Consider the problem stated at the beginning of section 4.5, which involves choosing the distribution of posteriors $F$ that maximizes

$$(1 - \alpha) \int_0^1 w(\pi) \frac{1-\pi}{1-\pi_0} dF(\pi) + \alpha \left( \int_0^1 w(\pi) \frac{\pi}{\pi_0} dF(\pi) - c \left( \int_0^1 w(\pi) \left( \frac{\pi}{\pi_0} - \frac{1-\pi}{1-\pi_0} \right) dF(\pi) \right) \right)$$

subject to the constraints that $F$ is (i) Bayes-consistent, i.e.,

$$\int_0^1 \pi dF(\pi) = \pi_0,$$

and (ii) implementable, i.e.,

$$\int_0^1 w(\pi) \left( \frac{\pi}{\pi_0} - \frac{1-\pi}{1-\pi_0} \right) dF(\pi) \leq C(\bar{m}|L).$$

To make further progress without imposing additional assumptions on cost functions, I decompose the problem into two steps. In the first step, I solve the problem for a given value of the gap between the expected surplus claimed by $L$ and $H$, $x = \int_0^1 w(\pi) \left( \frac{\pi}{\pi_0} - \frac{1-\pi}{1-\pi_0} \right) dF(\pi) \in [0, w(1) - w(0)]$, henceforth referred to as the surplus gap, and derive the optimal distribution of posteriors conditional on $x$, denoted $F_x^*$. Note that the size of the surplus gap $x$ is a measure of information transmission and thus the Lagrange multiplier associated with the constraint for $x$ can be interpreted as the shadow value of information transmission. In the second step, I use $F_x^*$ to determine the optimal surplus gap $x^* \in [0, \min\{w(1) - w(0), C(\bar{m}|L)\}]$ that maximizes social welfare and infer the optimal distribution of posteriors $F^* = F_{x^*}^*$. 
Step 1: Optimal Communication Error For a Given Surplus Gap

A distribution of posteriors optimally induces the surplus gap $x \in (0, w(1) - w(0))$ if and only if it solves the following problem:

$$\max_{F \in \Delta([0,1])} \int_0^1 w(\pi) \left( (1 - \alpha) \frac{1 - \pi}{1 - \pi_0} + \alpha \frac{\pi}{\pi_0} \right) dF(\pi)$$

subject to

(i) $$\int_0^1 w(\pi) \left( \frac{\pi}{\pi_0} - \frac{1 - \pi}{1 - \pi_0} \right) dF(\pi) = x$$

(ii) $$\int_0^1 \pi dF(\pi) = \pi_0.$$ 

Letting $\lambda$ denote the Lagrange multiplier for the first constraint, we can rewrite this problem as

$$\max_{F \in \Delta([0,1])} \int_0^1 w(\pi) \left( (1 - \alpha) \frac{1 - \pi}{1 - \pi_0} + \alpha \frac{\pi}{\pi_0} \right) dF(\pi) + \lambda \left( x - \int_0^1 w(\pi) \left( \frac{\pi}{\pi_0} - \frac{1 - \pi}{1 - \pi_0} \right) dF(\pi) \right)$$

subject to

$$\int_0^1 \pi dF(\pi) = \pi_0.$$ 

or, equivalently,

$$\max_{F \in \Delta([0,1])} \int_0^1 w(\pi) \left( 1 - \frac{(1 - \psi) \pi}{1 - (1 - \psi) \pi_0} \right) dF(\pi) + \lambda x$$

subject to

$$\int_0^1 \pi dF(\pi) = \pi_0,$$

where

$$\psi = \frac{\alpha - \lambda}{1 - (\alpha - \lambda)} \left/ \frac{\pi_0}{1 - \pi_0} \right..$$

Note that conditional on $\psi$, the solution to this problem is the same as in the case of proportional cost functions discussed in section 4.5 if the social MRS equals $\psi$. In what follows, I refer to a solution to problem 1 as $\psi$-optimal.

