I am writing this note in order to explain under what circumstances debt contracts are optimal; more importantly, to explain why debt contracts are optimal. For a sufficiently general setup, Gale and Hellwig ’85 show that in cases where verification is costly, debt contracts are optimal. I will try to keep the framework as minimal and simple as possible; I will only include those components which I believe to be essential. I will drop all the assumptions which are not needed to produce the result. I will present a setup with a continuous random variable, but similar results hold for all random variables. 

Disclaimer: Many proofs will be omitted.

0.1. Motivating Story.

I, in the role of the entrepreneur, am looking to take out a loan in order to pay for graduate school. I turn to a financial institution in order to borrow $200,000. All loans have a fixed repayment date which is set for five years after my planned graduation date. In order to determine whether I can take out a loan, the financial institution considers my accumulated future earnings, which is determined to be a random variable uniformly distributed with values between $80,000 and $350,000. On the day of repayment, I will know the realization of my earnings but the financial institution will not. However, there is an option of auditing my earnings at a cost of $40,000. If I am audited, my exact earnings will be known. All dollar amounts are stated in terms of present value. The financial institution and I are risk neutral.

There are two questions we need to ask. First, can I finance my education? Second, how will repayment be structured?

Some observations from the story:

(1) If my future earnings were a random variable with values between $250,000 and $350,000 the question would have an easy answer. I would borrow the $200,000 and return $200,000 in every future state of the world. This is possible because the amount I would need to repay is less than my minimal possible future earnings.

(2) If my future earnings were a random variable uniformly distributed with values between $80,000 and $300,000 the question would have an easy answer. My future earnings have an expectation of $180,000 which is less
than what I need to repay and so there is no contract which finances my education.  

(3) If my future earnings were a random variable uniformly distributed with values between $80,000 and $420,000 the first question would have an easy answer. I could finance my education by the following contract: we agree that I am audited in every state of the world. After accounting for the cost of auditing, I am left with a random variable uniformly distributed with values between $40,000 and $380,000. This random variable has expectation greater than $200,000 and so I can repay the loan.

I will now translate the story into a theoretical model. After analyzing the general setup, I will return to the motivating example and answer both motivating questions.

1. THE GENERAL SETUP

Data:
- $X$ - the returns of a project, a continuous random fully supported on the interval $[\alpha, \beta]$ where $0 \leq \alpha < \beta \leq \infty$. $F$ will denote the CDF associated to $X$.
- $l$ - the level of investment needed to carry out project.
- $\xi : [\alpha, \beta] \rightarrow \mathbb{R}$ - the cost of verification. $\xi(x)$ is the cost of verification in state of the world $x$. For convenience, I will assume that $\xi$ is bounded.\footnote{This is not to say that from my perspective higher education is not of positive NPV. Over my lifetime, higher education may be of positive NPV; however, from the point of view of the financial institution my education is not of positive NPV as my earnings are truncated five years after graduation.}

Assumptions:
- (1) The entrepreneur and firm are risk neutral.
- (2) The financial market is competitive.
- (3) $\alpha < l$ and thus the entrepreneur cannot repay $l$ in the worst state of the world.
- (4) $l < E[X]$.
- (5) $\xi > 0$.

Constraints:
- The project is non-transferable. If this were not the case the entrepreneur would sell the project to the firm for $E[X] - l$.
- The entrepreneur cannot have negative consumption. Alternatively, this can be viewed as limited liability.

2. CONTRACTS

Definition 1. A contract is a pair of $dF_x$-measurable functions $(g, s)$

- $g : [\alpha, \beta] \rightarrow \mathbb{R}$ describes the amount the entrepreneur agrees to pay the firm in each state of the world.
- $s : [\alpha, \beta] \rightarrow \{0, 1\}$, $s^{-1}(1)$ are those states in which there is verification.

