The Impact of Trade on Intra-Industry Reallocations and Aggregate Industry Productivity, Marc J. Melitz (2003)

Mainpoint
Melitz departs from Krugman by allowing firms to have heterogeneous productivity. The equilibrium is characterized as a stationary distribution of existing firm's productivity. To summarize the whole distribution, Melitz uses the measure of average/aggregate. Key results are: increase in size leads to increase in M, the number of varieties. Opening to trade is like increasing in country size, but M does not affect firm variables, only affect P, Q, R, and aggregate profit. The same number of firms in each country produce at the same output levels and earn the same profits as they did in the closed economy. We are interested to see what happen to both aggregate and firm level variables in a trade equilibrium compared to autarky.

Goals of this paper:

- Explain both firm level empirical facts (how productivity is related to export status, revenue, output, profits, price)

  -How does trade affect firm level profits compared to before trade?

  -How does trade affect firm level revenue, compared to before trade?

  -How does trade affect firm level output and price?

- Explain aggregate level empirical facts

  -How does aggregate price, revenue, output, and profits change?

Define the equilibrium:
An equilibrium is characterized by the aggregate variables $\{M, P, Q, R, \Pi\}$,the firm level variables $\{p(\widetilde{\rho}), r(\widetilde{\rho}), q(\widetilde{\rho}), \pi(\widetilde{\rho})\}$, the distribution of productivity level of surviving firms $\mu(\varphi)$.In the equilibrium we are looking for, $\mu(\varphi)$ is stationary, and all the aggregate variables are constant.

Aggregate
The aggregate variables are $\{M, P, Q, R, \Pi\}$. The last four variables are fixed exogenously by M.

Firm level
Since R is fixed regardless of trade or autarky, we can easily compare the market share of firms before and after trade. He shows the inequality that firm lose domestic sells but sales abroad more than make up the loss (see it from writing firm level revenue as function of cutoffs and compare)

Note: he defines "profits" as variable profits – total revenue minus total variable costs.

Preferences
CES, with degree of substitutability measure as $0 < \rho < 1$, $\rho = \frac{\sigma-1}{\sigma}$, where σ is elasticity of substitution

$$\begin{aligned} U &= Q = \left[\int_{\omega\in\Omega} q(\omega)^{\rho} d\omega\right]^{\frac{1}{\rho}} \\ \text{where } 0 &< \rho < 1 \text{ (degree of substitutability measure)} \\ \sigma &= \frac{1}{1-\rho} > 1 \text{ (elasticity of substitution)} \end{aligned}$$

The characteristic of CES preference is that consumers demand can be analysed as aggregate good Q associated with price index and revenue

$$P = \left[\int_{\omega\in\Omega} p(\omega)^{1-\sigma} d\omega\right]^{\frac{1}{1-\sigma}}$$

Demand for individual varieties are

$$q(\omega) = Q\left[\frac{p(\omega)}{P}\right]^{-\sigma}$$

Revenue for each goods are

$$\begin{aligned} r(\omega) &= p(\omega)q(\omega) \\ &= p(\omega)Q\left[\frac{p(\omega)}{P}\right]^{-\sigma} \\ &= p(\omega)^{1-\sigma}PQP^{\sigma-1} \\ &= R\left[\frac{p(\omega)}{P}\right]^{1-\sigma} \end{aligned}$$

Aggregate revenue is

$$R = \int_{\omega\in\Omega} r(\omega)d\omega$$

Production

- Single input: labor, inelastically supplied and homogenous
- Constant marginal cost and with fixed cost.
- labor demand:

$$l = f + \frac{q}{\rho}$$

- where ρ is the productivity parameter, a random variable which firms don't know when they make the fixed cost

Market Structure
Monopolistic Competition ⇒firms only compete across industries
CES preference plus monopolistic competition implies that:
Firms face isoelastic demand curves with elasticity of demand σ, therefore they charge a constant markup of $\frac{1}{\rho}$:

$$P = \frac{1}{\rho}MC = \frac{\sigma}{\sigma-1}MC$$

We can see that the demand curve has elasticity demand σ calculating the demand elasticity

$$\begin{aligned} \frac{\partial q(\omega)}{\partial p(\omega)}\frac{p(\omega)}{q(\omega)} &= -\sigma\frac{Q}{P}\left[\frac{p(\omega)}{P}\right]^{-\sigma-1} p(\omega)\frac{1}{Q}\left[\frac{p(\omega)}{P}\right]^{\sigma} \\ &= -\sigma \end{aligned}$$