Our next goal is to describe how the set of $\psi$-optimal distributions depends on $\psi$, $w$, and $\pi_0$. In particular, we need to know when a $\psi$-optimal distribution can be uninformative, perfectly informative or partially informative, respectively. To that end, define the following sets:

$$\Pi_U(\psi, w) = \{ \pi \in [0,1] | \nu(\pi | \psi, w) = \hat{\nu}(\pi | \psi, w) \}$$

$$\Pi_I(\psi, w) = \{ \pi \in [0,1] | \hat{\nu}(\pi | \psi, w) \text{ is linear in a neighborhood around } \pi \}$$

Note that $\Pi_U(\psi, w)$ is a closed set and that $\{0,1\} \subseteq \Pi_U(\psi, w)$. $\Pi_I(\psi, w)$ is empty if and only if $\nu(\pi | \psi, w)$ is globally strictly concave; otherwise, $\Pi_I(\psi, w)$ is a union of open intervals.
The results above imply that an uninformative distribution is \( \psi \)-optimal if and only if \( \pi_0 \in \Pi_U(\psi, w) \) and it is uniquely \( \psi \)-optimal if and only if \( \pi \in \Pi_U(\psi, w) \backslash \Pi_I(\psi, w) \). An informative distribution, on the other hand, is \( \psi \)-optimal if and only if \( \pi_0 \in \Pi_I(\psi, w) \). To describe the \( \psi \)-optimal distribution in this case, let \( t(\pi|\pi_0) \) denote the line that is tangent to \( \hat{\nu}(\pi|\psi) \) at \( \pi_0 \in \Pi_I(\psi, w) \) and let

\[
\Pi^*(\psi|\pi_0, w) = \begin{cases} 
\{ \pi \in [0, 1] | \nu(\pi|\psi, w) = t(\pi|\pi_0) \} & \text{if } \pi_0 \in \Pi_I(\psi, w), \\
\{ \pi_0 \} & \text{if } \pi_0 \notin \Pi_I(\psi, w). 
\end{cases}
\]

Then, given a surplus function \( w \) and a prior \( \pi_0 \in (0, 1) \), a Bayes-consistent \( F \) is \( \psi \)-optimal if and only if \( \text{Supp}(F) \subseteq \Pi^*(\psi|\pi_0, w) \). Finally, define

\[
\Pi^*_1(\psi|\pi_0, w) = \Pi^*(\psi|\pi_0, w) \cap [0, \pi_0]
\]

and

\[
\Pi^*_2(\psi|\pi_0, w) = \Pi^*(\psi|\pi_0, w) \cap [\pi_0, 1],
\]

and for \( i = 1, 2 \), let

\[
\pi^*_i(\psi|\pi_0, w) \equiv \min\{ \Pi^*_i(\psi|\pi_0, w) \}
\]

\[
\bar{\pi}^*_i(\psi|\pi_0, w) \equiv \max\{ \Pi^*_i(\psi|\pi_0, w) \}.
\]

If \( \pi_0 \in \Pi_I(\psi, w) \backslash \Pi_U(\psi, w) \), then the least informative \( \psi \)-optimal distribution is determined by the binary support \( \{ \pi^*_1(\psi|\pi_0, w), \pi^*_2(\psi|\pi_0, w) \} \) and the most informative \( \psi \)-optimal distribution is determined by the binary support \( \{ \bar{\pi}^*_1(\psi|\pi_0, w), \bar{\pi}^*_2(\psi|\pi_0, w) \} \).

The above allows us to describe the set of \( \psi \)-optimal distributions of posteriors. Now, we are interested in how \( \Pi^*(\psi|\pi_0, w) \) depends on \( \psi \). Obviously, \( \Pi^*(\psi|\pi_0, w) \) depends crucially on the shape of \( \nu \), which is determined by \( w \) and \( \psi \). In particular, if \( \nu \) is locally strictly convex at \( \pi_0 \), then an uninformative distribution cannot be \( \psi \)-optimal. Likewise, if an uninformative distribution is not \( \psi \)-optimal, then \( \nu \) must be strictly smaller than \( \hat{\nu} \) at \( \pi_0 \), which in turn implies that \( \nu \) must be locally strictly convex for some \( \pi \) in an open interval around \( \pi_0 \). Lemma 1 states the relationship between the curvature of \( \nu \) on the one hand and the technological value of information and \( \psi \) on the other hand.

