then, we allowed the Voter to use a probabilistic voting rule and saw that the Voter could achieve a higher level of compliance.

This raises the question: Does the Voter's ability to achieve compliance crucially depend on his ability to give probabilistic incentives? Arguably not. Specifically, there is an interesting extension of the game in which the Voter can achieve the same level of compliance while only using deterministic voting rules (i.e., pure strategies).

Consider a modified game in which the Voter has the possibility of learning the true state. That is, prior to the Voter's electoral decision, Nature chooses whether or not to inform the Voter of the true state. Ex ante, the Ideologue assigns probability π to the event that the Voter learns the true state and this probability is "transparent" to the players.

We restrict the Voter to use pure strategies. Nonetheless, the Voter has a richer strategy set. Since he can now learn the true state, he can offer two types of incentives: informed incentives and uninformed incentives.

Informed Incentives: Say the Voter offers *informed incentives* to choose policy p at a state ω if the Voter's strategy takes the following form: If the true state is ω and the Voter is informed, he reelects the Ideologue if and only if she chooses policy p.

We have already seen the use of informed incentives, in the Benchmark case. (See Section 3.1.) These incentives are state contingent and so do not conflict with one another. But note, the Voter can only use informed incentives if Nature does indeed inform him of the true state. So, if at the state ω informed incentives by themselves are to induce the Ideologue to choose the Voters' ideal point, then $-x_V(\omega)^2 + \pi B \geq 0$.

Uninformed Incentives: Say the Voter offers *uninformed incentives* to choose policy p if the Voter's strategy takes the following form: If the Voter is uninformed, he reelects the Ideologue if she chooses policy p.

We have already seen the use of uninformed incentives in Section 3.2. These incentives are not state contingent and so may conflict with one another. Again, the Voter can only use uninformed incentives if Nature does not inform him. So, if at the state ω uninformed incentives are to induce the Ideologue to choose the Voter's ideal point, then $-x_V(\omega)^2 + (1-\pi)B \geq 0$.

Uninformed and Informed Incentives: The Voter can combine uninformed and informed incentives. Doing so allows the Voter to avoid the conflict that was associated with using uniformed incentives alone. Indeed, by using both of these incentives simultaneously, the Voter can induce the Ideologue to choose his ideal point whenever it is contained in

$$[-\sqrt{B}, -\sqrt{(1-\pi)B}] \cup [\sqrt{(1-\pi)B}, \sqrt{B}].$$

To see this, fix two states with $\sqrt{B} \geq x_V(\omega_1) > x_V(\omega_2) \geq \sqrt{(1-\pi)B}$. Suppose that the Voter only offers (i) informed incentives to choose $x_V(\omega_1)$ (resp. $x_V(\omega_2)$) at the state ω_1 (resp. ω_2) and (ii) uninformed incentives to choose $x_V(\omega_1)$ and $x_V(\omega_2)$. Further, suppose that the true state is ω_1. If the Ideologue chooses the Voter's ideal point at this state, her expected payoffs (namely, $-x_V(\omega_1)^2 + B$) are positive. As such, she has no incentive to choose her own ideal point. (If she chooses her own ideal point, she is certain not to be reelected.) She also has no incentive to choose $x_V(\omega_2)$. If the Voter is uninformed, she is reelected if she chooses either of $x_V(\omega_1)$ or $x_V(\omega_2)$. By choosing $x_V(\omega_2)$, she gets a policy closer to her ideal point, but at the cost of forgoing the informed incentives. Because $x_V(\omega_2)$ is sufficiently far from zero, the benefits associated with choosing this more-preferred policy are outweighed by the costs of forgoing the informed incentives.

Now let's consider a voting rule that provides both informed and uninformed incentives. Specifically, consider a voting rule that provides informed incentives to choose the Voter's ideal point whenever it is contained in

$$[-\sqrt{B}, -\sqrt{(1-\pi)B}] \cup [-\sqrt{\pi B}, \sqrt{\pi B}] \cup [\sqrt{(1-\pi)B}, \sqrt{B}],$$

and provides uninformed incentives to choose policies in

$$[-\sqrt{B}, -\sqrt{(1-\pi)B}] \cup [\sqrt{(1-\pi)B}, \sqrt{B}].$$

The arguments above suggest that this voting rule induces the Ideologue to choose the Voter's ideal point whenever it is contained within

$$[-\sqrt{B}, -\sqrt{(1-\pi)B}] \cup [-\sqrt{\pi B}, \sqrt{\pi B}] \cup [\sqrt{(1-\pi)B}, \sqrt{B}].$$

Notice that when $\pi \geq \frac{1}{2}$ this fact implies that the Voter can achieve $[-\sqrt{B}, \sqrt{B}]$-compliance. Indeed, the Voter can do just as well as in Proposition 3.1.

Proposition 3.2 *Suppose that $\pi \in [\frac{1}{2}, 1]$. Then there is a pure strategy Bayesian equilibrium in which the Ideologue's strategy, viz. s_P^*, is as defined in Proposition 3.1.*

The proof can be found in Appendix B. There we also describe equilibrium for the case where $\pi \in [0, \frac{1}{2})$.

4 Pragmatists

Up to this point, we considered a model with an Ideologue. We stated that, if an equilibrium is C-compliant, then $C \subseteq [-\sqrt{B}, \sqrt{B}]$. Moreover, there is some equilibrium that is

$[-\sqrt{B}, \sqrt{B}]$-compliant.

Now consider the case where the Politician is a Pragmatist whose preferences are positively correlated with the Voter's. Intuitively, it seems that, when ideal points are aligned in this way, the Voter's electoral control problem should be particularly easy. But we will see that, even under these circumstances, the Voter may be better off with an Ideologue. Indeed, this may be the case, even if the Pragmatist's preferences are arbitrarily close to perfectly positively correlated with the Voter's.

To see the basic problem, we begin with a particular specification of the Pragmatist—one in which the Pragmatist's ideal point is always k units above the Voter's ideal point. In particular, for each $\omega \in \Omega$, $x_P(\omega) = x_V(\omega) + k$, where $k \in (0, \sqrt{B}]$. So, conditional on the Voter knowing the Pragmatist's ideal point is p, he also knows his own ideal point is $p - k$. But, importantly, because the Voter does not know the true state, he does not know the Pragmatist's ideal point. As such, he does not know the Pragmatist's cost of choosing any given policy over her own ideal point. This is where the problem arises.

Suppose that, at each state, the Voter can induce the Pragmatist to choose his ideal point. Then, for each policy p, the Voter must offer electoral incentives to choose p over $p + k$: If not, there will be some state at which the Pragmatist prefers to choose her own ideal point, viz. $p + k$, over the Voter's ideal point, viz. p. So, the electoral incentives for choosing the policy p must be higher than the electoral incentives for choosing the policy $p + k$. But, the Voter also wants the Pragmatist to choose his ideal point when it is $p + k$. So the electoral incentives for choosing $p + k$ must be higher than the electoral incentives for choosing $p + 2k$. More generally, for each m, the electoral incentives for choosing policy $p + mk$ must be higher than the electoral incentives for choosing $p + (m+1)k$. (And, of course, the minimal electoral incentives required to choose $p + mk$ over $p + (m+1)k$ do not depend on m.) But this cannot be, since the electoral incentives are bounded from above by B.

The next result formalizes this idea.

Proposition 4.1 *Fix some set $\{p, p+k, \ldots, p+mk\}$. If (s_P^*, s_V^*) is $\{p, p+k, \ldots, p+mk\}$-compliant, then $\frac{B-k^2}{k^2} \geq m$.*

To prove Proposition 4.1, we will make use of the following remark.

Remark 4.1 *Fix a Bayesian Equilibrium, viz. (s_P^*, s_V^*), and a state $\omega \in \Omega$ with $s_P^*(\omega)(x_V(\omega)) = 1$. Then, for each $p \in \mathbb{R}$,*

$$[s_V^*(x_V(\omega)) - s_V^*(p)]\, B \geq k^2 - (p - x_V(\omega) - k)^2 .$$

In particular,

$$\left[s_V^*\left(x_V\left(\omega\right)\right) - s_V^*\left(x_V\left(\omega\right)+k\right)\right]B \geq k^2,$$

and so

$$s_V^*\left(x_V\left(\omega\right)\right) \geq \frac{k^2}{B}.$$

Proof. Fix some state ω so that $s_P^*\left(\omega\right)$ assigns probability one to $x_V\left(\omega\right)$. Then, for each $p \in \mathbb{R}$,

$$\begin{aligned} \mathbb{E}u_P\left(\omega, s_P^*, s_V^*\right) &= -k^2 + s_V^*\left(x_V\left(\omega\right)\right)B \\ &\geq -\left(p - x_V\left(\omega\right) - k\right)^2 + s_V^*\left(p\right)B = \mathbb{E}u_P\left(\omega, p, s_V^*\right). \end{aligned}$$

This gives the first inequality. Taking $p = x_V\left(\omega\right) + k$ gives the second inequality. Noting that $1 \geq s_V^*\left(x_V\left(\omega\right)+k\right)$ gives the final inequality. ■

Proof of Proposition 4.1. Fix some equilibrium $\left(s_P^*, s_V^*\right)$ that is $\{p, p+k, \ldots, p+mk\}$-compliant. Applying Remark 4.1, we have that

$$\left[s_V^*\left(p\right) - s_V^*\left(p+k\right)\right]B \geq k^2$$

$$\left[s_V^*\left(p+k\right) - s_V^*\left(p+2k\right)\right]B \geq k^2$$

$$\cdots$$

$$\left[s_V^*\left(p+mk\right) - s_V^*\left(p+\left(m+1\right)k\right)\right]B \geq k^2.$$

So, certainly,

$$\left[s_V^*\left(p\right) - s_V^*\left(p+\left(m+1\right)k\right)\right]B \geq \left(m+1\right)k^2.$$

Now note that $1 \geq \left[s_V^*\left(p\right) - s_V^*\left(p+\left(m+1\right)k\right)\right]$. It follows that

$$B \geq \left[s_V^*\left(p\right) - s_V^*\left(p+\left(m+1\right)k\right)\right]B \geq \left(m+1\right)k^2,$$

or

$$\frac{B}{k^2} \geq m+1.$$

The result now follows immediately. ■

Proposition 4.1 says that there is a limit on the ability of the Voter to achieve compliance from a Pragmatist. The reason is that the Voter would like to give the Pragmatist incentives to choose policies $p, p+k, p+2k$, and so on. But, because the Voter does not know the actual state, the Voter does not know the Pragmatist's cost of choosing any one of these policies. As such, he must offer higher and higher electoral incentives to choose lower and lower policies.[5] He cannot do so because electoral incentives are bounded.

4.1 A Second Limitation on Compliance

Proposition 4.1 provides one limitation on the ability of the Voter to achieve compliance. However, it is not the end of the story. We begin by highlighting a second important limitation. This is a key step for identifying some of the tradeoffs the Voter faces—i.e., for understanding which equilibrium is "best" from the Voter's perspective. (We elaborate on that in the next subsection.)

Consider a simple numerical example. Take $B = 11$ and $k = 1$. The set $[0,6] \cup \{7,8,9,10\}$ satisfies the requirements of Proposition 4.1. (Each $\{p, p+k, \ldots, p+mk\}$ contained in $[0,6] \cup \{7,8,9,10\}$ has $m \leq 10$.) The limitation examined in Proposition 4.1 focused on the Voter providing incentives to choose 0 over 1, 1 over 2, and so on. However, if the Voter achieves $[0,6] \cup \{6,7,8,9,10\}$ compliance, then he must also provide the Pragmatist with incentives to choose 0 over, say, .5, i.e., to choose a policy p over a policy $q \in (p, p+k)$. We will see that he cannot do so for each policy $p \in [0,6]$.

Here is the idea of the proof: Suppose we found an equilibrium, viz. (s_P^*, s_V^*) that is $[0,6] \cup \{7,8,9,10\}$-compliant. The key step is that, for each $p \in [0,6]$, s_V^* is differentiable at p and, in particular, the derivative is $-\frac{2}{11} = -\frac{2k}{B}$. (See Lemma 4.1 below. This arises from the requirement that the Voter must offer electoral incentives to choose a policy p over a policy $q \in (p, p+k)$.) But then, we have

$$\frac{s_V^*(6) - s_V^*(0)}{6-0} = -\frac{2}{11},$$

or

$$s_V^*(0) - s_V^*(6) = \frac{12}{11}.$$

But, this cannot be, since $1 \geq s_V^*(0)$ and $s_V^*(6) \geq 0$.

Let us give this argument more generally. Fix an interval $[\underline{p}, \overline{p}] \subseteq \mathbb{R}$. We call $\overline{p} - \underline{p}$ the **length of the interval** $[\underline{p}, \overline{p}]$.

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[5] Of course, this is because the Pragmatist's ideal point is biased upward (i.e., $k > 0$). If k were negative, the Voter would have to offer lower and lower electoral incentives to choose lower and lower policies. An analogous limitation would arise.

Proposition 4.2 *If* (s_P^*, s_V^*) *is* $[\underline{p}, \overline{p}]$*-compliant, then the length of* $[\underline{p}, \overline{p}]$ *is less than or equal to* $\frac{B-k^2}{2k}$.

The key to Proposition 4.2 is the following lemma.

Lemma 4.1 *Fix a Bayesian equilibrium, viz.* (s_P^*, s_V^*)*, that is* $[\underline{p}, \overline{p}]$*-compliant. Then, for each* $p \in [\underline{p}, \overline{p}]$*,* s_V^* *is differentiable at* p *and, specifically, the derivative of* s_V^* *at* p *is* $-\frac{2k}{B}$.

Proof. Fix some (s_P^*, s_V^*) that is $[\underline{p}, \overline{p}]$-compliant. Then, by Remark 4.1,

$$s_V^*(p) - s_V^*(q) \geq -\frac{(p-q)^2}{B} - \frac{2k(p-q)}{B},$$

for each $p, q \in [\underline{p}, \overline{p}]$.