We can see that the firm's price rule would be the constant markup above unit variable cost by the following reasonong: monopolistic firms set MR=MC. We know that:

Normalize wage to 1 ⇒ $w = 1$

CES preference and Monopolistic competition assumption immediate gives us the following

firm level variables:

$$
\begin{aligned}
\text{(price)} \quad p(\omega) &= p(\rho) = \frac{1}{\rho\varphi} \\
\text{(revenue)} \quad r(\omega) &= r(\rho) = R\left[P\rho\varphi\right]^{\sigma-1} \\
\text{(profit)} \quad \pi(\omega) &= \pi(\rho) = r(\rho) - l(\rho) \\
&= \underbrace{\frac{r(\rho)}{\sigma}}_{\text{TR-variable cost or production labor costs}} - \underbrace{f}_{\text{fixed cost but here interpreted as investment labor costs}} \\
\text{(ratio of output)} \quad \frac{q_1(\omega)}{q_2(\omega)} &= \left(\frac{\varphi_1}{\varphi_2}\right)^{\sigma} \\
\text{(ratio of revenue)} \quad \frac{r_1(\omega)}{r_2(\omega)} &= \left(\frac{\varphi_1}{\varphi_2}\right)^{\sigma-1}
\end{aligned}
$$

Observations immediate from CES and monopolistic competitoin assumption:

- high productivity firms obtain more profits, revenues than low productivity firms
- high productivity firms produce more output than low productivity firms
- price of a good produced by hihg productivity firms is lower than low productivity firms
- firms who have the same technology obtain the same revenue, price, and profits

Define Aggregate Variables
We want to define an average productivity so that we can summarize the distribution of productivity and talk about M, which is the equilibrium variable that we want.
Given that in equilibrium, a mass M of firms exist. We rewrite the aggregate price index

$$
\begin{aligned}
P &= \left[\int_{\omega\in\Omega} p(\omega)^{1-\sigma} d\omega\right]^{\frac{1}{1-\sigma}} \\
&= \left[\int_0^{\infty} p(\varphi)^{1-\sigma} M\mu(\varphi) d\varphi\right]^{\frac{1}{1-\sigma}} \\
&= M^{\frac{1}{1-\sigma}} \left[\int_0^{\infty} p(\varphi)^{1-\sigma} \mu(\varphi) d\varphi\right]^{\frac{1}{1-\sigma}}
\end{aligned}
$$

Recall that

$$p(\varphi) = \frac{1}{\rho\varphi}$$

Define our weighted average of firm productivity levels:

$$\widetilde{\varphi} = \left[\int_0^{\infty} \varphi^{\sigma-1} \mu(\varphi) d\varphi\right]^{\frac{1}{\sigma-1}}$$

So

$$p(\widetilde{\varphi}) = \rho^{-1} \left[\int_0^{\infty} \varphi^{\sigma-1} \mu(\varphi) d\varphi\right]^{\frac{1}{1-\sigma}}$$

So the aggregate price index can be written as:

$$\begin{aligned} P &= M^{\frac{1}{1-\sigma}}\left[\int_0^\infty p(\varphi)^{1-\sigma}\mu(\varphi)d\varphi\right]^{\frac{1}{1-\sigma}} \\ &= M^{\frac{1}{1-\sigma}}\left[\int_0^\infty (\rho\varphi)^{\sigma-1}\mu(\varphi)d\varphi\right]^{\frac{1}{1-\sigma}} \\ &= M^{\frac{1}{1-\sigma}}\rho^{-1}\left[\int_0^\infty \varphi^{\sigma-1}\mu(\varphi)d\varphi\right]^{\frac{1}{1-\sigma}} \\ &= M^{\frac{1}{1-\sigma}}p(\widetilde{\varphi}) \end{aligned}$$