\footnote{Let us assume that I will pay the cost of the audit - see Remark \footnote{It would be enough to assume that $\xi \in L^1(dF_x)$.}}
Definition 2. A contract \((g, s)\) is said to be incentive compatible if for every state of the world \(x\), and every alternative state \(y\) in the state of the world \(x\), the entrepreneur does not improve her payoff by reporting that the state of the world is \(y\). For this to be possible, \(y\) must be a state which is not verified. The condition is then

\[
\forall x \in [\alpha, \beta] \forall y \in s^{-1}(0) \quad x - g(x) - s(x)\xi(x) \geq x - g(y)
\]

Remark 3. In the formulation of the incentive compatible condition I have implicitly assumed that it is the entrepreneur who is faced with the costs of verification. One can alternatively consider a setup in which it is the financial institution that faces the costs of verification. The two setups are equivalent. See Appendix for details.

Lemma 4. A contract \((g, s)\) is incentive compatible iff the following two conditions hold

1. \(g\) is constant on \(s^{-1}(0)\). I will denote this constant by \(c_g\).
2. \(\forall x \in s^{-1}(1) \quad g(x) + \xi(x) \leq c_g\).

Given a contract \((g, s)\), not necessarily incentive compatible, and a state of the world \(x\), we can read from the contract the state of the world which will be reported by the entrepreneur. Overlooking questions of existence, the entrepreneur will report some \(r(x) \in \arg\min_{y \in \{x\} \cup s^{-1}(0)} g(y) + s(y)\xi(y)\). We will assume that if the entrepreneur is indifferent she will report the true state of the world. The following lemma shows us that we lose nothing by restricting ourselves to incentive compatible contracts.

Lemma 5. Let \((g, s)\) be any contracts. There exists a contract \((\hat{g}, \hat{s})\) which is incentive compatible and is outcome equivalent to \((g, s)\), i.e., in every state of the world \(x\), \(\hat{g}(x) = g(r(x))\) and \(x - \hat{g}(x) + \hat{s}(x)\xi(x) = x - g(r(x)) - s(r(x))\xi(r(x))\).

Definition 6. A contract \((g, s)\) is said to be non-negative if in every state of the world the entrepreneur is left with non-negative consumption:

\[
\forall x \quad x - g(x) - s(x)\cdot\xi(x) \geq 0
\]

3. Utility

Let \((g, s)\) be an incentive compatible, non-negative contract.

- The utility of the firm, denoted by \(U_F\), from the contract \((g, s)\) is its expected payoff

\[
U_F(g, s) = E[g] = \int_{\alpha}^{\beta} g(x) \, dF(x)
\]
The utility of the entrepreneur, denoted by $U_E$, from the contract $(g, s)$ is her expected payoff

$$U_E(g, s) = E\left[ x - g(x) - s(x)\xi(x) \right]$$

$$= \int_{\alpha}^{\beta} x - g(x) - s(x)\xi(x) dFx$$

$$= E[X] - U_2(g, s) - \int_{s^{-1}(1)} \xi(x) dFx$$

(3.1)

4. QUESTIONS REVISITED

We now return to our questions: can the project be financed? Can we do so optimally, if so, then how?

The first question asks whether there exist an incentive compatible non-negative contract $(g, s)$ s.t. $U_F(g, s) \geq l$.

Assuming that the project can be financed, the second question we are faced with is finding

$$\text{Argmax} U_F(g, s)$$

subject to the constraints:

(1) $(g, s)$ is incentive compatible and non-negative.

(2) $U_F(g, s) \geq l$.

Using the last form of the entrepreneur’s utility (equation 3.1), the problem is equivalent to minimizing the sum of the expected payment to the firm and the expected cost of verification. We now rewrite the problem as

$$\text{Argmin} \left\{ U_F(g, s) + \int_{s^{-1}(1)} \xi(x) dFx \right\}$$

subject to the constraints:

(1) $(g, s)$ is incentive compatible and non-negative.

(2) $U_2(g, s) \geq l$

**Definition 7.** A contract $(g, s)$ is said to be feasible if it is incentive compatible, non-negative and finances the project.