Fix some $p \in [\underline{p}, \overline{p}]$ and some sequence $\{q_n\}_{n \in \mathbb{N}}$, so that $q_n \in [\underline{p}, \overline{p}] \backslash \{p\}$ and $\lim_{n \to \infty} q_n = p$. We have that, for each q_n,

$$s_V^*(p) - s_V^*(q_n) \geq -\frac{(p-q_n)^2}{B} - \frac{2k(p-q_n)}{B}$$

and

$$s_V^*(q_n) - s_V^*(p) \geq -\frac{(q_n-p)^2}{B} - \frac{2k(q_n-p)}{B}.$$

Putting these two together, we have

$$\frac{(q_n-p)^2}{B} - \frac{2k(q_n-p)}{B} \geq s_V^*(q_n) - s_V^*(p) \geq -\frac{(q_n-p)^2}{B} - \frac{2k(q_n-p)}{B}.$$

It follows that

$$\frac{|q_n-p|}{B} - \frac{2k}{B} \geq \frac{s_V^*(q_n) - s_V^*(p)}{q_n-p} \geq -\frac{|q_n-p|}{B} - \frac{2k}{B}.$$

Now, fix some open set U with $-\frac{2k}{B} \in U$. Then, there exists some N so that, for each $n \geq N$,

$$\frac{s_V^*(q_n) - s_V^*(p)}{q_n-p} \in U.$$

It follows that

$$\lim_{n \to \infty} \frac{s_V^*(q_n) - s_V^*(p)}{q_n-p} = -\frac{2k}{B},$$

as required. ■

Proof of Proposition 4.2. Fix some (s_P^*, s_V^*) that is C-compliant and suppose that $[\underline{p}, \overline{p}] \subseteq C$. By Lemma 4.1, for each $p \in [\underline{p}, \overline{p}]$, s_V^* is differentiable at p and the derivative at

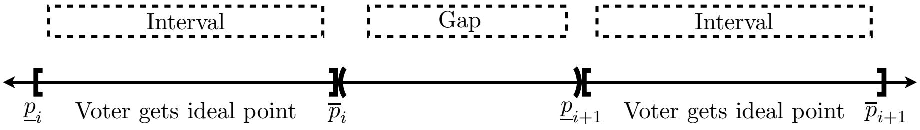

Figure 4.1

p is $-\frac{2k}{B}$. It follows that there exists some $a \in \mathbb{R}$, so that, for each $p \in [\underline{p}, \overline{p}]$, $s_V^*(p) = a - \frac{2k}{B}p$. Given that, on the domain $[\underline{p}, \overline{p}]$, the slope of s_V^* is $-\frac{2k}{B}$, we must have

$$\frac{s_V^*(\overline{p}) - s_V^*(\underline{p})}{\overline{p} - \underline{p}} = -\frac{2k}{B},$$

or

$$\overline{p} - \underline{p} = \frac{B}{2k}[s_V^*(\underline{p}) - s_V^*(\overline{p})].$$

Note, by Remark 4.1, $s_V^*(\overline{p})\,B \geq k^2$. Moreover, $1 \geq s_V^*(\underline{p})$. As such,

$$\frac{B - k^2}{B} = 1 - \frac{k^2}{B} \geq [s_V^*(\underline{p}) - s_V^*(\overline{p})].$$

With this,

$$\begin{aligned} \frac{B - k^2}{2k} &\geq \frac{B}{2k}[s_V^*(\underline{p}) - s_V^*(\overline{p})] \\ &= \overline{p} - \underline{p}, \end{aligned}$$

as required. ■

4.2 Equilibria: Intervals and Gaps

Proposition 4.2 implies that any C-compliant equilibrium (for $C \neq \emptyset$) is made up of intervals and gaps. Refer to Figure 4.1. When the Voter's ideal point is contained in one of the intervals, the Pragmatist chooses his (i.e., the Voter's) ideal point. When the Voter's ideal point is contained in one of the gaps, she does not.

Which equilibrium is "best" from the Voter's perspective? Two factors are important here. First, where are the intervals (and gaps) located relative to the Voter's prior? Second, how big are the gaps?

Begin with the first consideration—i.e., the location of the intervals (and gaps). This is straightforward. All else equal, the Voter would prefer that the Pragmatist choose his ideal point when it is contained in an interval to which he, ex ante, assigns high probability. But, of course, where two intervals are located affects the size of the gap. That is, the location of the interval depends, in part, on the size of the gap, and so the Voter may indeed face tradeoffs here.

Now, turn to the second consideration—i.e., the size of the gaps. All else equal, the Voter would prefer to shrink the size of the gaps, since the Pragmatist does not choose the Voter's ideal point on this set. But, again, the Voter faces tradeoffs—in particular, there are two distinct reasons the Voter may prefer to increase the size of the gap. First, as suggested above, to shrink the size of the gap, the Voter must also shrink the size of adjacent intervals. Second, the size of the gap determines the types of incentives the Voter can give the Pragmatist on the gap. In particular, we will see that, by increasing the size of the gap, the Voter may be able to induce the Pragmatist to choose better policies when the Voter's ideal point falls within the gap. Now we turn to an analysis of these issues.

4.3 Tradeoffs

To better understand some of the tradeoffs mentioned above, let us fix intervals, viz. $[\underline{p}_1, \overline{p}_1]$ and $[\underline{p}_2, \overline{p}_2]$, with $\underline{p}_2 > \overline{p}_1$. Take $[\underline{p}_1, \overline{p}_1]$ to have length $\frac{B-k^2}{2k}$. We do not specify the length of $[\underline{p}_2, \overline{p}_2]$, so it can even be degenerate. Proposition 4.2 implies that if an equilibrium, viz. (s_P^*, s_V^*), is $[\underline{p}_1, \overline{p}_1] \cup [\underline{p}_2, \overline{p}_2]$-compliant, it is not $[\underline{p}_1, \underline{p}_2]$-compliant. That is, there must be a gap between $[\underline{p}_1, \overline{p}_1]$ and $\underline{p}_2$. How large must the gap be? How large should the gap be?

Begin with the question of how large the gap must be. The answer depends on the length of the interval $[\underline{p}_2, \overline{p}_2]$. Lemma 4.1 gives that there exists some $a \geq 0$ so that, for each $p \in [\underline{p}_2, \overline{p}_2]$, $s_V^*(p) = a - \frac{2k}{B}p$. Let us construct this function as sparingly as possible. In this case, $s_V^*(\overline{p}_2) = \frac{k^2}{B}$. (Refer to Remark 4.1.) As such,

$$s_V^*(\underline{p}_2) = \frac{2k}{B}(\overline{p}_2 - \underline{p}_2) + \frac{k^2}{B}.$$

This says that when the length of the interval $[\underline{p}_2, \overline{p}_2]$ is larger, the electoral incentives from choosing $s_V^*(\underline{p}_2)$ must be larger. At the same time, the gap must increase as we increase the electoral incentives from choosing $s_V^*(\underline{p}_2)$. This is a consequence of Lemma C1 in Appendix C. (See the discussion in Appendix C.a.) So, in this sense, the size of the gap may need to be larger when the length of the interval $[\underline{p}_2, \overline{p}_2]$ is larger.

In the above argument, we constructed the function s_V^* so that $s_V^*(\overline{p}_2) = \frac{k^2}{B}$. Remark 4.1 tells us that we must have $s_V^*(\overline{p}_2) \geq \frac{k^2}{B}$. However, importantly, the Voter may not be able to choose $s_V^*(\overline{p}_2) = \frac{k^2}{B}$. Specifically, suppose that (s_P^*, s_V^*) is also $[\underline{p}_1, \overline{p}_1] \cup [\underline{p}_2, \overline{p}_2] \cup [\underline{p}_3, \overline{p}_3]$-compliant, for $\underline{p}_3 > \overline{p}_2$. Then, the Voter may want to increase $s_V^*(\overline{p}_2)$ precisely so that the Voter can minimize the gap between $\overline{p}_2$ and $\underline{p}_3$. That is, the Voter must jointly determine the size of the gap between $\overline{p}_1$ and $\underline{p}_2$ versus the size of the gap between $\overline{p}_2$ and $\underline{p}_3$.

Now consider the question of how large the gap should be. Will the Voter always prefer the smallest gap? In light of the discussion above, we see that the answer may be no. The Voter may want to increase the size of the gap, so that the adjacent intervals are of "full length," i.e., of length $\frac{B-k^2}{2k}$. In the particular case where the Voter wants each of the intervals to be of full length, the gap must be at least $k + \sqrt{B}$. (See Proposition 4.3 to come.) But we will see, the Voter may have a second incentive to increase the size of the gap. By doing so, he may be able to improve his welfare, when his ideal point is within the gap. Specifically, by increasing the size of the gap beyond $k + \sqrt{B}$, he may be able to give the Pragmatist better incentives within the gap.

To better understand this last point, consider a $\bigcup_i [\underline{p}_i, \overline{p}_i]$-compliant equilibrium, where the length of each interval is $\frac{B-k^2}{2k}$. Here, we can precisely characterize the smallest possible gap. Specifically, we have:

Proposition 4.3 *Fix a collection of disjoint intervals* $[\underline{p}_i, \overline{p}_i]$*, each of length* $\frac{B-k^2}{2k}$.

(i) If (s_P^*, s_V^*) *is* $\bigcup_i [\underline{p}_i, \overline{p}_i]$*-compliant, then* $\underline{p}_j - \overline{p}_i \geq k + \sqrt{B}$*, whenever* $\underline{p}_j > \overline{p}_i$.

(ii) Suppose that, $\underline{p}_j - \overline{p}_i \geq k + \sqrt{B}$*, whenever* $\underline{p}_j > \overline{p}_i$*. Then, there exists some* $\bigcup_i [\underline{p}_i, \overline{p}_i]$*-compliant equilibrium.*

The proof can be found in Appendix C. Let us focus on part (ii). Refer to Figure 4.1 and consider sets $[\underline{p}_1, \overline{p}_1]$, $[\underline{p}_2, \overline{p}_2], \ldots$. Suppose each interval has length $\frac{B-k^2}{2k}$ and, in particular, each $\underline{p}_{i+1} - \overline{p}_i = k + \sqrt{B}$. Let us construct a $\bigcup_i [\underline{p}_i, \overline{p}_i]$-compliant equilibrium, viz. (s_P^*, s_V^*). Lemma 4.1 says that, if (s_P^*, s_V^*) is $\bigcup_i [\underline{p}_i, \overline{p}_i]$-compliant and $p \in \bigcup_i [\underline{p}_i, \overline{p}_i]$, then s_V^* must be differentiable at p and, specifically, the derivative of $s_V^*(p)$ must be $-\frac{2k}{B}$. Using the length of each interval, we have that, if $p \in [\underline{p}_i, \overline{p}_i]$, we must set

$$s_V^*(p) = 1 + \frac{2k}{B}(\underline{p}_i - p).$$

Indeed, let us set $s_V^*(p)$ accordingly. If p is not contained in any of the sets $[\underline{p}_i, \overline{p}_i]$, we set $s_V^*(p) = 0$.

Figure 4.2 depicts the Pragmatist's best responses, when the Voter plays the strategy s_V^*. In particular, if the Voter's ideal point is contained in one of the sets $[\underline{p}_i, \overline{p}_i]$, it is a best

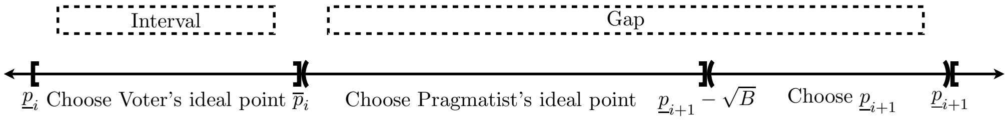

Figure 4.2

response for the Pragmatist to choose the Voter's ideal point. If, however, the Voter's ideal point is contained in one of the sets $(\overline{p}_i, \underline{p}_{i+1} - \sqrt{B}]$, it is a best response for the Pragmatist to choose her own ideal point. Finally, if the Voter's ideal point is contained in one of the sets $(\underline{p}_{i+1} - \sqrt{B}, \underline{p}_{i+1})$, it is a best response for the Pragmatist to choose $\underline{p}_{i+1}$.

The constructed equilibrium is made up of intervals and gaps, as it should be. The Pragmatist chooses the Voter's ideal point, only when it falls in the intervals. But this does not imply that the Voter necessarily wants to shrink the size of the gaps, i.e., so that each interval falls exactly $k + \sqrt{B}$ units from the adjacent intervals. That is, even if the Voter does want to induce the Pragmatist to choose his ideal point (only) over intervals of "full length," i.e., of length $\frac{B-k^2}{2k}$, the Voter may not want to minimize the gap between two adjacent intervals. This may be the case for two reasons.

First, the Voter may prefer a larger gap, so that he can achieve compliance on a set of policies that, ex ante, he thinks is more likely to contain his actual ideal point. For instance, suppose $B = 100$, $k = 1$, and the support of the Voter's prior is $(x_V)^{-1}([0, 49.5] \cup [70.5, 120])$. The Voter can achieve $[0, 49.5]$-compliance. Doing so means that he cannot achieve compliance on the interval $(49.5, 60.5)$. He can achieve $[0, 49.5] \cup [60.5, 110]$-compliance. But he can also achieve $[0, 49.5] \cup [70.5, 120]$-compliance and this would improve his expected payoffs.