Now write R, Q, Π, in terms of this $\tilde{\varphi}$

[Q]: Using

$$\begin{aligned} \text{(CES aggregator)} \quad Q &= \left[\int_0^\infty q(\varphi)^\rho M\mu(\varphi)d\varphi\right]^{\frac{1}{\rho}} \\ \text{(Ratio of firm output)} \quad \frac{q(\varphi)}{q(\tilde{\varphi})} &= \left(\frac{\varphi}{\tilde{\varphi}}\right)^\sigma \Rightarrow q(\varphi) = \left(\frac{\varphi}{\tilde{\varphi}}\right)^\sigma q(\tilde{\varphi}) \end{aligned}$$

$$\begin{aligned} Q &= M^{\frac{1}{\rho}}\left[\int_0^\infty q(\varphi)^\rho\mu(\varphi)d\varphi\right]^{\frac{1}{\rho}} \\ &= M^{\frac{\sigma}{\sigma-1}}\left[\int_0^\infty q(\varphi)^{\frac{\sigma-1}{\sigma}}\mu(\varphi)d\varphi\right]^{\frac{\sigma}{\sigma-1}} \\ &= M^{\frac{\sigma}{\sigma-1}}\left[\int_0^\infty \left(\left(\frac{\varphi}{\tilde{\varphi}}\right)^\sigma q(\tilde{\varphi})\right)^{\frac{\sigma-1}{\sigma}}\mu(\varphi)d\varphi\right]^{\frac{\sigma}{\sigma-1}} \\ &= M^{\frac{\sigma}{\sigma-1}}q(\tilde{\varphi})\left(\frac{1}{\tilde{\varphi}}\right)^\sigma\left[\int_0^\infty \varphi^{\sigma-1}\mu(\varphi)d\varphi\right]^{\frac{\sigma}{\sigma-1}} \\ &= M^{\frac{\sigma}{\sigma-1}}q(\tilde{\varphi})\left(\frac{1}{\tilde{\varphi}}\right)^\sigma\tilde{\varphi}^\sigma \\ &= M^{\frac{\sigma}{\sigma-1}}q(\tilde{\varphi}) \end{aligned}$$

[R]: Using

$$\begin{aligned} R &= PQ \\ r(\tilde{\varphi}) &= p(\tilde{\varphi})q(\tilde{\varphi}) \end{aligned}$$

gives:

$$\begin{aligned} R &= M^{\frac{1}{1-\sigma}}p(\widetilde{\varphi})M^{\frac{\sigma}{\sigma-1}}q(\tilde{\varphi}) \\ &= M^{\frac{-1+\sigma}{\sigma-1}}r(\widetilde{\varphi}) \\ &= Mr(\widetilde{\varphi}) \end{aligned}$$

or, using definition of R and ratio of firm level revenue

$$
\begin{aligned}
R &= \int r(\varphi) M\mu(\varphi) d\varphi \\
\frac{r(\varphi)}{r(\tilde{\varphi})} &= \left(\frac{\varphi}{\tilde{\varphi}}\right)^{\sigma-1}
\end{aligned}
$$

gives

$$
\begin{aligned}
R &= \int r(\varphi) M\mu(\varphi) d\varphi \\
&= \int r(\tilde{\varphi}) \left(\frac{\varphi}{\tilde{\varphi}}\right)^{\sigma-1} M\mu(\varphi) d\varphi \\
&= M r(\tilde{\varphi}) \left(\frac{1}{\tilde{\varphi}}\right)^{\sigma-1} \left[\left(\int \varphi^{\sigma-1}\mu(\varphi) d\varphi\right)^{\frac{1}{\sigma-1}}\right]^{\sigma-1} \\
&= M r(\tilde{\varphi}) \left(\frac{1}{\tilde{\varphi}}\right)^{\sigma-1} \tilde{\varphi}^{\sigma-1} \\
&= M r(\tilde{\varphi})
\end{aligned}
$$