5. IDENTIFYING THE PROBLEM

I would like to claim that the problem we are facing is that of transferring funds from the entrepreneur to the firm. In light of this we should consider costly verification, not as a hurdle, but as a mechanism which allows us to transfer funds. Consider a world without the possibility of verification. In this world, the most that the entrepreneur would be able to transfer under an incentive compatible, non-negative contract is $\alpha$ (the minimal realization of $X$). Costly verification is a mechanism which expands the space of incentive compatible, non-negative, contracts and thus allows us to finance additional projects (see motivating story, Observation 3 above).
We should think of \( \int_{s^{-1}(1)} \xi(x) \, dF_x \) as the cost of transferring \( U_F(g, s) \) from the entrepreneur to the firm. I offer the following lemma and corollary to justify my claim.

**Lemma 8.** Let \((g, s)\) be any feasible contract. Assume that \( U_F(g, s) > l \). There exists a feasible contract \((\bar{g}, \bar{s})\) s.t.

1. \( U_F(\bar{g}, \bar{s}) = l \).
2. \( U_E(\bar{g}, \bar{s}) = U_E(g, s) + (U_F(g, s) - l) \).

**Corollary 9.** Let \((g, s)\) be a feasible contract. \((g, s)\) is optimal iff

1. \( U_F(g, s) = l \).
2. For every feasible contract \((\tilde{g}, \tilde{s})\), \( \int_{s^{-1}(1)} \xi(x) \, dF_x \leq \int_{\tilde{s}^{-1}(1)} \xi(x) \, dF_x \).

By Corollary 9, we can reformulate the maximization problem as follows:

\[
\text{Argmin}_{(g, s) \text{ feasible}} \left\{ \int_{s^{-1}(1)} \xi(x) \, dF_x \right\}
\]

6. **OPTIMAL CONTRACTS**

We now set our sights on finding feasible contracts and on finding optimal contracts. I have suggested above that our primary concern is finding a feasible contract. In practice, I will first classify optimal contracts. I would like to explain why I am attaching the problem in this order. The answer is that the space of contracts is very complicated and I would not know where to start looking for a feasible contract. The characterization of optimal contracts leads us to a very nice, 1-parameter, subspace. By restricting attention to this subspace I can easily compute whether the project can be financed, and if so I can then find an optimal contract.

I was told that some people like to know where they are headed and so I will now give away the result. The proof will follow.

**Definition 10.** A Standard Debt Contract \((SDC)\) is a contract \((g, s)\) characterized by \( x_0 \in [\alpha, \beta] \) satisfying

1. \( s^{-1}(0) = [x_0, \beta] \) i.e. a state of the world \( y \) is verified iff \( y < x_0 \).
2. \( c_g = x_0 \) i.e. in high states of the world the entrepreneur pays \( x_0 \) (which is the maximal possible payment that a non-negative, incentive compatible contract can offer).
3. \( \forall y < x_0 \quad g(y) + \xi(y) = y \) i.e. the maximal possible payment is made in low states of the world.

An \( SDC \) is uniquely defined by \( x_0 = \min (s^{-1}(0)) \) and we will denote the contract by \( SDC(x_0) \).

**Claim 11.** \( \forall x_0 \) SDC \((x_0)\) is incentive compatible and non-negative.

**Theorem 12.** (MAIN THEOREM) If \((g, s)\) is feasible and optimal then it is an SDC.
7. PROOF OF THE MAIN THEOREM

There are two ways in which this proof can be understood. The first is constructive. I start with some feasible contract, make piecemeal corrections and end up with an SDC. The second, is to restrict in stages the space in which an optimal feasible contract can lie, until we have restricted ourselves to the space of SDC’s.

The proof will proceed as follows: there will be three stages which correspond to the three conditions in the definition of an SDC. At each stage I will start with a feasible contract and alter the contract in a way which reduces verification costs. I will show that the new, altered, contract is feasible. I will show that, had we started with a contract which is not an SDC, the alterations will lead us to a contract with strictly lower verification cost; therefore, only an SDC can be optimal. In short: SDC’s minimize verification costs for a given level of repayment.

Remark 13. As I suggested in the introduction there is a dual approach, a dual proof to the theorem. Here we are looking to minimize verification costs for a given level of repayment. The dual approach looks to maximize repayment for a given level of verification costs. SDC’s maximize repayment for a given level of verification costs.