Second, the Voter may prefer a larger gap so that he can induce the Pragmatist to choose "better policies" within the gap. To see this, refer back to the equilibrium constructed in Figure 4.2. When the Voter's ideal point is contained in the interval $(\underline{p}_{i+1} - \sqrt{B}, \underline{p}_{i+1} - k)$, the Pragmatist chooses $\underline{p}_{i+1}$. If $\sqrt{B}$ is large relative to k, this may be particularly bad from the Voter's perspective. In this case, he would prefer that the Pragmatist choose her own ideal point versus the policy $\underline{p}_{i+1}$. When the gap is exactly $k + \sqrt{B}$, this is not possible. By giving the Pragmatist sufficient incentives to induce her to choose her own ideal point in this range, the Voter also insures that the Pragmatist will not choose his (i.e., the Voter's) ideal point when it is exactly $\overline{p}_i$. That is, there is a conflict between (i) giving the Pragmatist incentives to choose the Voter's ideal point when it is $\overline{p}_i$ and (ii) giving the Pragmatist incentives to choose her own ideal point when the Voter's ideal point is contained in $(\underline{p}_{i+1} - \sqrt{B}, \underline{p}_{i+1} - k)$. Specifically:

Lemma 4.2 *Fix sets* $[\underline{p}_i, \overline{p}_i]$ *and* $[\underline{p}_{i+1}, \overline{p}_{i+1}]$ *each of length* $\frac{B-k^2}{2k}$. *Suppose further that* $\underline{p}_{i+1} - \overline{p}_i = k + \sqrt{B}$. *If* (s_P^*, s_V^*) *is* $[\underline{p}_i, \overline{p}_i] \cup [\underline{p}_{i+1}, \overline{p}_{i+1}]$*-compliant, then* $s_V^*(\omega)(x_P(\omega)) = 0$ *whenever* $x_V(\omega) \in (\underline{p}_{i+1} - \sqrt{B}, \underline{p}_{i+1} - k)$.

Fix a $[\underline{p}_i, \overline{p}_i] \cup [\underline{p}_{i+1}, \overline{p}_{i+1}]$-compliant equilibrium, and suppose the gap is exactly $k + \sqrt{B}$. Lemma 4.2 says that, in this case, the Voter cannot induce the Pragmatist to choose her own ideal point when $x_V(\omega) \in (\underline{p}_{i+1} - \sqrt{B}, \underline{p}_{i+1} - k)$, i.e., when the Voter would prefer that the Pragmatist does so. Specifically, by giving the Pragmatist sufficient incentives to choose her own ideal point when $x_V(\omega) \in (\underline{p}_{i+1} - \sqrt{B}, \underline{p}_{i+1} - k)$, the Voter disturbs the Pragmatist's incentives to choose $\overline{p}_i$ when that is the Voter's ideal point. However, when the gap is larger, i.e., when $\underline{p}_{i+1} - \sqrt{B}$ is large relative to $\overline{p}_i$, this conflict need not occur. Now, the Voter may be able induce the Pragmatist to choose her (i.e, the Pragmatist's) ideal point when the Voter's ideal point is contained in $(\underline{p}_{i+1} - \sqrt{B}, \underline{p}_{i+1} - k)$, without disturbing the Pragmatist's incentives for $\{\overline{p}_i\}$-compliance. Lemma C8 in Appendix C formalizes this idea.

4.4 Robustness

We have seen that, in a very simple setting, the Voter may fail to induce the Pragmatist to choose his ideal point—specifically, this may occur when the Pragmatist's ideal point is always k units above the Voter's. This raises the question, does the Voter's failure to achieve compliance crucially depend on the particular Pragmatist studied? Arguably not.

The Voter's compliance problem may become even more difficult with different types of Pragmatists. In particular, this can occur because, with other types of Pragmatists, the Voter may have less information. For instance, the Voter may know that the Pragmatist's ideal point is always k units away from his own, but may not know if it is above or below his own. Likewise, the Voter may know the Pragmatist's ideal point is above his own, but not know exactly how far apart they are.

Consider a simple example of this first type—the Voter knows the distance between his ideal point and the Pragmatist's, but does not know if the Pragmatist's ideal point is above or below his own. Specifically, suppose the space is given by $\Omega = \Omega_1 \times \Omega_2$ where $\Omega_2 = \{\overline{\omega}_2, \underline{\omega}_2\}$. The Voter's ideal point is a surjective random variable $x_V : \Omega \to \mathbb{R}$, where $x_V(\omega_1, \overline{\omega}_2) = x_V(\omega_1, \underline{\omega}_2)$ for all $\omega_1 \in \Omega_1$. This means that the Voter's ideal point is only sensitive to changes in ω_1. The Pragmatist's ideal point is also a surjective random variable, viz. $x_P : \Omega \to \mathbb{R}$, that takes the following form. For each $\omega_1 \in \Omega_1$, $x_P(\omega_1, \overline{\omega}_2) = x_V(\omega_1, \overline{\omega}_2) + k$ and $x_P(\omega_1, \underline{\omega}_2) = x_V(\omega_1, \underline{\omega}_2) - k$, where $k \in (0, \sqrt{B}]$. That is, the Politician's ideal point is the same as the Voter's, up to a constant k. Whether the Politician's ideal point is greater or less than the Voter's depends on the realization of ω_2.

(Note, here, so long as the variance of x_V is strictly positive, the Voter's and Pragmatist's ideal points are positively correlated. Moreover, k may be close to zero—so that the ideal points are almost perfectly correlated.)

Let's understand the additional compliance problem that arises here. Fix a policy p. There are states $(\omega_1, \overline{\omega}_2)$ and $(\omega_1', \underline{\omega}_2)$ so that (i) at the state $(\omega_1, \overline{\omega}_2)$, the Voter's ideal point is p and the Politician's ideal point is $p + k$ and (ii) at the state $(\omega_1', \underline{\omega}_2)$, the Voter's ideal point is $p + k$ and the Politician's ideal point is p. (We know that there are such states, since x_V is surjective.) We will argue that the Voter cannot induce the Pragmatist to choose his ideal point at both of these states. Specifically, by offering the Pragmatist incentives to choose the policy p at the state $(\omega_1, \overline{\omega}_2)$, the Voter ensures that the Pragmatist also prefers p over $p + k$ when the state is $(\omega_1', \underline{\omega}_2)$.

To see this, fix a strategy of the Voter, viz. s_V^*, and first consider the state $(\omega_1, \overline{\omega}_2)$. Suppose that, at this state, the Voter can induce the Pragmatist to choose his ideal point. Then, it must be the case that, at this state, the Pragmatist's expected payoffs from choosing the Voter's ideal point (the policy p) are greater than her expected payoffs from choosing her own ideal point (the policy $p+k$). That is, we must have that $-k^2 + s_V^*(p)\,B \geq s_V^*(p+k)\,B$. Since $k > 0$, this can occur only if $s_V^*(p) > s_V^*(p+k)$. But, now, consider the state $(\omega_1', \underline{\omega}_2)$—where the Pragmatist's ideal point is p. Here, the Pragmatist's expected payoffs from choosing her own ideal point are $s_V^*(p)\,B$. These expected payoffs are greater than $s_V^*(p+k)\,B$, and so certainly greater than her expected payoffs from choosing the Voter's ideal point, viz. $-k^2 + s_V^*(p+k)\,B$.

Indeed, an analogous argument holds more generally.

Proposition 4.4 *Fix some* $k \in (0, \sqrt{B}]$ *and consider a Pragmatist with, for each* $\omega_1 \in \Omega_1$, $x_P(\omega_1, \overline{\omega}_2) = x_V(\omega_1, \overline{\omega}_2) + k$ *and* $x_P(\omega_1, \underline{\omega}_2) = x_V(\omega_1, \underline{\omega}_2) - k$.

(i) For each $p \in \mathbb{R}$, *there exist states* $(\omega_1, \overline{\omega}_2), (\omega_1', \underline{\omega}_2) \in \Omega_1 \times \Omega_2$ *with* $x_V(\omega_1, \overline{\omega}_2) = x_P(\omega_1', \underline{\omega}_2) = p$.

(ii) For each Bayesian equilibrium, viz. (s_P^*, s_V^*), *if* $x_V(\omega_1, \overline{\omega}_2) = x_P(\omega_1', \underline{\omega}_2) = p$, *then* $s_P^*(\omega_1, \overline{\omega}_2)(p) > 0$ *implies* $s_P^*(\omega_1', \underline{\omega}_2)(p) = 0$.

The proof of Proposition 4.4 is immediate from the discussion above, and so omitted.

To compare this situation to the Pragmatist studied in Sections 4.1–4.3, note the following corollary.

Corollary 4.1 *Fix some* $k \in (0, \sqrt{B}]$ *and consider a Pragmatist with, for each* $\omega_1 \in \Omega_1$, $x_P(\omega_1, \overline{\omega}_2) = x_V(\omega_1, \overline{\omega}_2) + k$ *and* $x_P(\omega_1, \underline{\omega}_2) = x_V(\omega_1, \underline{\omega}_2) - k$*. Then, for each* $p \in \mathbb{R}$*, there is no* $\{p, p+k\}$*-compliant equilibrium.*

This corollary shows that the compliance problem is worse with this latter type of Pragmatist. In particular, when $B \geq 2k^2$, there is always a $\{p, p+k\}$-compliant equilibrium for the Pragmatist studied in Sections 4.1–4.3.

5 Ideologues or Pragmatists?

Section 3 showed that the Voter can induce an Ideologue to choose his ideal point whenever it is within $\sqrt{B}$ of the Ideologue's ideal point. Section 4 showed that this may not be the case for a Pragmatist. The Voter may not be able to induce a Pragmatist to choose his ideal point, even if it is very close to the Pragmatist's own ideal point. This the raises the question: Is the Voter better off with an Ideologue or with a Pragmatist?

We focus on the case where the Pragmatist's ideal point is always k units above the Voter's. Section 4.4 suggests that it may be easier for the Voter to achieve compliance when faced with this type of Pragmatist (versus a Pragmatist whose ideal point may be above or below the Voter's). Yet, even here, we will see that the Voter may sometimes prefer the Ideologue. Why?

Notice, the Voter faces a trade-off. On the one hand, with the Ideologue, the Voter can achieve $[-\sqrt{B}, \sqrt{B}]$-compliance. If $2\sqrt{B} > \frac{B-k^2}{2k}$, the Voter cannot do so with the Pragmatist. So for this range of ideal points the Voter may find an Ideologue more attractive than a Pragmatist. On the other hand, with the Ideologue, the Voter cannot achieve his ideal point whenever it lies further from zero than $\sqrt{B}$, whereas with the Pragmatist doing so may be possible.

This tradeoff implies that the Voter is sometimes better off with an Ideologue and sometimes better off with a Pragmatist, depending on the prior. To formalize this idea, let us introduce some terminology. Suppose that a particular Ideologue is associated with a (finite) benefit of reelection B. Call this Ideologue a B-Ideologue. Likewise, consider a Pragmatist with a (finite) benefit of reelection B and whose ideal point is always k units above the Voter's, with $k \in (0, \sqrt{B}]$. Call this Pragmatist a (k, B)-Pragmatist.

Definition 5.1 *Fix a prior* $\mu \in \mathcal{M}(\Omega)$. *Say the Voter* ***prefers a*** $\mathbf{B_0}$***-Ideologue to a*** $(\mathbf{k_1}, \mathbf{B_1})$***-Pragmatist (given*** μ***)*** *if the following holds: There exists some equilibrium, viz.* (s_P^*, s_V^*), *of the game with the* B_0*-Ideologue so that*

$$\int_\Omega \mathbb{E}u_V(\omega, s_P^*)\, d\mu \geq \int_\Omega \mathbb{E}u_V(\omega, r_P^*)\, d\mu,$$

for each equilibrium (r_P^*, r_V^*) *of the game with a* (k_1, B_1)*-Pragmatist.*

Proposition 5.1 *Suppose* x_V *is continuous.*

- *(i) Fix some* $k > 0$. *There exists a non-empty, open interval* $U \subseteq \mathbb{R}_+$ *so that the following holds: for each* $B \in U$, *there exists a prior* $\mu \in \mathcal{M}(\Omega)$, *such that the Voter prefers the* B*-Ideologue over the* (k, B)*-Pragmatist.*
- *(ii) Fix some* $B > 0$. *There exists a non-empty open interval* $U \subseteq \mathbb{R}_+$ *so that, the following holds: for each* $k \in U$, *there exists a prior* $\mu \in \mathcal{M}(\Omega)$, *such that the Voter prefers the* B*-Ideologue over the* (k, B)*-Pragmatist.*

Part (i) says that for each $k > 0$ there is a level of electoral rewards such that the Voter prefers the Ideologue to that Pragmatist. Importantly, this means that the Voter may prefer an Ideologue even when his preferences are arbitrarily close to perfectly positively correlated with the Pragmatist's. Part (ii) says that, for any $B > 0$ there is a $k < \sqrt{B}$ such that the Voter prefers the Ideologue to the Pragmatist. That is, even if the rewards of office are arbitrarily small, the Ideologue may be better for the Voter than a Pragmatist whose preferences are positively correlated with the Voter's.

6 Ideologues versus Pragmatists

We have seen that the Voter may strictly prefer an Ideologue to a Pragmatist. This is because he may be able to better control the Ideologue. Why can the Voter exert such control? In the context of our model, the Voter is indifferent between reelecting and firing the Politician, at the time he makes his reelection decision. As such, he can credibly use his reelection rule to reward past behavior, without regard for the future.

What if there is a future value to a Politician, so that the Voter is no longer indifferent between reelecting and firing the Politician? In this case, can the Voter credibly use his reelection rule to reward past behavior? In particular, if the Voter has a choice between

Ideologues and Pragmatists, can the type of incentives identified in Section 3 be used to exert similar control over an Ideologue?

To gain intuition, think of an infinitely repeated version of the game that begins with an incumbent Ideologue. In each period, there are multiple candidates of each type that can be elected to a single office. Consider any period with an Ideologue in office. Can the Voter choose a reelection rule so that, in each period this Ideologue is in office, she chooses the Voter's ideal point whenever it is contained in $[-\sqrt{B}, \sqrt{B}]$? That is, can the Voter choose a reelection rule so that this Ideologue is indifferent between all policy choices in $[-\sqrt{B}, \sqrt{B}]$?