---

1The tangent exists since by construction, \( \hat{\nu}(\pi|\psi, w) \) is linear in a neighborhood around \( \pi_0 \) for any \( \pi_0 \in \Pi_I(\psi, w) \).
Lemma 1. \( \nu \) is locally concave (convex) at \( \pi \) if and only if there exists an open neighborhood \( \mathcal{N}(\pi) \) around \( \pi \) such that for all \( x \in \mathcal{N}(\pi) \), we have

\[
TVI(x|w) \left( 1 - (1 - \psi)x \right) \leq (\geq) 2(1 - \psi).
\]

Moreover, if

\[
TVI(\pi|w) (1 - (1 - \psi)\pi) < (>) 2(1 - \psi),
\]

then \( \nu \) is locally strictly concave (strictly convex) at \( \pi \in (0,1) \).

Proof. Consider an arbitrary belief \( \pi \in (0,1) \). \( \nu \) is locally concave if and only if for some open neighborhood \( \mathcal{N}(\pi) \) around \( \pi \in (0,1) \), we have \( \nu'(x)(y-x) \geq \nu(y) - \nu(x) \) for all \( x, y \in \mathcal{N}(\pi) \). Such a neighborhood exists if and only if \( \nu' \) is nonincreasing on some open neighborhood \( \mathcal{N}(\pi) \) around \( \pi \in (0,1) \), or equivalently,

\[
w''(x) (1 - (1 - \psi)x) \leq 2(1 - \psi)w'(x)
\]

for all \( x \in \mathcal{N}(\pi) \), which is equivalent to

\[
TVI(x|w) \left( 1 - (1 - \psi)x \right) \leq 2(1 - \psi).
\]

Moreover, \( \nu \) is locally strictly concave at \( \pi \) if

\[
\nu''(\pi) < 0,
\]

or, equivalently,

\[
TVI(\pi|w) (1 - (1 - \psi)\pi) < 2(1 - \psi).
\]

The proof for local convexity and strict convexity is analogous. \( \square \)

Corollary 1. If \( \psi \leq 1 - \frac{1}{\pi} \) for some \( \pi \in (0,1) \), then \( \nu \) is locally strictly concave on \( (\pi', 1) \) for some \( \pi' < \pi \). If \( \psi = 0 \), \( \nu \) is globally strictly concave if and only if \( TVI(\pi|w) < \frac{2}{1-\pi} \) almost everywhere on \( (0,1) \).

Proposition 1 uses the above results to describe how the set of \( \psi \)-optimal distributions of posteriors depends on the social MRS \( \psi \). In particular, it derives a lower and an upper bound for \( \psi \) such that a uninformative distribution is \( \psi \)-optimal if and only if \( \psi \) is smaller than the lower bound and a perfectly informative distribution is \( \psi \)-optimal if and only if \( \psi \) is larger than the upper bound. If \( \psi \)
lies strictly between these two bounds, then only a partially informative distribution is \( \psi \)-optimal, and the informativeness of the \( \psi \)-optimal distribution is strictly increasing in \( \psi \).

**Proposition 1.** A perfectly informative distribution of posteriors is \( \psi \)-optimal if and only if

\[
\psi \geq \sup_{\pi \in (0,1)} \left\{ \frac{w(\pi) - w(0)}{\pi} \left/ \frac{w(1) - w(\pi)}{1 - \pi} \right. \right\} \equiv \bar{\psi}(w) \in (0,1).
\]

An uninformative distribution of posteriors is \( \psi \)-optimal if and only if

\[
\psi \leq \inf_{\pi \in [0,1] \setminus \pi_0} \left\{ \frac{1 - \pi_0 (w(\pi) - w(\pi_0))}{\pi_0 (w(\pi) - w(\pi_0))} \left/ \frac{1 - \pi}{\pi} - w'(\pi_0)(\pi - \pi_0) \right. \right\} \equiv \psi(w, \pi_0) \in \left( 1 - \frac{1}{\pi_0}, \bar{\psi}(w) \right],
\]

If \( \psi \in (\psi(w, \pi_0), \bar{\psi}(w)) \), then a Bayes-consistent and partially informative distribution of posteriors with support \( \text{Supp}(F) \subseteq \Pi^*(\psi|\pi_0, w) \) is \( \psi \)-optimal.