Before beginning the proof, it is essential that we understand the role of verification. A correct understanding will allow us to identify and eliminate unnecessary verification costs. Verification in the low states of the world is what allows the entrepreneur to make higher payments in the high states of the world, without violating incentive compatible. If we can identify verification which does not help to increase payment in high states of the world, then we can cut costs. Assume for the moment that we have already determined which states will be verified, that is the function $s$ is given; we should ask ourselves - how much can the entrepreneur pay in the unverified states of the world? The upper bound of $c_g$ (the payment in unverified states) comes from the non-negativity constraint $c_g \leq \inf (s^{-1}(0))$.

Any verification in the states of the world $[\inf (s^{-1}(0)), \beta]$ does not help the entrepreneur to make higher payments, and as such is wasteful. The inequality also shows us that in order to allow the entrepreneur to make the payment of $c_g$ in the high states of the world we only need to verify the states $[\alpha, c_g]$.

Now to the proof!

7.1. **Step 1: Choice of $x_0$ and First Alteration.** Let $(g, s)$ be any feasible contract. I will assume that $s$ is not identically one (if it is then $x_0 = \beta$ and no alteration is needed). Following our understanding from above, I define $x_0 = \inf (s^{-1}(0))$.

We now alter the contract. Define $(g_1, s_1)$ as follows:

$$s_1(x) = \begin{cases} 1 & x < x_0 \\ 0 & x \geq x_0 \end{cases}$$

$$g_1(x) = \begin{cases} g(x) & x < x_0 \\ c_g & x \geq x_0 \end{cases}$$

**Claim 14.** $(g_1, s_1)$ is feasible.

**Proof.**
Non-Negativity: Let \( x \in [\alpha, \beta] \). If \( x < x_0 \) then \( g_1(x) + \xi(x) = g(x) + \xi(x) \leq x \) where the last inequality holds by non-negativity of \((g, s)\). By non-negativity of \((g, s)\) we know that for any \( x \in s^{-1}(0) \), \( c_g \leq x \) and so \( c_g \leq \inf \{s^{-1}(0)\} = x_0 \).

Incentive Compatibility: Let \( y < x_0 \). \( g(y) + \xi(y) = g(y) + \xi(y) \leq c_g = c_{g_2} \) where the inequality holds by incentive compatibility of \((g, s)\). By Lemma \(6\) this implies incentive compatibility.

Finances the Project: Let \( A := \{x | s_1(x) \neq s(x)\} \) the set of states of the world which were verified under the old contract but not under the new contract.

\[
U_F(g_1, s_1) = U_F(g, s) + \int_A c_g - (g(x) + \xi(x)) dFx
\]

(7.1)

\[
\geq U_F(g, s) + \int_A 0 dFx
\]

(7.2)

where (7.1) holds by incentive compatibility of \((g, s)\) and (7.2) holds by feasibility of \((g, s)\).

Claim 15. If \((g, s) \neq (g_1, s_1)\) then \( \int_{s_1^{-1}(1)} \xi(x) dFx < \int_{s^{-1}(1)} \xi(x) dFx \). Clarification - the equality of functions which I am referring to is that of functions in \(L^1(dFx)\) i.e. equality up to a set of measure zero.

Proof. Let \( A := \{x | s_1(x) \neq s(x)\} \). \((g, s) \neq (g_1, s_1)\) iff the set \( A \) is of positive measure. \( \xi \) is a positive function and so the set \( A \) is of positive measure iff \( \int_A \xi(x) dFx > 0 \). We observe that the last integral is the savings in verification offered by the new contract.

7.2. Step 2: Second Alteration. Let \((g, s)\) be any feasible contract s.t. \( s^{-1}(0) = [x_0, \beta] \) for some \( x_0 \). I will show that if \( c_g < x_0 \) we can alter the contract and save on the cost of verification. Indeed assume \( c_g < x_0 \). Let \( x_1 = \text{Max} \{\alpha, c_g\} \).