It seems that the answer to this question may be yes. To see this, suppose that in each period the Voter reelects an Ideologue in office using the probabilities associated with Proposition 3.1. The Voter assigns the remaining probability to electing a new Ideologue—i.e., one who has never been in office and who has the same ideal point as the Ideologue in office. Then, the (current) Ideologue's per-period expected payoff from choosing any policy in $[-\sqrt{B}, \sqrt{B}]$ is zero. With this, it is a best response for the Ideologue to choose a strategy that, in each period, plays in accordance with the $[-\sqrt{B}, \sqrt{B}]$-compliant equilibrium of the stage game.

Note that, under this reelection rule, the Voter always elects an Ideologue to office. (The elected Ideologue may be an incumbent or a challenger.) As such, in each period, the Voter obtains his ideal point, whenever it is contained in $[-\sqrt{B}, \sqrt{B}]$.

Of course, we have not provided a thorough analysis of the repeated game. Indeed, it may be possible for the Voter to induce the Ideologue in office to choose the Voter's ideal point beyond the interval $[-\sqrt{B}, \sqrt{B}]$, since reelection offers the Ideologue benefits beyond the current period.

Nonetheless, this example suggests that, when the Voter has a choice between an Ideologue and a Pragmatist, he may be able to credibly use his reelection rule to reward past behavior and achieve compliance.

Appendix A Proofs for Section 3.3

Lemma A1 *Fix metrizable spaces X, Y and measurable maps $f : X \to Y$ and $g : X \to Y$. Let E be a measurable subset of Y and construct $h : X \to Y$ so that*

$$h(x) = \begin{cases} f(x) & \text{if } f(x) \in E \\ g(x) & \text{otherwise.} \end{cases}$$

Then h is also measurable.

Proof. Fix a measurable set $F \subseteq Y$. Then the sets $E \cap F$ and $F \backslash (F \cap E)$ are both measurable. We have

$$h^{-1}(F) = h^{-1}(E \cap F) \cup h^{-1}(F \backslash (E \cap F)).$$

So, it suffices to show that $h^{-1}(E \cap F)$ and $h^{-1}(F \backslash (E \cap F))$ are both measurable. The former comes from the fact that $h^{-1}(E \cap F) = f^{-1}(E \cap F)$ and the fact that f is measurable. The latter comes from the fact that $h^{-1}(F \backslash (E \cap F)) = g^{-1}(F \backslash (E \cap F))$ and the fact that g is measurable. ■

Lemma A2 *Fix metrizable spaces X, Y and a measurable map $f : X \to Y$. Define $g : X \to \mathcal{M}(Y)$ so that $g(x)$ assigns probability one to $f(x)$. Then g is measurable.*

Proof. Immediate from Theorem 14.8 in Aliprantis and Border (1999). ■

Lemma A3 *Define $s_P : \Omega \to \mathcal{M}(\mathbb{R})$ as follows. If $x_V(\omega) \in [-\sqrt{B}, \sqrt{B}]$, then set $s_P(\omega)(x_V(\omega)) = 1$. If $x_V(\omega) < -\sqrt{B}$ (resp. $x_V(\omega) > \sqrt{B}$), then set $s_P(\omega)(-\sqrt{B}) = 1$ (resp. $s_P(\omega)(\sqrt{B}) = 1$). The map s_P is measurable.*

Proof. Construct a function $f : \Omega \to \mathbb{R}$ so that $f(\omega) = x_V(\omega)$ if $x_V(\omega) \in [-\sqrt{B}, \sqrt{B}]$, $f(\omega) = -\sqrt{B}$ if $x_V(\omega) < -\sqrt{B}$, and $f(\omega) = \sqrt{B}$ if $x_V(\omega) > \sqrt{B}$. Since each of the maps x_V, $\omega \mapsto -\sqrt{B}$, and $\omega \mapsto \sqrt{B}$ are measurable, Lemma A1 says that f is measurable. Now, the result follows from Lemma A2. ■

Lemma A4 *Consider a strategy s_V so that $s_V(p) = \frac{p^2}{B}$ when $p \in [-\sqrt{B}, \sqrt{B}]$ and $s_V(p) = 1$ otherwise. Then, s_V is measurable.*

Proof. Note, $p \mapsto \frac{p^2}{B}$ and $p \mapsto 1$ are continuous functions and so measurable. As such, the result follows from Lemma A1. ■

Appendix B Proofs for Section 3.4

Let us review the timing of this game: Nature chooses a state. The Ideologue observes the true state and then chooses a policy. The Voter observes the Ideologue's policy choice. Nature then chooses ι from $\{i, ni\}$, where i represents the decision to inform the Voter of the true state and ni represents the decision not to inform the Voter of this state. The Voter then decides whether or not to reelect the Ideologue.

Because the Voter has the potential to learn the true state, now a strategy of the Voter is a mapping from $\{i, ni\} \times \Omega \times \mathbb{R} \to \mathcal{M}(\{0,1\})$, which is constant on $\{ni\} \times \Omega \times \{p\}$, i.e., for each $p \in \mathbb{R}$. Once again, we write instead $s_V : \{i, ni\} \times \Omega \times \mathbb{R} \to [0,1]$, so that $s_V(\iota, \omega, p)$ gives the probability that the Voter reelects the Politician if Nature chooses ι, the true state is ω, and the Politician chooses p. (And, again, for each $p \in \mathbb{R}$, s_V is constant on $\{ni\} \times \Omega \times \{p\}$.)

Recall, the Politician assigns probability π to the Voter learning the true state, and this probability is transparent to the players. So, now

$$\mathbb{E}u_P(\omega, p, s_V) = -(p - x_P(\omega))^2 + \pi B(s_V(i, \omega, p)) + (1-\pi) B(s_V(ni, \omega, p)).$$

We begin by verifying a claim made in Section 3.1.

Lemma B1 *Fix a Bayesian Equilibrium* (s_P^*, s_V^*)*. For each* $\omega \in \Omega$*,* $s_P^*(\omega)(\mathbb{R} \backslash [-\sqrt{B}, \sqrt{B}]) = 0$*.*

Proof. Fix a state ω and suppose $s_P^*(\omega)(\mathbb{R} \backslash [-\sqrt{B}, \sqrt{B}]) > 0$. Note, for each $p \in \mathbb{R} \backslash [-\sqrt{B}, \sqrt{B}]$,

$$\begin{aligned} 0 &> -p^2 + B \\ &\geq -p^2 + \pi B(s_V^*(i, \omega, p)) + (1-\pi) B(s_V^*(ni, \omega, p)). \end{aligned}$$

Given this,

$$0 > \int_{-\infty}^{-\sqrt{B}} \mathbb{E}u_P(\omega, p, s_V^*)\, ds_P^*(\omega) + \int_{\sqrt{B}}^{\infty} \mathbb{E}u_P(\omega, p, s_V^*)\, ds_P^*(\omega). \tag{B1}$$

Now construct $\sigma_P \in \mathcal{M}(\mathbb{R})$ so that $\sigma_P(\mathbb{R} \backslash [-\sqrt{B}, \sqrt{B}) = 0$ and, for each event $E \subseteq$

$[-\sqrt{B},\sqrt{B}]$,

$$\sigma_P(E) = \begin{cases} s_P^*(\omega)(E \cup \mathbb{R} \backslash [-\sqrt{B},\sqrt{B}]) & \text{if } 0 \in E \\ s_P^*(\omega)(E) & \text{otherwise.} \end{cases}$$

It follows immediately from Equation B1 that

$$\mathbb{E}u_P(\omega,\sigma_P,s_V^*) > \mathbb{E}u_P(\omega,s_P^*,s_V^*),$$

contradicting that (s_P^*,s_V^*) is a Bayesian Equilibrium. ■

For the remainder of this section, we restrict attention to **pure-strategy equilibria**, i.e., equilibria where, for each $\omega \in \Omega$, $s_P^*(\omega)$ is a dirac measure and $s_V^*(\{i,ni\} \times \Omega \times \mathbb{R}) \subseteq \{0,1\}$. It will be useful to make use of a number of mathematical facts.

Remark B1 *For each $i = 1,2,\ldots$, fix metrizable disjoint sets X_i and maps $f_i : X_i \to \mathbb{R}$. Define $f : \bigcup_i X_i \to \mathbb{R}$ so that $f(x) = f_i(x)$ where $x \in X_i$. Then f is measurable if and only if each f_i is measurable.*

Lemma B2 *Fix metrizable spaces X and Y, and a measurable map $f : X \to Y$. Define a map $g : X \times Y \to [0,1]$ so that, for each $x \in X$, $g(x,f(x)) = 1$ and $g(x,y) = 0$ for $y \neq f(x)$. Then, g is measurable.*

Proof. It suffices to show that $(g)^{-1}(\{1\})$ is measurable in $X \times Y$. To see this, note that $(g)^{-1}(\{1\})$ is the graph of f. So the measurability of $(g)^{-1}(\{1\})$ follows from Theorems 4.43 and 4.44 in Aliprantis and Border (1999).) ■

Proof of Proposition 3.2. Let s_P^* be as defined in Proposition 3.1, i.e., $s_P^*(\omega)$ assigns probability one to $x_V(\omega)$ when $x_V(\omega) \in [-\sqrt{B},\sqrt{B}]$ and $s_P^*(\omega)$ assigns probability one to $-\sqrt{B}$ (resp. $\sqrt{B}$) if $-\sqrt{B} > x_V(\omega)$ (resp. $x_V(\omega) > \sqrt{B}$). Define s_V^* as follows: For each $\omega \in \Omega$, let $s_V^*(i,\omega,p) = 1$ if and only if $s_P^*(\omega)(p) = 1$ and let $s_V^*(ni,\omega,p) = 1$ if and only if $p^2 \geq \pi B$. We will show that (s_P^*,s_V^*) is indeed a Bayesian Equilibrium.

Begin with condition (i). Lemma A3 says that s_P^* is measurable. For the measurability of s_V^*, it suffices to show that the restrictions of s_V^* to $\{i\} \times \Omega \times \mathbb{R}$ and $\{ni\} \times \Omega \times \mathbb{R}$ are both measurable. (See Remark B1.) The former follows from Lemmata B2 and A1. The latter is immediate.

Now turn to condition (ii). We will show that, for each $\omega \in \Omega$ and each p with $s_p^*(\omega)(p) = 0$

$$\mathbb{E}u_P(\omega,s_p^*,s_v^*) \geq 0 \geq \mathbb{E}u_P(\omega,p,s_V^*).$$

Fix some $\omega \in \Omega$. First, consider the case where $\pi B > [x_V(\omega)]^2$. Here

$$\mathbb{E}u_P(\omega, s_P^*, s_V^*) = -x_V(\omega)^2 + \pi B > 0,$$

as required. Next suppose $[x_V(\omega)]^2 \geq \pi B$. Here

$$\mathbb{E}u_P(\omega, s_P^*, s_V^*) = \begin{cases} -B + B = 0 & \text{if } [x_V(\omega)]^2 \geq B \\ -x_V(\omega)^2 + B > 0 & \text{if } B > [x_V(\omega)]^2, \end{cases}$$

as required.

Fix some policy $p \in \mathbb{R}$ with $s_P^*(\omega)(p) = 0$. Note that if $\pi B > p^2$, then

$$0 \geq -p^2 = \mathbb{E}u_P(\omega, p, s_V^*).$$

If $p^2 \geq \pi B$, then

$$\begin{aligned} 0 &\geq -p^2 + \pi B \\ &\geq -p^2 + (1-\pi) B = \mathbb{E}u_P(\omega, p, s_V^*), \end{aligned}$$

where the second inequality comes from the fact that $\pi \geq \frac{1}{2}$. ■

We've seen that, in the case where $\pi \in [\frac{1}{2}, 1]$, there is a $[-\sqrt{B}, \sqrt{B}]$-compliant equilibrium and we constructed such an equilibrium in Proposition 3.2. In light of Lemma B1, the constructed equilibrium is the "best equilibrium" from the Voter's perspective. Now let's turn to the case where $\pi \in [0, \frac{1}{2})$. Here, there is no $[-\sqrt{B}, \sqrt{B}]$-compliant equilibrium (see the Online Appendix). This raises the question: in this case, what is the best equilibrium from the Voter's perspective?

In the main text, we verbally argued that the Voter can induce the Ideologue to choose his ideal point when it is contained in

$$[-\sqrt{B}, -\sqrt{(1-\pi) B}] \cup [-\sqrt{\pi B}, \sqrt{\pi B}] \cup [\sqrt{(1-\pi) B}, \sqrt{B}].$$

Indeed, this is the case and it may be the "best equilibrium" from the Voter's perspective. That said, it need not be the case that, from an ex ante perspective, this equilibrium is desirable from the perspective of the Voter.

Take the following example. Let $\Omega = \mathbb{R}$ and x_V be the identity map. Suppose $\pi = \frac{1}{36}$

and the Voter's prior μ is uniform on

$$[-\tfrac{1}{6}\sqrt{2B}, -\tfrac{1}{6}\sqrt{B}] \cup [\tfrac{1}{6}\sqrt{B}, \tfrac{1}{6}\sqrt{2B}].$$

Then, under the equilibrium discussed above, the Ideologue chooses the Voter's ideal point whenever it is contained in

$$[-\sqrt{B}, -\sqrt{\tfrac{35}{36}B}] \cup [-\tfrac{1}{6}\sqrt{B}, \tfrac{1}{6}\sqrt{B}] \cup [\sqrt{\tfrac{35}{36}B}, \sqrt{B}].$$

There are exactly two states that the Voter considers possible (i.e., that are in the support of μ) and, at which, the Politician chooses the Voter's ideal point—namely, $-\frac{1}{6}\sqrt{B}$ and $\frac{1}{6}\sqrt{B}$. But, there is an equilibrium where the Ideologue chooses the Voter's ideal point, whenever it is contained in the support of μ. Specifically, suppose that (i) at any state ω in the support of μ, the Voter offers informed incentives to choose $x_V(\omega)$ at ω and (ii) the Voter offers uninformed incentives to choose any policy in the support of μ. It is readily verified that this strategy induces an equilibrium.