The correspondence \( \Pi^*_1(\psi|\pi_0, w) \) is upper hemicontinuous and decreasing in \( \psi \), while the correspondence \( \Pi^*_2(\psi|\pi_0, w) \) is upper hemicontinuous and increasing in \( \psi \). Moreover, \( \Pi^*_1(\psi|\pi_0, w) \) is strictly decreasing in the sense that for all \( \psi' > \psi \), \( \Pi^*_1(\psi|\pi_0, w) > 0 \) implies \( \Pi^*_1(\psi'|\pi_0, w) < \Pi^*_1(\psi|\pi_0, w) \) and \( \Pi^*_1(\psi|\pi_0, w) = 0 \) implies \( \Pi^*_1(\psi'|\pi_0, w) = \{0\} \). Also, \( \Pi^*_2(\psi|\pi_0, w) \) is strictly increasing in the sense that for all \( \psi' > \psi \), \( \Pi^*_2(\psi|\pi_0, w) < 1 \) implies \( \Pi^*_2(\psi'|\pi_0, w) > \Pi^*_2(\psi|\pi_0, w) \) and \( \Pi^*_2(\psi|\pi_0, w) = 1 \) implies \( \Pi^*_2(\psi'|\pi_0, w) = \{1\} \).

**Proof.** First, recall that there always exists a \( \psi \)-optimal distribution of posteriors with at most binary support. Now consider an arbitrary Bayes-consistent support \( \{\pi_1, \pi_2\} \in [0, \pi_0) \times (\pi_0, 1] \) and recall that if the social MRS is \( \psi \), the social value associated with that support is given by

\[
l(\pi_0|\pi_1, \pi_2, \psi) = \nu(\pi_1|\psi, w) + (\pi_0 - \pi_1) \frac{\nu(\pi_2|\psi, w) - \nu(\pi_1|\psi, w)}{\pi_2 - \pi_1}.
\]

The binary support \( \{\pi_1, \pi_2\} \) is \( \psi \)-optimal if and only if it maximizes the social value \( l(\pi_0|\pi_1, \pi_2, \psi) \) and thus if and only if \( l(\pi|\pi_1, \pi_2, \psi) \geq \nu(\pi|\psi, w) \) for all \( \pi \in [0, 1] \). In particular, this implies that the perfectly informative support \( \{0, 1\} \) is \( \psi \)-optimal if and only if \( l(\pi|0, 1, \psi) \geq \nu(\pi|\psi, w) \) for all \( \pi \in (0, 1) \). For a given value of \( \pi \), this is true if and only if

\[
l(\pi|0, 1, \psi) = \nu(0|\psi, w) + \pi \left( \nu(1|\psi, w) - \nu(0|\psi, w) \right) \geq \nu(\pi|\psi, w)
\]