Define \((g_1, s_1)\) as follows

\[
s_1(x) = \begin{cases} 1 & x < x_1 \\ 0 & x \geq x_1 \end{cases}
\]

\[
g_1(x) = \begin{cases} g(x) & x < x_1 \\ x_1 & x \geq x_1 \end{cases}
\]

Claim 16. \((g_1, s_1)\) is feasible.

Proof.

Non-Negativity: Let \( x \in [\alpha, \beta] \). If \( x < x_1 \) then \( g_1(x) + \xi(x) = g(x) + \xi(x) \leq x \) where the inequality holds by non-negativity of \((g, s)\). For \( x \geq x_1 \) we have \( g_1(x) = c_{g_1} = x_1 \leq x \).

\(^5\)I am using the following result: let \( \mu \) be a positive measure on \([\alpha, \beta]\), let \( A \subset [\alpha, \beta] \) be a measurable set let \( h \) be a positive measurable function. \( \mu(A) > 0 \) iff \( \int_A h d\mu > 0 \).

\(^6\)In fact, we know that \( c_g > \alpha \). By incentive compatibility we know that \( \forall y \in s^{-1}(1) \ g(y) + \xi(y) \leq c_g \). \( c_g > 0 \) and so \( \forall y \in s^{-1}(1) \ g(y) \leq c_g \) and so \( \forall x \in [\alpha, \beta] \ g(x) \leq c_g \). If \( c_g \leq \alpha \) then Assumption \(3\) would contradict feasibility.
Incentive Compatibility: Let \( y < x \). \( g_1(y) + \xi(y) = g(y) + \xi(y) \leq c_g = c_{g_1} \) where the inequality holds by incentive compatibility of \((g, s)\). By Lemma 4 this implies incentive compatibility.

Finances the Project: Let \( A := \{x | s_1(x) \neq s(x)\} = [x_1, x_0] \) the set of states of the world which were verified under the old contract but not under the new contract.

\[
U_F(g_1, s_1) = U_F(g, s) + \int_A c_g - (g(x) + \xi(x)) \, dFx \\
\geq U_F(g, s) + \int_A 0 \, dFx \\
\geq l
\]

where (7.3) holds by incentive compatibility of \((g, s)\) and (7.4) holds by feasibility of \((g, s)\).

Claim 17. If \((g, s) \neq (g_1, s_1)\) then \( \int_{s_1^{-1}(1)} \xi(x) \, dFx < \int_{s^{-1}(1)} \xi(x) \, dFx \).

7.3. Step 3: Third and Final Alteration. The philosophy here is different than that of the first two alterations. To repeat myself, we are willing to pay for verification because it allows us to transfer funds from the entrepreneur to the firm. If in some states of the world we are not transferring as much as possible (subject to non-negativity and incentive compatibility) then we can transfer additional funds from the entrepreneur to the firm free of charge. Once we transfer the additional funds, free of charge, we know that the entrepreneur has transferred more than the minimal required amount \( l \). If this is the case then we have spent too much on verification; we can reduce the verification slightly, transfer a little less to the firm and still finance the project.

Let \((g, s)\) be any feasible contract s.t. \( s^{-1}(0) = [x_0, \beta] \) for some \( x_0 \) and such that \( c_g = x_0 \).

Define \((g_1, s_1)\) as follows

\[
s_1 = s \\
g_1(x) = \begin{cases} x - \xi(x) & x < x_0 \\ x_0 & x \geq x_0 \end{cases}
\]

Claim 18. \((g_1, s_1)\) is feasible.

Proof.

Non-negativity: Follows from definition.

Incentive Compatibility: Let \( y < x_0 \). \( g_1(y) + \xi(y) = y < x_0 = c_{g_1} \). By Lemma 4 this implies incentive compatibility.
Finances the Project: Let $A := \{y \mid g_1(y) \neq g(y)\}$ the set of states of the world for which the old contract did not offer maximal possible repayment.

$$U_F(g_1, s_1) = U_F(g, s) + \int_A (y - \xi(y)) - g(y) \, dFy$$

(7.5)

$$\geq U_F(g, s) + \int_A 0 \, dFx$$

(7.6)

where (7.5) holds by non-negativity of $(g, s)$ and (7.6) holds by feasibility of $(g, s)$.