A similar phenomenon holds more generally. The Voter achieves such compliance, by changing the range of policies for which he offers both informed and uninformed incentives—shifting it toward the center. This is the basis for the result to come. But first, we need to formalize the idea of a "best equilibrium."

We will consider a criterion that we call state-by-state optimality. Say a pure strategy profile, viz. (s_P^*, s_V^*), is **state-by-state optimal** if it is a pure-strategy Bayesian equilibrium and there does not exist a pure-strategy Bayesian equilibrium, viz. (r_P^*, r_V^*), with (i) $u_V(\omega, r_P^*(\omega)) \geq u_V(\omega, s_P^*(\omega))$ for each $\omega \in \Omega$, and (ii) $u_V(\omega, r_P^*(\omega)) > u_V(\omega, s_P^*(\omega))$ for some $\omega \in \Omega$. (Notice that the criterion is defined only for pure-strategy equilibria.) State-by-state optimality is a weak dominance criterion. If (s_P^*, s_V^*) is state-by-state optimal, then there is some $\mu \in \mathcal{M}(\Omega)$ such that (s_P^*, s_V^*) maximizes ex ante expected payoffs under μ.

Proposition B1 *Fix $\pi \in [0, \frac{1}{2})$. If (s_P^*, s_V^*) is a state-by-state optimal equilibrium, then it must take one of the following forms:*

(i) There is a policy $p_* \in [0, \sqrt{(1-2\pi)B})$ so that the maximal compliance induced by (s_P^*, s_V^*) is

$$[-\sqrt{p_*^2 + \pi B}, -p_*] \cup [p_*, \sqrt{p_*^2 + \pi B}].$$

(ii) There is a policy $p_* \in [\sqrt{(1-2\pi)B}, \sqrt{(1-\pi)B}]$ *so that the maximal compliance induced by* (s_P^*, s_V^*) *is*

$$[-\sqrt{p_*^2+\pi B}, -p_*] \cup [-\sqrt{p_*^2-(1-2\pi)B}, \sqrt{p_*^2-(1-2\pi)B}] \cup [p_*, \sqrt{p_*^2+\pi B}].$$

Refer back to the discuss in Section 3.2. There we studied the case where $\pi = 0$. Here, part (i) corresponds to the Moderate Two-Sided Voting Rule and part (ii) corresponds to the Extreme Two-Sided Voting Rule.

The proof of Proposition B1 is somewhat involved and so relegated to the Online Appendix.

Appendix C Proofs for Section 4.3

In Section 4.3, we discussed two reasons the Voter may "prefer" to increase the gap. The first reason was so that he can increase the size of the interval. The second reason was so that he can be made better off on the gap. This section formalizes the two arguments. In particular, we begin with the question of how the gap relates to the size of the interval. Then, we turn to the proof of Proposition 4.3, which constructs equilibria for the case of intervals of "full length." Finally, we turn to the question of whether we can improve upon Proposition 4.3, by increasing the size of the gap.

a. Increasing the Size of the Interval: Fix two intervals $[\underline{p}, \overline{p}]$ and $[\underline{q}, \overline{q}]$, where $\underline{q} > \overline{p}$. Further, take the length of $[\underline{p}, \overline{p}]$ to be $\frac{B-k^2}{2k}$. Now, let's ask: If we can find a $[\underline{p}, \overline{p}] \cup [\underline{q}, \overline{q}]$-compliant equilibrium, viz. (s_P^*, s_V^*), what is the minimum distance between $\underline{q}$ and $\overline{p}$? The answer will depend on the length of the interval $[\underline{q}, \overline{q}]$. To see this, first note that

$$s_V^*(\underline{q}) = s_V^*(\overline{q}) + \frac{2k}{B}(\overline{q} - \underline{q}).$$

(Here, we use Lemma 4.1.) So, holding $s_V^*(\overline{q})$ fixed, increasing the length of the interval increases the electoral incentives the Voter must offer the Pragmatist to induce her to choose the Voter's ideal point when it is $\underline{q}$. However, as $s_V^*(\underline{q})$ increases, there are greater electoral incentives for choosing $\overline{p}$ over $\underline{q}$, and so we need the minimum gap between $[\underline{p}, \overline{p}]$ and $[\underline{q}, \overline{q}]$ to be larger. This last claim is formalized in the following Lemma.

Lemma C1 *Fix some interval* $[\underline{p}, \overline{p}]$ *of length* $\frac{B-k^2}{2k}$*, and some* $q > \overline{p}$*. If* (s_P^*, s_V^*) *is* $[\underline{p}, \overline{p}] \cup \{q\}$*-compliant then:*

(i) $(q-\overline{p})^2-2k(q-\overline{p})+k^2 \geq s_V^*(q)B$, *and*

(ii) $(q-\overline{p}) \geq 2k$.

Let us comment on Lemma C1. Begin with a situation where $(q-\overline{p})^2-2k(q-\overline{p})+k^2 = s_V^*(q)B$. If we must increase $s_V^*(q)$, then certainly we must increase the left-hand side. For a given k, the only way to do so is to change the gap between q and $\overline{p}$. Indeed, part (ii) says that this "change of the gap" must be an "increase the gap."

To prove Lemma C1, let us begin with a consequence of Lemma 4.1.

Lemma C2 *Fix an interval* $[\underline{p},\overline{p}]$ *with length* $\frac{B-k^2}{2k}$. *If* (s_P^*, s_V^*) *is* $[\underline{p},\overline{p}]$*-compliant, then, for each* $p \in [\underline{p},\overline{p}]$, $s_V^*(p) = 1+\frac{2k}{B}(\underline{p}-p)$.

Proof. By Lemma 4.1, for each $p \in [\underline{p},\overline{p}]$, $s_V^*(p) = a-\frac{2k}{B}p$, for some a. (Here, a is constant across $p \in [\underline{p},\overline{p}]$.) We will show that $a = 1+\frac{2k}{B}\underline{p}$.

To see this, note that

$$\frac{s_V^*(\overline{p})-s_V^*(\underline{p})}{\overline{p}-\underline{p}} = -\frac{2k}{B}.$$

Since $\overline{p}-\underline{p} = \frac{B-k^2}{2k}$, this gives that

$$s_V^*(\overline{p})-s_V^*(\underline{p}) = -\frac{2k}{B}\frac{B-k^2}{2k} = -\frac{B-k^2}{B}.$$

Recall, $1 \geq s_V^*(\underline{p})$, so that

$$s_V^*(\overline{p}) = s_V^*(\underline{p}) - \frac{B-k^2}{B} \leq \frac{k^2}{B}.$$

Now, by Remark 4.1, $s_V^*(\overline{p}) \geq \frac{k^2}{B}$, so that, in fact,

$$s_V^*(\overline{p}) = \frac{k^2}{B}.$$

With this,

$$s_V^*(\underline{p}) = \frac{k^2}{B} + \frac{B-k^2}{B} = 1.$$

As such,

$$s_V^*(\underline{p}) = a - \frac{2k}{B}\underline{p} = 1,$$

from which the claim follows. ■

Proof of Lemma C1. Fix an equilibrium, viz. (s_P^*, s_V^*), that is $[\underline{p}, \overline{p}] \cup \{q\}$-compliant. Consider a state ω with $x_V(\omega) = \overline{p}$. By Remark 4.1 and Lemma C2,

$$-k^2 + B + 2k(\underline{p} - \overline{p}) \geq -(q - \overline{p} - k)^2 + s_V^*(q)B.$$

Using the fact established above, i.e., that the length of the interval $[\underline{p}, \overline{p}]$ is $\frac{B-k^2}{2k}$, we get that

$$0 \geq -(q - \overline{p} - k)^2 + s_V^*(q)B.$$

Part (i) follows immediately.

For Part (ii), begin by using Remark 4.1. This gives that $s_V^*(q)B \geq k^2$. So,

$$\begin{aligned} -(q - \overline{p} - k)^2 + s_V^*(q)B &\geq -(q - \overline{p} - k)^2 + k^2 \\ &= -(q - \overline{p})^2 + 2k(q - \overline{p}). \end{aligned}$$

Using the fact that

$$0 \geq -(q - \overline{p} - k)^2 + s_V^*(q)B,$$

it follows that

$$0 \geq -(q - \overline{p})^2 + 2k(q - \overline{p}),$$

or $(q - \overline{p}) \geq 2k$. ∎

b. Proof of Proposition 4.3: It suffices to show the following two lemmata:

Lemma C3 *Fix intervals* $[\underline{p}_i, \overline{p}_i]$ *and* $[\underline{p}_{i+1}, \overline{p}_{i+1}]$, *with* $\underline{p}_{i+1} > \overline{p}_i$. *Suppose, further, that the length of these intervals is* $\frac{B-k^2}{2k}$. *If* (s_P^*, s_V^*) *is* $[\underline{p}_i, \overline{p}_i] \cup [\underline{p}_{i+1}, \overline{p}_{i+1}]$*-compliant, then* $(\underline{p}_{i+1} - \overline{p}_i) \geq k + \sqrt{B}$.

Lemma C4 *Fix a collection of intervals* $[\underline{p}_i, \overline{p}_i]$, *each of length* $\frac{B-k^2}{2k}$. *Suppose further that, given two such intervals* $[\underline{p}_i, \overline{p}_i]$ *and* $[\underline{p}_{i+1}, \overline{p}_{i+1}]$, $(\underline{p}_{i+1} - \overline{p}_i) \geq k + \sqrt{B}$. *Then, there exists some* $\bigcup_i [\underline{p}_i, \overline{p}_i]$*-compliant equilibrium.*

We begin with Lemma C3. We will first need the following result.

Lemma C5 *Fix intervals* $[\underline{p}_i, \overline{p}_i]$ *and* $[\underline{p}_{i+1}, \overline{p}_{i+1}]$, *with* $\underline{p}_{i+1} > \overline{p}_i$. *Suppose, further, that the length of these intervals is* $\frac{B-k^2}{2k}$. *If* (s_P^*, s_V^*) *is* $[\underline{p}_i, \overline{p}_i] \cup [\underline{p}_{i+1}, \overline{p}_{i+1}]$*-compliant, then* $(\underline{p}_{i+1} - \overline{p}_i)^2 \geq 2k(\underline{p}_{i+1} - \underline{p}_i)$.

Proof. Fix some state $\omega \in (x_V)^{-1}(\overline{p}_i)$. Then,

$$\begin{aligned}\mathbb{E}u_P(\omega, s_P^*, s_V^*) &= -k^2 + B + 2k(\underline{p}_i - \overline{p}_i) \\ &\geq -(\underline{p}_{i+1} - \overline{p}_i - k)^2 + B = \mathbb{E}u_P(\omega, \underline{p}_{i+1}, s_V^*),\end{aligned}$$

where the first line follows from Lemma C2 and the second from Lemma C1(ii). With this,

$$2k(\underline{p}_i - \overline{p}_i) \geq -(\underline{p}_{i+1} - \overline{p}_i)^2 + 2k(\underline{p}_{i+1} - \overline{p}_i),$$

or

$$(\underline{p}_{i+1} - \overline{p}_i)^2 \geq 2k(\underline{p}_{i+1} - \underline{p}_i),$$

as required. ■

Lemma C6 *Fix intervals* $[\underline{p}_i, \overline{p}_i]$ *and* $[\underline{p}_{i+1}, \overline{p}_{i+1}]$, *with* $\underline{p}_{i+1} > \overline{p}_i$. *Suppose, further, that the length of* $[\underline{p}_i, \overline{p}_i]$ *is* $\frac{B-k^2}{2k}$. *Then,* $(\underline{p}_{i+1} - \overline{p}_i)^2 \geq 2k(\underline{p}_{i+1} - \underline{p}_i)$ *if and only if* $(\underline{p}_{i+1} - \overline{p}_i) \geq k + \sqrt{B}$.

Proof. Begin with the fact that $(\underline{p}_{i+1} - \overline{p}_i)^2 \geq 2k(\underline{p}_{i+1} - \underline{p}_i)$. Note that $(\underline{p}_{i+1} - \overline{p}_i)^2 \geq 2k(\underline{p}_{i+1} - \underline{p}_i)$ if and only if

$$(\underline{p}_{i+1} - \overline{p}_i)^2 - 2k(\underline{p}_{i+1} - \overline{p}_i) - \left(B - k^2\right) \geq 0. \tag{C1}$$

Differentiating the left-hand side of Equation C1 with respect to $(\underline{p}_{i+1} - \overline{p}_i)$ shows that the left-hand side is increasing in $(\underline{p}_{i+1} - \overline{p}_i)$ if and only if $(\underline{p}_{i+1} - \overline{p}_i)$ is greater than k. So, if $(\underline{p}_{i+1} - \overline{p}_i)^2 \geq 2k(\underline{p}_{i+1} - \underline{p}_i)$, then either $(\underline{p}_{i+1} - \overline{p}_i)$ is less than

$$\frac{2k - \sqrt{4k^2 + 4\left(B - k^2\right)}}{2} = k - \sqrt{B}$$

or at least

$$\frac{2k + \sqrt{4k^2 + 4\left(B - k^2\right)}}{2} = k + \sqrt{B}.$$

Recall that $\underline{p}_{i+1} > \overline{p}_i \geq \underline{p}_i$. So, $(\underline{p}_{i+1} - \overline{p}_i)^2 \geq 2k(\underline{p}_{i+1} - \underline{p}_i)$ implies

$$\underline{p}_{i+1} - \overline{p}_i \geq 2k \frac{\underline{p}_{i+1} - \underline{p}_i}{\underline{p}_{i+1} - \overline{p}_i} \geq 2k.$$

Since $(\underline{p}_{i+1} - \overline{p}_i) \geq 2k$, it follows that $(\underline{p}_{i+1} - \overline{p}_i)$ cannot be less than $k - \sqrt{B}$. That is,

$(\underline{p}_{i+1} - \overline{p}_i)$ must be at least $k + \sqrt{B}$.