or equivalently,

\[
\psi \geq \frac{w(\pi) - w(0)}{\pi} \left/ \frac{w(1) - w(\pi)}{1 - \pi} \right.
\]
Thus, a perfectly informative signal solves problem (1) if and only
\[
\psi \geq \sup_{\pi \in (0,1)} \left\{ \frac{w(\pi) - w(0)}{\pi} / \frac{w(1) - w(\pi)}{1 - \pi} \right\} \equiv \tilde{\psi}(w).
\]
Note that \(\tilde{\psi}(w)\) is independent of the prior and \(\tilde{\psi}(w) \in (0,1)\) since \(w\) is strictly increasing, weakly convex and not linear. In contrast, an uninformative support \(\{\pi_0\}\) is \(\psi\)-optimal if and only if \(\nu(\pi_0|\psi, w) = \tilde{\nu}(\pi_0|\psi, w)\), or, equivalently, if and only if \(\nu\) is everywhere weakly smaller than the line that is tangent to \(\nu\) at \(\pi_0\), i.e., if and only if for all \(\pi \in [0,1]\), we have
\[
\nu(\pi|\psi, \pi_0) \equiv \nu(\pi_0|\psi, w) + \nu'(\pi_0|\psi, w)(\pi - \pi_0) \geq \nu(\pi|\psi, \pi_0),
\]
which is equivalent to
\[
\psi \leq -\frac{1 - \pi_0}{\pi_0} (w(\pi) - w(\pi_0)) \frac{\pi - \pi_0}{1 - \pi_0} - w'(\pi_0)(\pi - \pi_0) \equiv \psi(w, \pi_0|\pi) \quad \forall \pi \neq \pi_0.
\]
It follows that in this case, an uninformative distribution of posteriors is \(\psi\)-optimal if and only if
\[
\psi \leq \inf_{\pi \in [0,1]\backslash\pi_0} \left\{ -\frac{1 - \pi_0}{\pi_0} (w(\pi) - w(\pi_0)) \frac{\pi - \pi_0}{1 - \pi_0} - w'(\pi_0)(\pi - \pi_0) \right\} \equiv \psi(w, \pi_0) \in (1 - \frac{1}{\pi_0}, \tilde{\psi}(w)].
\]
Note that if \(\psi \leq 1 - \frac{1}{\pi_0}\), then by Corollary 1, \(\nu\) is strictly concave above \(\pi_0\). Note that in this case, a necessary condition for a partially informative distribution to be \(\psi\)-optimal is that for some \(\pi_1 \in [0, \pi_0]\), \(\nu(\pi_1|\psi, w)\) strictly exceeds the tangent to \(\nu\) to at \(\pi_0\), i.e.,
\[
\nu(\pi_1|\psi, w) > \nu(\pi_0|\psi, w) - (\pi_0 - \pi_1)\nu'(\pi_0|\psi, w).
\]
However, this requires
\[
(\pi_0 - \pi_1)w'(\pi_0)(1 - (1 - \psi)\pi_0) > (w(\pi_0) - w(\pi_1))(1 - (1 - \psi)\pi_1),
\]
which is a contradiction since \((\pi_0 - \pi_1)w'(\pi_0) \geq w(\pi_0) - w(\pi_1) > 0\) as well as \(1 - (1 - \psi)\pi_0 \leq 0\) and \((1 - (1 - \psi)\pi_0) < (1 - (1 - \psi)\pi_1).\) Thus, if \(\psi \leq 1 - \frac{1}{\pi_0}\), the \(\psi\)-optimal distribution must be uninformative.

Finally, consider an arbitrary \(\psi^* \in (\psi(w, \pi_0), \tilde{\psi}(w))\) and suppose \(\{\pi_1^*, \pi_2^*\}\) is a \(\psi\)-optimal support given \(\psi^*\), which implies \(1 \geq \pi_2^* > \pi_0 > \pi_1^* \geq 0\). Then, we must have
\[
\nu(\pi|\psi^*, w) \leq l(\pi|\pi_1^*, \pi_2^*, \psi^*) \quad \forall \pi \in [0,1].
\]
Now, consider $\psi^{**} > \psi^*$ and let $\{\pi_1^{**}, \pi_2^{**}\}$ denote a $\psi -$optimal binary support given $\psi^{**}$. Note that
\[
\frac{\partial \nu(\pi|\psi, w)}{\partial \psi} = \pi w(\pi)
\]
is strictly increasing in $\pi$. As a result, convexity of $w$ implies that for any $\pi_2 > \pi_1$, the change in the slope of $l(\pi|\pi_1, \pi_2, \psi)$ that results from an increase in $\psi$,
\[
\frac{\partial}{\partial \psi} \left( \frac{\nu(\pi_2|\psi, w) - \nu(\pi_1|\psi, w)}{\pi_2 - \pi_1} \right) = \frac{\pi_2 w(\pi_2) - \pi_1 w(\pi_1)}{\pi_2 - \pi_1} = w(\pi_1) + \pi_2 \frac{w(\pi_2) - w(\pi_1)}{\pi_2 - \pi_1},
\]
is strictly increasing in both $\pi_1$ and $\pi_2$. In particular, this implies that
\[
\nu(\pi|\psi^{**}, w) < l(\pi|\pi_1^{**}, \pi_2^{**}, \psi^{**}) \quad \forall \pi \in (\pi_1^{*}, \pi_2^{*}).
\]
Hence, $\pi_1^{**} \leq \pi_1^{*}$ as well as $\pi_2^{**} \geq \pi_2^{*}$, which implies that the correspondence $\Pi_1^*(\psi|\pi_0, w)$ is decreasing in $\psi$, while the correspondence $\Pi_2^*(\psi|\pi_0, w)$ is increasing in $\psi$.