Claim 19. If $(g, s) \neq (g_1, s_1)$ then $U_F(g_1, s_1) > l$.

Proof. Assume $(g, s) \neq (g_1, s_1)$ i.e. $A := \{y \mid g_1(y) \neq g(y)\}$ is of positive measure. By definition $A$ is the set of states of the world in which $(g, s)$ did not offer maximal payment and so on the set $A$ we know that $g_1 > g$. As in the previous cases

$$U_F(g_1, s_1) = U_F(g, s) + \int_A g_1(y) - g(y) \, dFy$$

(7.7)

$$\geq l + \int_A g_1(y) - g(y) \, dFy$$

(7.8)

where (7.5) holds by feasibility of $(g, s)$ and (7.6) holds by $\int_A g_1(x) - g(x) \, dFx > 0$.

The function which will appear in the proof of Claim 22 will play a prominent role in what remains of this note. As such, I would like to take the time to define it and state its most basic and important property.

Definition 20. Let $H : [\alpha, \beta] \rightarrow \mathbb{R}$ be define as follows:

$$H(x_0) : = U_F(SDC_{x_0})$$

$$= \int_{x_0}^{x_0} x - \xi(x) \, dFx + \int_{x_0}^{\beta} x_0 \, dFx$$

$$= \int_{\alpha}^{\alpha} x - \xi(x) \, dFx + x_0 (1 - F(x_0))$$

Lemma 21. $H$ is continuous. Moreover, if $\xi$ is continuous then $H$ is differentiable.

Claim 22. If $(g, s) \neq (g_1, s_1)$ then $(g, s)$ is not optimal.

Proof. Assume $(g, s) \neq (g_1, s_1)$. Let $x_0 = \min_{s^{-1}(0)}$. By Claim 19 $H(x_0) > l$. By Lemma 21 there exists $x_1 < x_0$ s.t. $H(x_1) > l$. The verification cost of $SDC(x_1)$ is strictly lower than that of $(g, s)$, hence $(g, s)$ is not optimal. \[\square\]

\[\footnote{This uses the fact that $H(\alpha) < l$ therefore we can find such an $x_1 \in (\alpha, \beta)$.}\]
This concludes the proof of the main theorem.

8. THE OPTIMAL CONTRACT

We now set out to answer the questions we stated in Section 4: can we finance the project? Can we do so optimally, and if so, then how? Using Theorem 12, we have reduced the original problem to an easier one. The project can be financed if there exists \( x_0 \in [\alpha, \beta] \) s.t. \( H(x_0) \geq l \).

**Theorem 23.** If the project can be financed then there is a unique solution to the optimization problem

\[
\text{Argmax } \max_{(g,s) \text{ feasible}} U_E(g,s)
\]

By Corollary 9 and Theorem 12, the solution must be of the form \( SDC(x_0) \) for some constant \( x_0 \in [\alpha, \beta] \) which is characterized by the following:

1. \( x_0 \) is feasible.
2. \( \forall x \neq y \in [\alpha, \beta] \) if \( y \) is feasible then \( U_E(SDC(x_0)) > U_E(SDC(y)) \).

8.1. **Proof of Theorem 23.** I will call a constant \( x \in [\alpha, \beta] \) feasible if \( SDC(x) \) is feasible.

Assume that the project can be financed. For two arbitrary feasible constants \( y, z \in [\alpha, \beta] \) it is difficult to compare \( U_1(SDC(y)) \) to \( U_1(SDC(z)) \). There are, however, some cases which allow for easy comparison. To see this we write the entrepreneur’s utility as

\[
U_E(SDC(y)) = E[X] - H(y) - \int_{s^{-1}(1)}^{\beta} \xi(x) dFx
\]

\[
= E[X] - H(y) - \int_y^{\beta} \xi(x) dFx
\]