Conversely, suppose that $(\underline{p}_{i+1} - \overline{p}_i) \geq k + \sqrt{B}$. Note, $k + \sqrt{B} \geq 2k$. So,

$$\begin{aligned}(\underline{p}_{i+1} - \overline{p}_i)((\underline{p}_{i+1} - \overline{p}_i) - 2k) &\geq (k + \sqrt{B})(k + \sqrt{B} - 2k) \\ &= B - k^2.\end{aligned}$$

This establishes that

$$(\underline{p}_{i+1} - \overline{p}_i)^2 - 2k(\underline{p}_{i+1} - \overline{p}_i) - \left(B - k^2\right) \geq 0,$$

as required. ■

Proof of Lemma C3. Immediate from Lemmata C5-C6. ■

We now turn to the proof of Lemma C4.

Proof of Lemma C4. Fix a collection of intervals as in the statement of the Lemma, and let $C = \bigcup_i [\underline{p}_i, \overline{p}_i]$. There are a countable number of such intervals. (See Proposition 5.3 in Krantz (2004)). As such, C is measurable, and so $\mathbb{R} \backslash C$ is also measurable.

Define a strategy s_V^* as follows: If $p \in \mathbb{R} \backslash C$, set $s_V^*(p) = 0$. If $p \in C$, then there exists exactly one interval i with $p \in [\underline{p}_i, \overline{p}_i] \subseteq C$. We set $s_V^*(p) = 1 + \frac{2k}{B}(\underline{p}_i - p)$. Note, the strategy s_V^* is well-defined. In particular, $s_V^*(p) \in [0,1]$. (Here, we use the fact that, if $p \in [\underline{p}_i, \overline{p}_i]$, then $p - \underline{p}_i \leq \frac{B-k^2}{2k}$.) Also, notice that s_V^* is measurable: For each i, we can construct a map $f_i : [\underline{p}_i, \overline{p}_i] \mapsto [0,1]$, that is the restriction of s_V^* to $[\underline{p}_i, \overline{p}_i]$. Each f_i is continuous, and so measurable. Now, fix a measurable set E in $[0,1]$ and note that $(s_V^*)^{-1}(E)$ is either $\bigcup_i (f_i)^{-1}(E)$ or $\bigcup_i (f_i)^{-1}(E) \cup (\mathbb{R} \backslash C)$. In either case, $(s_V^*)^{-1}(E)$ is the countable union of measurable sets, and so measurable.

Now we turn to constructing a strategy s_P^*. We will show that we can construct s_P^* so that (s_P^*, s_V^*) is a C-compliant equilibrium. We break the construction up into three steps: the case where $\omega \in (x_V)^{-1}(C)$, the case where $\omega \in (x_V)^{-1}(\mathbb{R} \backslash C) \cap (x_P)^{-1}(\mathbb{R} \backslash C)$, and finally the case when $(x_V)^{-1}(\mathbb{R} \backslash C) \cap (x_P)^{-1}(C)$. For each case, we show that $\mathbb{E}u_P(\omega, s_P^*, s_V^*) \geq \mathbb{E}u_P(\omega, p, s_V^*)$ for all $p \in \mathbb{R}$. Then, we conclude the proof by showing that the constructed s_P^* is measurable. This will establish that we have a Bayesian equilibrium. The fact that the equilibrium is C-compliant will follow from the construction.

Case I: Fix some state $\omega \in (x_V)^{-1}(C)$. Set $s_P^*(\omega)(x_V(\omega)) = 1$. We show that, for each $p \in \mathbb{R}$, $\mathbb{E}u_P(\omega, s_P^*, s_V^*) \geq \mathbb{E}u_P(\omega, p, s_V^*)$.

Given that $\omega \in (x_V)^{-1}(C)$, there is some i, so that $x_V(\omega) \in [\underline{p}_i, \overline{p}_i]$. Note

$$\mathbb{E}u_P(\omega, s_P^*, s_V^*) = -k^2 + B + 2k(\underline{p}_i - x_V(\omega)) \geq 0.$$

Fix some policy $p \in \mathbb{R}$. If $p \in \mathbb{R} \backslash C$, we certainly have that

$$\mathbb{E}u_P(\omega, s_P^*, s_V^*) \geq 0 \geq \mathbb{E}u_P(\omega, p, s_V^*).$$

So, suppose that $p \in C$. Then, there exists some j so that $p \in [\underline{p}_j, \overline{p}_j]$. If $i = j$,

$$\begin{aligned} \mathbb{E}u_P(\omega, s_P^*, s_V^*) &= -k^2 + B + 2k(\underline{p}_i - x_V(\omega)) \\ &\geq -(p - x_V(\omega))^2 - k^2 + B + 2k(\underline{p}_i - x_V(\omega)) \\ &= -(p - x_V(\omega) - k)^2 + B + 2k(\underline{p}_i - p) \\ &= \mathbb{E}u_P(\omega, p, s_V^*), \end{aligned}$$

as required. Next suppose that $j < i$. We have already seen that $\mathbb{E}u_P(\omega, s_P^*, s_V^*) \geq \mathbb{E}u_P(\omega, \underline{p}_i, s_V^*)$. So, to show that $\mathbb{E}u_P(\omega, s_P^*, s_V^*) \geq \mathbb{E}u_P(\omega, p, s_V^*)$, it suffices to show that $\mathbb{E}u_P(\omega, \underline{p}_i, s_V^*) \geq \mathbb{E}u_P(\omega, p, s_V^*)$. But, this follows since

$$\begin{aligned} \mathbb{E}u_P(\omega, \underline{p}_i, s_V^*) &= -(\underline{p}_i - x_V(\omega) - k)^2 + B \\ &\geq -(p - x_V(\omega) - k)^2 + B + 2k(\underline{p}_j - p) \\ &= \mathbb{E}u_P(\omega, p, s_V^*). \end{aligned}$$

Finally, suppose $j > i$. Using Lemma C6, here, we have that

$$(\underline{p}_{i+1} - x_V(\omega))^2 \geq (\underline{p}_{i+1} - \overline{p}_i)^2 \geq 2k(\underline{p}_{i+1} - \underline{p}_i). \tag{C2}$$

So, using Equation C2,

$$\begin{aligned} \mathbb{E}u_P(\omega, s_P^*, s_V^*) &= -k^2 + B + 2k(\underline{p}_i - x_V(\omega)) \\ &\geq -(\underline{p}_{i+1} - x_V(\omega))^2 - k^2 + B + 2k(\underline{p}_{i+1} - x_V(\omega)). \end{aligned}$$

This implies

$$\begin{aligned}
\mathbb{E}u_P(\omega, s_P^*, s_V^*) &\geq -(\underline{p}_{i+1} - x_V(\omega))^2 - k^2 + B + 2k(\underline{p}_{i+1} - x_V(\omega)) \\
&= -(\underline{p}_{i+1} - x_V(\omega) - k)^2 + B \\
&\geq -(\underline{p}_j - x_V(\omega) - k)^2 + B \\
&\geq -(p - x_V(\omega) - k)^2 + B + 2k(\underline{p}_j - p) \\
&= \mathbb{E}u_P(\omega, p, s_V^*),
\end{aligned}$$

where the third line uses the fact that $\underline{p}_j > \underline{p}_{i+1} \geq x_v(\omega) + k$.

Case II: Fix some state $\omega \in (x_V)^{-1}(\mathbb{R}\backslash C) \cap (x_P)^{-1}(\mathbb{R}\backslash C)$. Note, there exists some i so that $x_P(\omega) \in (\overline{p}_i, \underline{p}_{i+1})$. (Here, we allow that $\overline{p}_i$ may be $-\infty$ and $\underline{p}_{i+1}$ may be ∞.) In fact, there exists some i so that $x_P(\omega) \in (\overline{p}_i, \underline{p}_{i+1})\backslash(\max\{\overline{p}_i, \underline{p}_i + k\}, \overline{p}_i + k)$. To see this last claim, suppose $\overline{p}_i < x_P(\omega) \leq \overline{p}_i + k$. Then $x_V(\omega) \leq \overline{p}_i$. Using the fact that $x_V(\omega) \in \mathbb{R}\backslash C$, this implies that $x_V(\omega) < \underline{p}_i$. It follows that $x_P(\omega) \leq \underline{p}_i + k$, as stated.

Consider the following three (disjoint) intervals: $(\overline{p}_i, \underline{p}_i + k)$, $(\overline{p}_i + k, \underline{p}_{i+1} - \sqrt{B}]$, and $(\underline{p}_{i+1} - \sqrt{B}, \underline{p}_{i+1})$. Note, the first of these may be empty, i.e., if $\underline{p}_i + k \leq \overline{p}_i$. The second of these however is non-empty, since $\underline{p}_{i+1} - \overline{p}_i \geq k + \sqrt{B}$. Of course, the latter is non-empty. Also note that the union of these intervals is $(\overline{p}_i, \underline{p}_{i+1})\backslash(\max\{\overline{p}_i, \underline{p}_i + k\}, \overline{p}_i + k)$.

Set $s_P^*(\omega)(\underline{p}_i) = 1$, if $x_P(\omega) \in (\overline{p}_i, \underline{p}_i + k)$. Set $s_P^*(\omega)(x_P(\omega)) = 1$, if $x_P(\omega) \in (\overline{p}_i + k, \underline{p}_{i+1} - \sqrt{B}]$. Finally, set $s_P^*(\omega)(\underline{p}_{i+1}) = 1$, if $x_P(\omega) \in (\underline{p}_{i+1} - \sqrt{B}, \underline{p}_{i+1})$.

We now turn to show that $\mathbb{E}u_P(\omega, s_P^*, s_V^*) \geq \mathbb{E}u_P(\omega, p, s_V^*)$, for each $p \in \mathbb{R}$. In fact, it suffices to show that, for each $p \in [\underline{p}_i, \underline{p}_{i+1}]$, $\mathbb{E}u_P(\omega, s_P^*, s_V^*) \geq \mathbb{E}u_P(\omega, p, s_V^*)$. If $p < \underline{p}_i$, there exists some policy q with $s_V^*(q) \geq s_V^*(p)$ and $(q - x_P(\omega))^2 \leq (p - x_P(\omega))^2$. So, if $\mathbb{E}u_P(\omega, s_P^*, s_V^*) \geq \mathbb{E}u_P(\omega, q, s_V^*)$, then certainly $\mathbb{E}u_P(\omega, s_P^*, s_V^*) \geq \mathbb{E}u_P(\omega, p, s_V^*)$. Likewise, if $p > \underline{p}_{i+1}$, then $\mathbb{E}u_P(\omega, s_P^*, s_V^*) \geq \mathbb{E}u_P(\omega, \underline{p}_{i+1}, s_V^*)$ implies that $\mathbb{E}u_P(\omega, s_P^*, s_V^*) \geq \mathbb{E}u_P(\omega, p, s_V^*)$.

First suppose that $\underline{p}_i + k > \overline{p}_i$, and specifically $\underline{p}_i + k > x_P(\omega) > \overline{p}_i$. Consider some $p \in [\underline{p}_i, \overline{p}_i]$, and note that

$$\mathbb{E}u_P(\omega, p, s_V^*) = -(p - x_P(\omega))^2 + B + 2k(\underline{p}_i - p),$$

so

$$\frac{d\mathbb{E}u_P(\omega, p, s_V^*)}{dp} = -2(p - x_P(\omega)) - 2k.$$

Note that $\underline{p}_i + k > x_P(\omega)$, and so $\mathbb{E}u_P(\omega, p, s_V^*)$ is strictly decreasing over the range $[\underline{p}_i, \overline{p}_i]$.

As such,

$$\mathbb{E}u_P\left(\omega, s_P^*, s_V^*\right) = \mathbb{E}u_P(\omega, \underline{p}_i, s_V^*) \geq \mathbb{E}u_P\left(\omega, p, s_V^*\right)$$

for each $p \in [\underline{p}_i, \overline{p}_i]$. Now, fix some $p \in (\overline{p}_i, \underline{p}_{i+1})$ and note that

$$\begin{aligned}
\mathbb{E}u_P\left(\omega, s_P^*, s_V^*\right) &= -(\underline{p}_i - x_P\left(\omega\right))^2 + B \\
&\geq -k^2 + B \\
&\geq 0 \\
&\geq \mathbb{E}u_P\left(\omega, p, s_V^*\right),
\end{aligned}$$

where the second line uses the fact that $\underline{p}_i + k > x_P\left(\omega\right) > \underline{p}_i$ and the third line uses the fact that $B \geq k^2$. Finally, using the fact just established, i.e., $\mathbb{E}u_P\left(\omega, s_P^*, s_V^*\right) \geq 0$, we have that

$$\begin{aligned}
\mathbb{E}u_P\left(\omega, s_P^*, s_V^*\right) &\geq 0 \\
&= -(k + \sqrt{B} - k)^2 + B \\
&\geq -(\underline{p}_{i+1} - \underline{p}_i - k)^2 + B \\
&\geq -(\underline{p}_{i+1} - x_P\left(\omega\right))^2 + B \\
&= \mathbb{E}u_P(\omega, \underline{p}_{i+1}, s_V^*),
\end{aligned}$$

where the third line uses the fact that $\underline{p}_{i+1} - \underline{p}_i \geq \underline{p}_{i+1} - \overline{p}_i \geq k + \sqrt{B}$ and the fourth line uses the fact (already established) that $\underline{p}_{i+1} \geq \underline{p}_i + k > x_P\left(\omega\right)$.