Next, note that if $\pi_1^* > 0$ and $\nu(\hat{\pi}_1|\psi^{**}, w) > l(\pi|\pi_1^{*}, \pi_2^{*}, \psi^{**})$ for some $\hat{\pi}_1 < \pi_1^{*}$, then we must have $\pi_1^{**} < \pi_1^{*}$. To see that this is true, suppose, by way of contradiction, that $\pi_1^{**} = \pi_1^{*}$. Since $\{\pi_1^{*}, \pi_2^{*}\}$ is $\psi -$optimal, we must have $\nu(\pi_2^{**}|\psi^{**}, w) \geq l(\pi_0|\pi_1^{*}, \pi_2^{*}, \psi^{**})$. But then, $\nu(\hat{\pi}_1|\psi^{**}, w) > l(\pi|\pi_1^{*}, \pi_2^{*}, \psi^{**})$ implies
\[
l(\pi_0|\hat{\pi}_1, \pi_2^{**}, \psi^{**}) > l(\pi_0|\pi_1^{*}, \pi_2^{*}, \psi^{**}),
\]
a contradiction to $\pi_1^{**} = \pi_1^{*}$. By an analogous argument, if $\pi_2^{*} < 1$ and $\nu(\hat{\pi}_2|\psi^{**}, w) > l(\pi_0|\pi_1^{*}, \pi_2^{*}, \psi^{**})$ for some $\hat{\pi}_2 > \pi_2^{*}$, then we must have $\pi_2^{**} > \pi_2^{*}$.

Now suppose $\pi_1^{*} > 0$. Note that
\[
\frac{\partial \nu'(\pi_1^{*}|\psi, w)}{\partial \psi} = w(\pi_1^{*}) + \pi_1^{*} w'(\pi_1^{*}) < w(\pi_1^{*}) + \pi_2^{*} w(\pi_2^{*}) - w(\pi_1^{*}) = \frac{\partial}{\partial \psi} \left( \frac{\nu(\pi_2^{*}|\psi, w) - \nu(\pi_1^{*}|\psi, w)}{\pi_2^{*} - \pi_1^{*}} \right).
\]
Thus, for any $\psi^{**} > \psi^*$, there exists a $\hat{\pi}_1 \in (0, \pi_1^{*})$ such that $\nu(\hat{\pi}_1|\psi^{**}, w) > l(\pi_0|\pi_1^{*}, \pi_2^{*}, \psi^{**})$, which implies that $\pi_1^{**} < \pi_1^{*}$. An analogous argument holds for $\pi_2^{**}$ if $\pi_2^{*} < 1$. Finally, upper hemicontinuity of $\Pi_1^*(\psi|\pi_0, w)$ and $\Pi_2^*(\psi|\pi_0, w)$ follows straight from the fact that $\nu$ varies continuously in both $\pi$ and $\psi$.

\[\square\]

Conditions (2) and (3) have nice intuitive interpretations: $\tilde{\psi}(w|\pi)$ is the ratio of the surplus loss incurred by low types relative to the surplus gain incurred by high types that results from replacing a signal that is degenerate at $\pi$ by a perfectly informative signal with support $\{0, 1\}$. $\tilde{\psi}(w|\pi)$ can
thus be interpreted as the marginal rate of transformation (MRT) of surplus claimed by high types for surplus claimed by low types that is associated with making a signal that is degenerate at \( \pi \) perfectly informative. Stated differently, it is the price of a unit of the high type’s claim on surplus in terms of units of the low type’s claim on surplus associated with perfect information revelation.