Written in this form, we see that \( U_E(SDC(y)) > U_E(SDC(z)) \) if \( H(y) + \int_y^{\beta} \xi(x) dFx < H(z) + \int_y^{\beta} \xi(x) dFx \). Verification cost are increasing, that is, \( \int_y^{\beta} \xi(x) dFx < \int_y^{\beta} \xi(x) dFx \) iff \( y < z \); therefore, if we have feasible constants \( y, z \) s.t. \( y < z \) and \( H(y) \leq H(z) \) then \( U_E(SDC(y)) > U_E(SDC(z)) \). To finish the proof I will show that there exists a feasible constant \( x_0 \in [\alpha, \beta] \) s.t. for every feasible constant \( y \in [\alpha, \beta] \)

1. \( x_0 < y \)
2. \( H(x_0) \leq H(y) \)

I would like to define \( x_0 = \inf(H^{-1}(l)) \) but to be able to do so I need to first show that the set \( H^{-1}(l) \) is non-empty. The project can be financed, therefore, by Theorem 12, there exists \( y \in [\alpha, \beta] \) s.t. \( H(y) \geq l \). By Assumption 3, \( H(\alpha) < l \). By Lemma 21, \( H \) is continuous functions. Applying the Intermediate Value Theorem, there exists some \( z \in [\alpha, y] \) s.t. \( H(z) = l \). Having shown that \( H^{-1}(l) \) in non-empty, I define \( x_0 = \inf(H^{-1}(l)) \). By continuity, \( H(x_0) = l \). To finish I need to show that for every \( y \), if \( y \) is feasible then \( x \leq y \). Let \( x \neq y \in [\alpha, \beta] \) be feasible.

\[\text{The following are conditions for optimality amongst feasible SDC’s.}\]
$H(\alpha) < l \leq H(y)$, therefore, there exists $z \in [\alpha, y]$ s.t. $H(z) = l$. By definition $x_0 \leq z$ and so $x_0 \leq y$ as required.

9. Motivating Story - Calculations

I now return to the motivating story with which I opened this note to check whether I can finance my education and if so how can I do so optimally.

The example uses the following specifications:

- $X \sim U[8, 35]$. For convenience, I have assumed that $X$ is a continuous random variable. For $x \in [8, 35]$ we have $F(x) = \frac{x - 8}{27}$, $f(x) = \frac{1}{27}$.
- $l = 20$.
- $\xi(x) = 4$.

For these specifications, we can rewrite the function $H$ (see Definition 20) as follows:

$$H(y) := U_F(SDC_y) = \int_{8}^{y} x - \xi(x) \, dF(x) + y (1 - F(y))$$

$$= \int_{8}^{y} (x - 4) \frac{dx}{27} + y \left(1 - \frac{y - 8}{27}\right)$$

$$= \frac{1}{27} \left[\frac{x^2}{2} - 4x\right]_{8}^{y} + y \left(\frac{35 - y}{27}\right)$$

$$= \frac{1}{54} \left[-y^2 + 62y\right]$$

The function $H$ has a unique maximum at the point $y = 31$. $H(31) = \frac{961}{54} \approx 17.8$. As a consequence the highest loan I can repay is $\frac{961}{54}$. In order to finance my education I would need to repay a loan of $20 > \frac{961}{54}$. If I were to enroll in a different program, one for which I would only need to borrow 15, I would be able to take out the necessary loan. In this case the optimal contract is an SDC and can be calculated as follows: first we find the solutions to the equation $H(y) = 15$, then we take the smallest such solution. In our case the solutions are the roots of the equation $y^2 - 62y + 790$. This equation has roots $31 \pm 3\sqrt{19}$. The smallest root (which is the only one in the interval $[8, 35]$) is $31 - 3\sqrt{19} \approx 17.92$. In this case the financial institution and I will agree on a debt contract with face value $\approx 17.92$. This contract has an interest rate of $\frac{17.92 - 1}{15} \approx 19.5$. I will default on the contract in all states of the world which are lower than 17.92; this will occur with probability $\frac{17.92 - 8}{35 - 8} \approx 0.37$.