Next, suppose that $x_P\left(\omega\right) \in (\overline{p}_i + k, \underline{p}_{i+1} - \sqrt{B}]$. Here, $\mathbb{E}u_P\left(\omega, s_P^*, s_V^*\right) = 0$. It suffices to show that, for each $p \in [\underline{p}_i, \underline{p}_{i+1}]$, $0 \geq \mathbb{E}u_P\left(\omega, p, s_V^*\right)$. Consider first the case where $p \in [\underline{p}_i, \overline{p}_i]$. Recall, here,

$$\frac{d\mathbb{E}u_P\left(\omega, p, s_V^*\right)}{dp} = -2\left(p - x_P\left(\omega\right)\right) - 2k.$$

Now, $x_P\left(\omega\right) > \overline{p}_i + k \geq p + k$, for each $p \in [\underline{p}_i, \overline{p}_i]$. So, increasing p (over the range $[\underline{p}_i, \overline{p}_i]$) increases $\mathbb{E}u_P\left(\omega, p, s_V^*\right)$. This gives that $\mathbb{E}u_P\left(\omega, \overline{p}_i, s_V^*\right) \geq \mathbb{E}u_P\left(\omega, p, s_V^*\right)$, for each such p. Moreover,

$$\begin{aligned}
\mathbb{E}u_P\left(\omega, \overline{p}_i, s_V^*\right) &= -(\overline{p}_i - x_P\left(\omega\right))^2 + k^2 \\
&\leq -k^2 + k^2,
\end{aligned}$$

where the second line uses the fact that $x_P(\omega) > \overline{p}_i + k$. So,

$$\mathbb{E}u_P(\omega, s_P^*, s_V^*) = 0 \geq \mathbb{E}u_P(\omega, \overline{p}_i, s_V^*) \geq \mathbb{E}u_P(\omega, p, s_V^*),$$

for each $p \in [\underline{p}_i, \overline{p}_i]$. Certainly we have that, for each $p \in (\overline{p}_i, \underline{p}_{i+1})$,

$$\mathbb{E}u_P(\omega, s_P^*, s_V^*) = 0 \geq \mathbb{E}u_P(\omega, p, s_V^*).$$

Finally, note that

$$\begin{aligned} \mathbb{E}u_P(\omega, s_P^*, s_V^*) &= -(\underline{p}_{i+1} - \underline{p}_{i+1} + \sqrt{B})^2 + B \\ &\geq -(\underline{p}_{i+1} - x_P(\omega))^2 + B \\ &= \mathbb{E}u_P(\omega, \underline{p}_{i+1}, s_V^*), \end{aligned}$$

where the second line uses the fact that $\underline{p}_{i+1} - \sqrt{B} \geq x_P(\omega)$.

Finally, consider the case where $x_P(\omega) \in (\underline{p}_{i+1} - \sqrt{B}, \underline{p}_{i+1})$. Here,

$$\begin{aligned} \mathbb{E}u_P(\omega, s_P^*, s_V^*) &= -(\underline{p}_{i+1} - x_P(\omega))^2 + B \\ &\geq -(\underline{p}_{i+1} - \underline{p}_{i+1} + \sqrt{B})^2 + B \\ &= 0. \end{aligned}$$

So, certainly,

$$\mathbb{E}u_P(\omega, s_P^*, s_V^*) = 0 \geq \mathbb{E}u_P(\omega, p, s_V^*),$$

for each $p \in (\overline{p}_i, \underline{p}_{i+1}]$. Consider $p \in [\underline{p}_i, \overline{p}_i]$ and recall that $\mathbb{E}u_P(\omega, \overline{p}_i, s_V^*) \geq \mathbb{E}u_P(\omega, p, s_V^*)$. (Here we use the fact that $x_P(\omega) \geq \underline{p}_{i+1} - \sqrt{B} \geq \overline{p}_i + k$.) So, it suffices to show that

$$\mathbb{E}u_P(\omega, s_P^*, s_V^*) = 0 \geq \mathbb{E}u_P(\omega, \overline{p}_i, s_V^*).$$

But this is immediate, since

$$\begin{aligned} \mathbb{E}u_P(\omega, \overline{p}_i, s_V^*) &= -(\overline{p}_i - x_P(\omega))^2 + k^2 \\ &\leq -(\overline{p}_i - \underline{p}_{i+1} + \sqrt{B})^2 + k^2 \\ &= -(\underline{p}_{i+1} - \overline{p}_i - \sqrt{B})^2 + k^2 \\ &\leq -(k + \sqrt{B} - \sqrt{B})^2 + k^2 \\ &= 0, \end{aligned}$$

where the second line uses the fact that $x_P(\omega) \geq \underline{p}_{i+1} - \sqrt{B} \geq \overline{p}$ and the fourth line uses the fact that $\underline{p}_{i+1} - \overline{p}_i \geq k + \sqrt{B}$.

Case III: Fix some state $\omega \in (x_V)^{-1}(\mathbb{R} \backslash C) \cap (x_P)^{-1}(C)$. Note, there exists some i so that $x_P(\omega) \in [\underline{p}_i, \overline{p}_i]$. In fact, since $x_V(\omega) \in \mathbb{R} \backslash C$, $x_P(\omega) < \overline{p}_i + k$. Put differently, $x_P(\omega) \in [\underline{p}_i, \overline{p}_i] \cap [\underline{p}_i, \underline{p}_i + k)$. We make use of this fact below.

Set $s_V^*(\omega)(\underline{p}_i) = 1$. We will show that, for each $p \in \mathbb{R}$, $\mathbb{E}u_P(\omega, s_P^*, s_V^*) \geq \mathbb{E}u_P(\omega, p, s_V^*)$. In fact, it suffices to show that $\mathbb{E}u_P(\omega, s_P^*, s_V^*) \geq \mathbb{E}u_P(\omega, p, s_V^*)$, for each $p \in [\underline{p}_i, \underline{p}_{i+1}]$. To see this, note that, if $p < \underline{p}_i$, then $\mathbb{E}u_P(\omega, \underline{p}_i, s_V^*) > \mathbb{E}u_P(\omega, p, s_V^*)$. (Here, we use the fact that $x_P(\omega) \geq \underline{p}_i > p$ and $s_V^*(\omega)(\underline{p}_i) \geq s_V^*(\omega)(p)$.) Likewise, if $p > \underline{p}_{i+1}$, then $\mathbb{E}u_P(\omega, \underline{p}_{i+1}, s_V^*) > \mathbb{E}u_P(\omega, p, s_V^*)$. (Here, we use the fact that $p > \underline{p}_{i+1} \geq x_P(\omega)$ and $s_V^*(\omega)(\underline{p}_{i+1}) \geq s_V^*(\omega)(p)$.)

Consider policies $p \in [\underline{p}_i, \overline{p}_i]$. For any such policy p, we have

$$\frac{d\mathbb{E}u_P(\omega, s_P^*, s_V^*)}{dp} = -2(p - x_V(\omega) - k) - 2k.$$

Since, for each such policy, $p > x_V(\omega)$, it follows that $\mathbb{E}u_P(\omega, s_P^*, s_V^*)$ is decreasing over the range of policies $[\underline{p}_i, \overline{p}_i]$. So certainly $\mathbb{E}u_P(\omega, \underline{p}_i, s_V^*) > \mathbb{E}u_P(\omega, p, s_V^*)$, for each $p \in (\underline{p}_i, \overline{p}_i]$. Likewise, fix some policy $p \in (\overline{p}_i, \underline{p}_{i+1})$ and note that

$$\begin{aligned}
\mathbb{E}u_P(\omega, s_P^*, s_V^*) &= -(\underline{p}_i - x_P(\omega))^2 + B \\
&> -(\underline{p}_i - \underline{p}_i - k)^2 + B \\
&\geq 0 \\
&\geq \mathbb{E}u_P(\omega, p, s_V^*),
\end{aligned}$$

where the second line uses the fact that $\underline{p}_i + k > x_P(\omega)$, the third line uses the fact that $B \geq k^2$, and the fourth line uses the fact that $s_V^*(p) = 0$. Finally, consider the policy $\underline{p}_{i+1}$.

We have that

$$\begin{aligned}
\mathbb{E}u_P(\omega, s_P^*, s_V^*) &= -(\underline{p}_i - x_P(\omega))^2 + B \\
&> -(\underline{p}_i - \underline{p}_i - k)^2 + B \\
&= -k^2 + B \\
&\geq -[(\underline{p}_{i+1} - \underline{p}_i)(\underline{p}_{i+1} - \underline{p}_i - 2k)] - k^2 + B \\
&= -(\underline{p}_{i+1} - \underline{p}_i - k)^2 + B \\
&\geq -(\underline{p}_{i+1} - x_P(\omega))^2 + B \\
&= \mathbb{E}u_P(\omega, \underline{p}_{i+1}, s_V^*),
\end{aligned}$$

where the second and sixth lines use the fact that $\underline{p}_i + k > x_P(\omega)$ and the fourth line uses the fact that $\underline{p}_{i+1} - \underline{p}_i \geq \underline{p}_{i+1} - \overline{p}_i \geq k + \sqrt{B} \geq 2k$.

Conclusion of Proof: We have constructed a measurable function s_V^*. Given this function, we were able to construct a function s_P^* so that, for each state ω and each policy p, $\mathbb{E}u_P(\omega, s_P^*, s_V^*) \geq \mathbb{E}u_P(\omega, p, s_V^*)$. The function had the property that $s_P^*(\omega)(x_V(\omega)) = 1$, when $x_V(\omega) \in C$. So, to show that we have a C-compliant equilibrium, it suffices to show that s_P^* is measurable.

The proof will make use of Remark B1. We will construct functions, by restricting the domain of s_P^*, viz. Ω, to $(x_V)^{-1}(C)$, $(x_V)^{-1}(\mathbb{R}\backslash C) \cap (x_P)^{-1}(\mathbb{R}\backslash C)$, and $(x_V)^{-1}(\mathbb{R}\backslash C) \cap (x_P)^{-1}(C)$. Then, we will show that each of these maps are measurable, thereby establishing the result.

Case A: The restriction of the domain to $(\mathbf{x_V})^{-1}(\mathbf{C})$. Here, measurability follows from measurability of x_V and Lemma A2.

Case B: The restriction of the domain to $(\mathbf{x_V})^{-1}(\mathbb{R}\backslash\mathbf{C}) \cap (\mathbf{x_V})^{-1}(\mathbb{R}\backslash\mathbf{C})$. Define sets

$$\Omega_i = (x_P)^{-1}((\overline{p}_i, \underline{p}_{i+1}) \backslash (\max\{\overline{p}_i, \underline{p}_i + k\}, \overline{p}_i + k)),$$

and maps $f_i : \Omega_i \to \mathbb{R}$ so that $s_P^*(f_i(\omega)) = 1$. Now use the fact that each of the maps $\omega \mapsto \underline{p}_i$, $\omega \mapsto \underline{p}_{i+1}$, and x_P are measurable. Along with Lemma A1, this gives that each of the maps f_i are measurable. Now note that $(x_V)^{-1}(\mathbb{R}\backslash C) \cap (x_P)^{-1}(\mathbb{R}\backslash C)$ is the countable disjoint union of the sets Ω_i. So, the result follows from Remark B1 and Lemma A2.

Case C: The restriction of the domain to $(\mathbf{x_V})^{-1}(\mathbb{R}\backslash\mathbf{C}) \cap (\mathbf{x_V})^{-1}(\mathbf{C})$. Now define sets

$$\Omega_i = (x_P)^{-1}([\underline{p}_i, \overline{p}_i] \cap [\underline{p}_i, \underline{p}_i + k)),$$

and maps $f_i : \Omega_i \to \mathbb{R}$ so that $f_i(\Omega_i) = \{\underline{p}_i\}$. Now note that $(x_V)^{-1}(\mathbb{R}\backslash C) \cap (x_P)^{-1}(C)$ is the countable disjoint union of the sets Ω_i. So the result follows from Remark B1 and Lemma A2. ■

c. Proof of Lemma 4.2: The following lemma will be of use.

Lemma C7 *Fix $\underline{p}_{i+1} > \overline{p}_i + k$ and consider the function*

$$f_i(p) = -(p - \overline{p}_i - k)^2 + B - (\underline{p}_{i+1} - p)^2$$

(i) Suppose $2\sqrt{B} - k > \underline{p}_{i+1} - \overline{p}_i$. Then, $0 \geq f_i(\frac{1}{2}(\underline{p}_{i+1} + \overline{p}_i + k))$ if and only if, for each $p \in (\underline{p}_{i+1} - \sqrt{B} + k, \underline{p}_{i+1})$, $0 \geq f_i(p)$.

(ii) Suppose $\underline{p}_{i+1} - \overline{p}_i \geq 2\sqrt{B} - k$. Then, $0 \geq f_i(\underline{p}_{i+1} - \sqrt{B} + k)$ if and only if, for each $p \in (\underline{p}_{i+1} - \sqrt{B} + k, \underline{p}_{i+1})$, $0 \geq f_i(p)$.

Proof. Note,

$$\frac{df_i}{dp} = -2(p - \overline{p}_i - k) + 2(\underline{p}_{i+1} - p),$$

and, moreover, the second derivative of f_i with respect to p is -4. So, f_i is maximized at

$$\frac{\underline{p}_{i+1} + \overline{p}_i + k}{2}.$$

First consider the case where $2\sqrt{B} - k > \underline{p}_{i+1} - \overline{p}_i$. Here,

$$\underline{p}_{i+1} > \frac{\underline{p}_{i+1} + \overline{p}_i + k}{2} > \underline{p}_{i+1} - \sqrt{B} + k.$$

So, $0 \geq f_i(p)$, for each $p \in (\underline{p}_{i+1} - \sqrt{B} + k, \underline{p}_{i+1})$, if and only if $0 \geq f_i(\frac{1}{2}(\underline{p}_{i+1} + \overline{p}_i + k))$.