Since \( \psi \) is the social MRS between these two quantities, i.e., the social planner’s willingness to pay for a unit of the high type’s claim on surplus in terms of units of the low type’s claim on surplus, condition (2) states that a perfectly informative signal is \( \psi \)-optimal if and only if the social MRS at least weakly exceeds the MRT associated with perfect information revelation for all interior beliefs.

It is worth noting that this MRT only depends on the curvature of \( w \), but not on the prior \( \pi_0 \).

Similarly, \(-\psi(w, \pi_0|\pi)\) is the ratio of the marginal change in the low type’s claim on surplus relative to the marginal change in the high type’s claim on surplus that results from replacing an uninformative signal with support \( \{\pi_0\} \) by a binary signal that assigns an infinitesimal probability to a posterior \( \pi \neq \pi_0 \). \(-\psi(w, \pi_0|\pi)\) can thus be interpreted as the MRT of the high type’s claim on surplus for the low type’s claim on surplus that is associated with a marginal improvement in information transmission through a binary signal that assigns an infinitesimal probability to \( \pi \neq \pi_0 \). Stated differently, it is the price of a unit of the high type’s claim on surplus in terms of units of the low type’s claim on surplus associated with marginal information revelation. Thus, condition (3) states that an uninformative signal is optimal if and only if the social MRS between these two quantities is smaller than the MRT associated with marginal information revelation for any posterior belief \( \pi \neq \pi_0 \).

To summarize, the results above allow us to identify the set of optimal distributions of posteriors for any given value of the social MRS \( \psi \in [\psi(w, \pi_0), \bar{\psi}(w)] \) as the set of Bayes-consistent distributions over \( \Pi^*(\psi|\pi_0, w) \). Moreover, for any element of that set, we can determine decision-related payoffs and the surplus gap \( x \), which is a measure of information transmission. Given Proposition 1, we know that as \( \psi \) increases from \( \psi(w, \pi_0) \) to \( \bar{\psi}(w) \), the optimal distribution becomes strictly more informative and thus the surplus gap strictly increases from 0 to \( w(1) - w(0) \). In particular, Proposition 1 implies that there exists a correspondence \( x_\psi \) from \( \psi \) to \( x \) that is upper hemicontinuous and strictly increasing on \( [\psi(w, \pi_0), \bar{\psi}(w)] \). Moreover, for any \( x \in [0, w(1) - w(0)] \), there exists a unique \( \psi \) such that \( x \in x_\psi(\psi) \). Hence, we can determine an increasing and continuous function \( \psi_x : [0, w(1) - w(0)] \rightarrow [\psi(w, \pi_0), \bar{\psi}(w)] \), \( x \mapsto \psi_x(x|w, \pi_0) \) such that for every \( x \in [0, w(1) - w(0)] \), there exists a Bayes-consistent distribution \( F^*_x \) over \( \Pi(\psi_x(x|w, \pi_0)|\pi_0, w) \) such that \( F^*_x \) is the optimal distribution of posteriors conditional on inducing the surplus gap \( x \).
Step 2: Constrained Pareto Efficient Garbling Mechanisms

We can now turn to the characterization of constrained Pareto efficient garbling mechanisms. To that end, recall that the solution to step 1 implies a positive relationship between the surplus gap $x$ and the social MRS $\psi$ that results from efficient implementation of $x$. Moreover, the shadow value of information transmission associated with $\psi$,

$$\lambda = \alpha - \frac{\pi_0 \psi}{1 - \pi_0 (1 - \psi)},$$

is strictly decreasing in $\psi$ since

$$\frac{\partial \lambda}{\partial \psi} = -\frac{(1 - \pi_0) \pi_0}{(1 - \pi_0 (1 - \psi))^2} < 0.$$