Appendix A. Equivalent Setup

I would like to show that a setup in which the financial institution faces the cost of verification is equivalent to the setup above. I will refer to the setup in which the entrepreneur faces the costs of verification as setup 1, and the setup in which the financial institution faces the costs of verification as setup 2.

I will now re-define contracts as well as non-negativity and incentive-compatibility in each setup.

A.0.1. Contract.
Setup 1: A contract is a pair of $dF_x$-measurable functions $(g, s)$

- $g : [\alpha, \beta] \to \mathbb{R}$ describes the amount the entrepreneur agrees to pay the firm in each state of the world.
- $s : [\alpha, \beta] \to \{0, 1\}$, $s^{-1}(1)$ are those states in which there is verification.

Setup 2: A contract is a pair of $dF_x$-measurable functions $(h, s)$

- $h : [\alpha, \beta] \to \mathbb{R}$ describes the amount the entrepreneur agrees to pay the firm in each state of the world.
- $s : [\alpha, \beta] \to \{0, 1\}$, $s^{-1}(1)$ are those states in which there is verification.

A.0.2. Incentive-compatibility.

Setup 1: A contract $(g, s)$ is incentive compatible iff

\[ \forall x \in [\alpha, \beta] \forall y \in s^{-1}(0) \quad x - g(x) - s(x) \xi(x) \geq x - g(y) \]

Setup 2: A contract $(h, s)$ is incentive compatible iff

\[ \forall x \in [\alpha, \beta] \forall y \in s^{-1}(0) \quad x - h(x) \geq x - h(y) \]

A.0.3. Non-Negativity.

Setup 1: A contract $(g, s)$ is non-negative iff

\[ \forall x \quad x - g(x) - s(x) \cdot \xi(x) \geq 0 \]

Setup 2: A contract $(h, s)$ is non-negative iff

\[ \forall x \quad x - h(x) \geq 0 \]

To summarize:

<table>
<thead>
<tr>
<th>Contract</th>
<th>I-C</th>
<th>N-N</th>
</tr>
</thead>
<tbody>
<tr>
<td>Setup 1 $(g, s)$</td>
<td>$g \equiv c_g$ on $s^{-1}(0)$ \quad $\forall x \in s^{-1}(1) \quad g(x) + \xi(x) \leq c_g$</td>
<td>$\forall x : x - g(x) - s(x) \xi(x) \geq 0$</td>
</tr>
<tr>
<td>Setup 2 $(h, s)$</td>
<td>$h \equiv c_h$ on $s^{-1}(0)$ \quad $\forall x \in s^{-1}(1) \quad h(x) \leq c_g$</td>
<td>$\forall x : x - h(x) \geq 0$</td>
</tr>
</tbody>
</table>

A.1. Equivalence. To show that the two setups are equivalent I will construct a map, $\Phi$, from contracts in Setup 1 to contracts in Setup 2, and an inverse map, $\Psi$, from contracts in Setup 2 to contracts in Setup 1. The definition is as follows

\[ \Phi(g, s) = (g + s \cdot \xi, s) \]
\[ \Psi(h, s) = (h - s \cdot \xi, s) \]

Claim.

1. $\Psi \circ \Phi(g, s) = (g, s)$.
2. $\Phi \circ \Psi(h, s) = (h, s)$.
3. $(g, s)$ is incentive compatible in Setup 1 iff $\Phi(g, s)$ is incentive compatible in Setup 2.
4. $(h, s)$ is incentive compatible in Setup 2 iff $\Psi(h, s)$ is incentive compatible in Setup 1.
5. $(g, s)$ is non-negative in Setup 1 iff $\Phi(g, s)$ is non-negative in Setup 2.
(6) $(h, s)$ is non-negative in Setup 2 iff $\Psi(h, s)$ is non-negative in Setup 1.

(7) $(g, s)$ is outcome equivalent to $\Phi(g, s)$, i.e., for every state of the world $x$ the firm and entrepreneur receive the same utility from $(g, s)$ in Setup 1 as they do from $\Phi(g, s)$ in Setup 2.

(8) $(h, s)$ is outcome equivalent to $\Psi(h, s)$. 