Next consider the case where $\underline{p}_{i+1} - \overline{p}_i \geq 2\sqrt{B} - k$. Here, $f_i(\cdot)$ is decreasing over the set of policies $(\underline{p}_{i+1} - \sqrt{B} + k, \underline{p}_{i+1})$. So, $0 \geq f_i(p)$, for each $p \in (\underline{p}_{i+1} - \sqrt{B} + k, \underline{p}_{i+1})$, if and only if $0 \geq f_i(\underline{p}_{i+1} - \sqrt{B} + k)$. ■

Lemma C8 *Fix intervals $[\underline{p}_i, \overline{p}_i]$ and $[\underline{p}_{i+1}, \overline{p}_{i+1}]$ each of length $\frac{B-k^2}{2k}$. Also, fix a $[\underline{p}_i, \overline{p}_i] \cup [\underline{p}_{i+1}, \overline{p}_{i+1}]$-compliant equilibrium, viz. (s_P^*, s_V^*). If $s_V^*(\omega)(x_P(\omega)) > 0$ for some $\omega \in (x_V)^{-1}((\underline{p}_{i+1} - \sqrt{B}, \underline{p}_{i+1} - k))$, then $(\underline{p}_{i+1} - \overline{p}_i)$ must satisfy one of the following conditions:*

(i) $2\sqrt{B} - k \geq (\underline{p}_{i+1} - \overline{p}_i) \geq k + \sqrt{2B}$, or

(ii) $(\underline{p}_{i+1} - \overline{p}_i) \geq 2\sqrt{B}$.

Proof. Fix a $[\underline{p}_i, \overline{p}_i] \cup [\underline{p}_{i+1}, \overline{p}_{i+1}]$-compliant equilibrium, viz. (s^*_P, s^*_V), per the statement of the Lemma. Suppose there exists some state $\omega \in (x_V)^{-1}((\underline{p}_{i+1} - \sqrt{B}, \underline{p}_{i+1} - k))$ so that $s^*_V(\omega)(x_P(\omega)) > 0$. Write $p = x_P(\omega)$, and notice that $p \in (\underline{p}_{i+1} - \sqrt{B} + k, \underline{p}_{i+1})$. (Here we use the fact that the Pragmatist's ideal point is k units above the Voter's.) By Lemma C2, we must have that

$$\mathbb{E}u_P(\omega, p^*, s^*_V) = s^*_V(p)B \geq -(\underline{p}_{i+1} - p)^2 + B = \mathbb{E}u_P(\omega, \underline{q}, s^*_V),$$

and this implies that

$$s^*_V(p) \geq 1 - \frac{(\underline{p}_{i+1} - p)^2}{B}.$$

Now, consider a state, viz. $\overline{\omega}$, at which the Voter's ideal point is $\overline{p}_i$. We need that, when the Voter's ideal point is $\overline{p}_i$, the Pragmatist's expected payoffs from $\overline{p}_i$ are higher than her expected payoffs from each $p \in (\underline{p}_{i+1} - \sqrt{B} + k, \underline{p}_{i+1})$. Using Lemma C2, this is the requirement that, for each such p,

$$\mathbb{E}u_P(\overline{\omega}, \overline{p}_i, s^*_V) = -k^2 + k^2 \geq -(p - \overline{p}_i - k)^2 + B - (\underline{p}_{i+1} - p)^2 = \mathbb{E}u_P(\overline{\omega}, p, s^*_V). \quad \text{(C3)}$$

By Lemma C7, there are two cases to consider—where $2\sqrt{B} - k > (\underline{p}_{i+1} - \overline{p}_i)$ and where $(\underline{p}_{i+1} - \overline{p}_i) \geq 2\sqrt{B} - k$.

Case A: First consider the case where $2\sqrt{B} - k > (\underline{p}_{i+1} - \overline{p}_i)$. Using Proposition 4.3(i), $(\underline{p}_{i+1} - \overline{p}_i) > k + \sqrt{B}$. So, we can apply Lemma C7 and get that Equation C3 holds if and only if

$$0 \geq -\frac{1}{2}\left(\underline{p}_{i+1} - \overline{p}_i - k\right)^2 + B,$$

or if and only if

$$(\underline{p}_{i+1} - \overline{p}_i - k)^2 - 2B \geq 0.$$

As such, Equation C3 if and only if

$$(\underline{p}_{i+1} - \overline{p}_i)^2 - 2k(\underline{p}_{i+1} - \overline{p}_i) - \left(2B - k^2\right) \geq 0.$$

Using Proposition 4.3, $(\underline{p}_{i+1} - \overline{p}_i) \geq k + \sqrt{B} \geq 2k$. As such,

$$\begin{aligned}(\underline{p}_{i+1} - \overline{p}_i) &\geq \frac{2k + \sqrt{4k^2 + 8B - 4k^2}}{2} \\ &= k + \sqrt{2B}.\end{aligned}$$

From this it follows that $2\sqrt{B} - k \geq (\underline{p}_{i+1} - \overline{p}_i) \geq k + \sqrt{2B}$, as required.

Case B: Now consider the case where $(\underline{p}_{i+1} - \overline{p}_i) \geq 2\sqrt{B} - k$. Using Proposition 4.3(i), $(\underline{p}_{i+1} - \overline{p}_i) > k + \sqrt{B}$. So, we can apply Lemma C7 and get that Equation C3 holds if and only if

$$0 \geq -(\underline{p}_{i+1} - \overline{p}_i - \sqrt{B})^2 + B - (\sqrt{B} - k)^2,$$

or if and only if

$$(\underline{p}_{i+1} - \overline{p}_i)^2 - 2\sqrt{B}(\underline{p}_{i+1} - \overline{p}_i) + (\sqrt{B} - k)^2 \geq 0.$$

This gives that either $(\underline{p}_{i+1} - \overline{p}_i)$ is less than both $2\sqrt{B}$ and

$$\frac{2\sqrt{B} - \sqrt{4B - 4(\sqrt{B} - k)^2}}{2} = \sqrt{B} - \sqrt{B - (\sqrt{B} - k)^2}$$

or $(\underline{p}_{i+1} - \overline{p}_i)$ is greater than both $2\sqrt{B}$ and

$$\frac{2\sqrt{B} + \sqrt{4B - 4(\sqrt{B} - k)^2}}{2} = \sqrt{B} + \sqrt{B - (\sqrt{B} - k)^2}.$$

In the former case, we have

$$2\sqrt{B} > \sqrt{B} - \sqrt{B - (\sqrt{B} - k)^2} \geq (\underline{p}_{i+1} - \overline{p}_i) \geq 2\sqrt{B} - k.$$

This implies that

$$0 \geq -\sqrt{B - (\sqrt{B} - k)^2} \geq \sqrt{B} - k \geq 0.$$

The left-most inequality is strict when $k \neq 2\sqrt{B}$ and the right-most inequality is strict when $k \neq \sqrt{B}$. As such, this case cannot hold. In the latter case,

$$\begin{aligned}(\underline{p}_{i+1} - \overline{p}_i) &\geq 2\sqrt{B} \\ &\geq \max\{k + \sqrt{B}, 2\sqrt{B} - k, \sqrt{B} + \sqrt{B - (\sqrt{B} - k)^2}\}.\end{aligned}$$

(Here, we implicitly use Proposition 4.3(i).) This establishes the result. ■

Proof of Lemma 4.2. Suppose $\underline{p}_{i+1} - \overline{p}_i = k + \sqrt{B}$. Then $\max\{k + \sqrt{2B}, 2\sqrt{B}\} > (\underline{p}_{i+1} - \overline{p}_i)$. So the result follows from Lemma C8. ■

Appendix D Proof for Section 5

Lemma D1

(i) Fix some $k > 0$. There exists some non-empty open interval $U \subseteq \mathbb{R}_+$ so that, for each $B \in U$, $4k\sqrt{B} > B - k^2$.

(ii) Fix some $B > 0$. There exists some non-empty open interval $U \subseteq \mathbb{R}_+$ so that, for each $k \in U$, $4k\sqrt{B} > B - k^2$.

Proof. Begin with part (i). Fix some $k > 0$. Consider the non-empty open interval $(k^2, (2+\sqrt{5})^2 k^2)$. We will show that, for any B in this interval, B satisfies $4k\sqrt{B} > B - k^2$. This will establish the result.

Fix some B in this interval. Suppose, contra hypothesis, that $B - k^2 \geq 4k\sqrt{B}$. Then,

$$B - \frac{B}{(2+\sqrt{5})^2} > B - k^2 \geq 4k\sqrt{B} > \frac{4B}{(2+\sqrt{5})},$$

where the first and last inequalities use the fact that $(2 + \sqrt{5})^2 k^2 > B$. It follows that

$$B((2+\sqrt{5})^2 - 1) > 4B(2+\sqrt{5}),$$

which cannot hold.

Now turn to part (ii). Fix some $B > 0$. Consider the non-empty open interval $(\sqrt{B}(\sqrt{5}-2), \sqrt{B})$. We will show that, for any k in this interval, k satisfies $4k\sqrt{B} > B - k^2$. This will establish the result.

Fix some k in this interval. Suppose, contra hypothesis, that $B - k^2 \geq 4k\sqrt{B}$. Then, using the fact that $k > \sqrt{B}(\sqrt{5}-2)$, we have

$$B \geq 4k\sqrt{B} + k^2 > 4B(\sqrt{5}-2) + B(\sqrt{5}-2)^2.$$

From this it follows that

$$1 > 4(\sqrt{5}-2) + (\sqrt{5}-2)^2 = 1,$$

which cannot hold. ■

Proof of Proposition 5.1. Begin with part (i). Fix some $k > 0$. By Lemma D1(i), there exists some non-empty open interval $U \subseteq \mathbb{R}_+$ so that, for each $B \in U$, $4k\sqrt{B} > B - k^2$. Fix some $B \in U$ and choose $\mu[B] \in \mathcal{M}(\Omega)$ so that the support of $\mu[B]$ is $(x_V)^{-1}([-\sqrt{B}, \sqrt{B}])$. By Proposition 3.1, there exists some $[-\sqrt{B}, \sqrt{B}]$-compliant equilibrium, viz. (s_P^*, s_V^*), of the game with a B-Ideologue. For this equilibrium,

$$\int_\Omega \mathbb{E}u_V(\omega, s_P^*)\, d\mu[B] = 0.$$

Now, consider the game with a (k, B)-Pragmatist. Fix some Bayesian equilibrium of this game, viz. (r_P^*, r_V^*). Note, since $B \in U$,

$$2\sqrt{B} > \frac{B - k^2}{2k}.$$

So, by Lemma C3, there is a non-empty open set of policies, viz. V, contained in $[-\sqrt{B}, \sqrt{B}]$ so that $r_P^*(\omega)(x_V(\omega)) = 0$ for each ω with $x_V(\omega) \in V$. Since x_V is continuous, $(x_V)^{-1}(V)$ is open and so $\mu[B]((x_V)^{-1}(V)) > 0$. It follows that

$$\int_\Omega \mathbb{E}u_V(\omega, r_P^*)\, d\mu[B] < 0,$$

as required.

The proof of part (ii) is analogous. ■

References

Aliprantis, Charalambos D. and Kim C. Border. 1999. *Infinite Dimensional Analysis: A Hithchiker's Guide, 2nd edition.* Berlin, Germany: Springer-Verlag.

Ashworth, Scott. 2005. "Reputational Dynamics and Political Careers." *Journal of Law, Economics and Organization* 21(2):441–466.

Ashworth, Scott and Ethan Bueno de Mesquita. 2006. "Delivering the Goods: Legislative Particularism in Different Electoral and Institutional Settings." *Journal of Politics* 68(1):168–179.

Besley, Timothy. 2006. *Principled Agents: The Political Economy of Good Government.* Oxford: Oxford University Press.

Callander, Steven. 2008. "A Theory of Political Expertise." *Quarterly Journal of Political Science* 3(2):123–140.

Callander, Steven. 2009. "Searching for Good Policies." Northwestern typescript.

Callander, Steven and Simon Wilkie. 2007. "Lies, Damned Lies, and Political Campaigns." *Games and Economic Behavior* 60(August):262–286.

Canes-Wrone, Brandice and Kenneth W. Shotts. 2007. "When Do Elections Encourage Ideological Rigidity?" *American Political Science Review* 101(2):273–288.

Canes-Wrone, Brandice, Michael C. Herron and Kenneth W. Shotts. 2001. "Leadership and Pandering: A Theory of Executive Policymaking." *American Journal of Political Science* 45(July):532–550.

Crawford, Vincent P. and Joel Sobel. 1982. "Strategic Information Transmission." *Econometrica* 50(6):1431–1451.

Fox, Justin. 2007. "Government Transparency and Policymaking." *Public Choice* 131(April):23–44.

Gilligan, Thomas W. and Keith Krehbiel. 1987. "Collective Decisionmaking and Standing Committees: An Informational Rationale for Restrictive Amendment Procedures." *Journal of Law, Economics and Organization* 3(2):287–335.

Gilligan, Thomas W. and Keith Krehbiel. 1989. "Asymmetric Information and Legislative Rules with Heterogenous Committees." *American Journal of Political Science* 33(2):459–490.

Gilligan, Thomas W. and Keith Krehbiel. 1990. "Organization of Informative Committees by a Rational Legislature." *American Journal of Political Science* 34:531–564.

Krantz, Steven G. 2004. *Real Analysis and Foundations, Second Edition (Studies in Advanced Mathematics)*. Chapman and Hall.

Maskin, Eric and Jean Tirole. 2004. "The Politician and the Judge: Accountability in Government." *American Economic Review* 94(4):1034–1054.

Myerson, Roger B. 1993. "Effectiveness of Electoral Systems for Reducing Government Corruption: A Game-Theoretic Analysis." *Games and Economic Behavior* 5:118–132.

Persson, Torsten and Guido Tabellini. 2000. *Political Economics: Explaining Economic Policy*. Cambridge: MIT Press.

Schultz, Christian. 1996. "Polarization and Inefficient Policies." *Review of Economic Studies* 63(April):331–343.