It follows that for any value of the Pareto-weight $\alpha \in [0, 1]$, the gross marginal benefit of improving information transmission in an optimal way as given by

$$\lambda(x|\alpha, w, \pi_0) = \alpha - \frac{\pi_0 \psi_x(x|w, \pi_0)}{1 - \pi_0 (1 - \psi_x(x|w, \pi_0))}$$

is decreasing in $x$. Hence, we can state the second step problem as

$$(4) \quad \max_x \int_0^x \lambda(t|\alpha, w, \pi_0) dt - \alpha c(x)$$

s.t. \quad $0 \leq x \leq \min\{C(\bar{m}|L), w(1) - w(0)\}.$

Therefore, we can determine an optimal level of the surplus gap, $x^*(\alpha|\pi_0, w, c)$, as well as the associated optimal distribution of posteriors, $F^* = F^*_{x^*}$, by comparing the marginal value of information transmission $\lambda(x|\alpha, w, \pi_0)$ to the marginal social cost $\alpha c'(x)$. Moreover, the entire set of constrained Pareto efficient distribution of posteriors can be determined by varying $\alpha$ from 0 to 1. Note that since $\frac{\partial \lambda}{\partial \alpha} = 1 > c'(x) \in (0, 1)$, an increase in $\alpha$ results in a larger optimal surplus gap $x^*$ and thus induces a more informative distribution of posteriors. Hence, the largest solution to problem (4) for $\alpha = 0$ and the smallest solution for $\alpha = 1$, respectively, determine a lower and an upper bound for surplus gaps that are induced by constrained Pareto efficient garbling mechanisms.\(^2\) If the former strictly exceeds 0 and the latter is strictly smaller than $w(1) - w(0)$, then any constrained Pareto efficient garbling mechanism implements a partially informative equilibrium and thus each equilibrium without miscommunication is strictly Pareto dominated with miscommunication.

\(^2\)Recall that if $\alpha = 0$, the social planner maximizes the low type’s payoff. If there are multiple solutions to this problem that imply different values of the surplus $x$, the high type’s payoff is uniquely maximized by the largest value of $x$ among this set. Hence, only the largest solution to problem (4) for $\alpha = 0$ is constrained Pareto efficient. Analogously, if $\alpha = 1$, the social planner maximizes the high type’s payoff, and the low type’s payoff is uniquely maximized by the smallest solution to problem (4) for $\alpha = 1$. 

In particular, suppose that $C(m|L) \geq w(1) - w(0)$, i.e., the best separating equilibrium exists. Note that if
\[ c'(w(1) - w(0)) > \lambda(w(1) - w(0)|\alpha = 1, w, \pi_0) \]
or, equivalently,
\[
\frac{(1 - c'(w(1) - w(0)))}{1 - (1 - c'(w(1) - w(0)))} \frac{\pi_0}{1 - \pi_0} < \bar{\psi}(w),
\]
then marginal costs exceed the marginal benefit for sufficiently large levels of information transmission, i.e., for $x$ sufficiently close to $w(1) - w(0)$. Thus, condition (5) is sufficient for the best separating equilibrium not to be constrained Pareto efficient. Moreover, since $\lambda$ is decreasing, this condition is generically necessary and sufficient if $c$ is weakly convex$^3$, i.e, if the high type’s communication costs increase at least proportionally to the low type’s communication costs. In this case, a distribution of posteriors $F_x^*$ is constrained Pareto efficient if and only if $x \in [\underline{x}(w, \pi_0), \bar{x}(w, \pi_0, c)]$, where $\underline{x}(w, \pi_0)$ is determined by the most informative Bayes-consistent distribution over $\Pi^*(0|\pi_0, w)$ and $\bar{x}(w, \pi_0, c)$ is determined by the smallest value of $x$ that solves
\[
\frac{1 - \pi_0}{1 - \pi_0(1 - \psi_x(x|w, \pi_0))} = c'(x).
\]

$^3$In the non-generic case that $\lambda(x|\alpha = 1, w, \pi_0)$ is constant on some open interval directly below $w(1) - w(0)$, $c'(w(1) - w(0)) = \lambda(w(1) - w(0)|\alpha = 1, w, \pi_0)$ is necessary and sufficient.