[Π]: Using:

$$
\begin{aligned}
R &= \frac{r(\varphi)}{(P\rho\varphi)^{\sigma-1}} \\
\pi(\varphi) &= \frac{R}{\sigma}(P\rho\varphi)^{\sigma-1} - f = \frac{r}{\sigma} - f \\
P &= M^{\frac{1}{1-\sigma}} p(\widetilde{\varphi}) \\
r(\tilde{\varphi}) &= \int r(\varphi) M\mu(\varphi) d\varphi
\end{aligned}
$$

gives

$$
\begin{aligned}
\Pi &= \int_0^\infty \pi(\varphi) M\mu(\varphi) d\varphi \\
&= M \int_0^\infty \left(\frac{R}{\sigma}(P\rho\varphi)^{\sigma-1} - f\right)\mu(\varphi) d\varphi \\
&= \frac{1}{\sigma} \int_0^\infty r(\varphi) M\mu(\varphi) d\varphi - Mf \\
&= \frac{1}{\sigma} M r(\tilde{\varphi}) - Mf \\
&= M\pi(\tilde{\varphi})
\end{aligned}
$$

Thus we have our aggregate variables written in terms of the weighted average productivity levels. When it changes, we can easily say what happens to aggregate when the distribution changes. Thus we have reduced the heterogeneity problem into a representation agent problem. An industry with M firms and distribution $\mu(\varphi)$ that has the mean productivity level of $\tilde{\varphi}$ is like one single firm with productivity $\tilde{\varphi}$

List of Variables, written in terms of the average

$$\begin{aligned}
M &= \text{ mass of firms} \\
Q &= M^{\frac{\sigma}{\sigma-1}} q(\tilde{\varphi}) = \text{Total Output} \\
R &= Mr(\tilde{\varphi}) = \text{Total Revenue} \\
\Pi &= M\pi(\tilde{\varphi}) = \text{Total Profits} \\
P &= M^{\frac{1}{1-\sigma}} p(\widetilde{\varphi}) = \text{Aggregate price Index} \\
q(\varphi) &= \text{individual firm output with } \varphi \\
r(\varphi) &= \text{individual firm revenue produced wit } \varphi \\
\pi(\varphi) &= \text{individual firm profits produced with } \varphi \\
p(\varphi) &= \text{individual price of a variety produced with } \varphi \\
\mu(\varphi) &= \text{distribution of productivities} \\
\tilde{\varphi} &= \text{weighted average productivities} \\
q(\tilde{\varphi}) &= \text{output of the representative firm with productivity } \varphi = \tilde{\varphi} \\
r(\tilde{\varphi}) &= \text{revenue of the representative firm with productivity } \varphi = \tilde{\varphi} \\
\pi(\tilde{\varphi}) &= \text{profits of the representative firm with productivity } \varphi = \tilde{\varphi} \\
p(\tilde{\varphi}) &= \text{price of product produced with productivity } \varphi = \tilde{\varphi} \\
\varphi^* &= \text{cutoff of producing firms} \\
\bar{r} &= \text{average revenue of the industry} \\
\bar{\pi} &= \text{average profit of the industry}
\end{aligned}$$

Characterization of the equilibrium $\mu(\varphi)$

We need the free entry assumption to characterize the equilibrium distribution of productivities. It gives us two equations in φ^* and $\bar{\pi}$

- (ZCP) zero profit condition (the producing firm with the lowest productivity must earn 0 profits per period)–follows from exit assumption
- (FE) free-entry (there is no dynamics in the productivity, so firms enter by calculating the stream of profits. The value of the firm must therefore by 0 in equilibrium in order for firms to be indifferent between entering and exiting–this follows from free entry assumption

Timing: Firms make the fixed cost, then find out about their productivity. Some will decide to exit immediately. Firms that decide to stay will earn constant profits since their productivity stay constant, but have (1-δ) probability of surviving each period from negative shock. This reasoning implies that:

- there exist a cutoff φ^* where firms with $\varphi < \varphi^*$ exit and not produce.
- because some firms will decide to exit, the distribution of producing firms productivity must be truncated (conditional on successful entry)

$$\mu(\varphi) = \frac{g(\varphi)}{1 - G(\varphi^*)} \quad \text{for } \varphi \geq \varphi^* \text{ and } 0 \text{ for } \varphi < \varphi^*$$

where g(φ) is the initial distribution of firm productivity. Melitz then rewrote the aggregrate productivity level $\tilde{\varphi}$ as a function of the cutoff level $\varphi*$

$$\begin{aligned}
\tilde{\varphi} &= \left(\int_0^\infty \varphi^{\sigma-1}\mu(\varphi)d\varphi\right)^{\frac{1}{\sigma-1}} \\
&= \left(\int_{\varphi*}^\infty \varphi^{\sigma-1}\frac{g(\varphi)}{1-G(\varphi^*)}d\varphi\right)^{\frac{1}{\sigma-1}} \\
&= \left(\frac{1}{1-G(\varphi^*)}\int_{\varphi^*}^\infty \varphi^{\sigma-1}g(\varphi)d\varphi\right)^{\frac{1}{\sigma-1}} \\
&= \tilde{\varphi}(\varphi^*)
\end{aligned}$$

Then, using the definition of the ratio of revenue and profit (which follows by CES and constant marginal cost assumptions)

$$\begin{aligned}
\frac{r(\varphi)}{r(\tilde{\varphi})} &= \left(\frac{\varphi}{\tilde{\varphi}}\right)^{\sigma-1} \\
\pi(\varphi) &= \frac{r(\varphi)}{\sigma} - f
\end{aligned}$$

We can write the revenue and profit for the representative firm as a function of the cutoff

$$\begin{aligned}
\bar{r} &= r(\tilde{\varphi}) = r(\varphi^*)\left(\frac{\tilde{\varphi}(\varphi^*)}{\varphi^*}\right)^{\sigma-1} \\
\bar{\pi} &= \pi(\tilde{\varphi}) = \frac{r(\varphi^*)}{\sigma}\left(\frac{\tilde{\varphi}(\varphi^*)}{\varphi^*}\right)^{\sigma-1} - f
\end{aligned}$$

Since we know that the profit of the marginal firm must be 0,

$$\pi(\varphi^*) = 0 \Rightarrow \frac{r(\varphi^*)}{\sigma} = f$$

Plugging that into the equations for $\bar{r}$ and $\bar{\pi}$, we get

$$\begin{aligned}
\bar{r} &= f\sigma\left(\frac{\tilde{\varphi}(\varphi^*)}{\varphi^*}\right)^{\sigma-1} \\
\bar{\pi} &= f\left[\left(\frac{\tilde{\varphi}(\varphi^*)}{\varphi^*}\right)^{\sigma-1} - 1\right] = fk(\varphi^*)
\end{aligned}$$

$\bar{r}$ and $\bar{\pi}$ are interpreted as the expected revenue and profit conditional on successful entry

- When firm consider entering the industry, they consider the net value of entry, $v_{e.}$ At this point, they don't know their φ. They enter the industry if the the expected stream of profits upon successful entry less fixed cost is non-negative. At equilibrium, this value of entry must be zero, otherwise, firms will keep coming in. Thus we have another equation in φ^* and $\bar{\pi}$

$$\begin{aligned}
v_e &= (1-G(\varphi^*))\bar{v} - f = 0 \\
&\Rightarrow (1-G(\varphi^*))\frac{\bar{\pi}}{\delta} = f
\end{aligned}$$

The last equality follows from the fact that firms makes its entry decisions based on the entire stream of profits. Firms who enter must have value

$$\left\{v(\varphi) : \Sigma_{t=0}\pi(\varphi)(1-\delta)^t = \frac{\pi(\varphi)}{\delta} \geq 0\right\}$$

Equilibrium in a closed economy ($\bar{\pi}, \varphi^*$, all aggregate variables constant)

- Solve for $\bar{\pi}$ and φ^* from our two equations

$$\text{ZCP:} \qquad \bar{\pi} = fk(\varphi^*)$$
$$\text{FE:} \qquad \bar{\pi} = \frac{f\delta}{1 - G(\varphi^*)}$$

where

$$k(\varphi^*) = \left(\frac{\tilde{\varphi}(\varphi^*)}{\varphi^*}\right)^{\sigma-1} - 1$$

The FE equation is increasing in φ, whereas the ZCP equation is decreasing in φ. So ($\bar{\pi}$, φ^*) is uniquely determined. But we haven't fully characterized the equilibrium yet. We still need to another equation to pin down M, the number of producing firm.

- Find the equilibrium (constant) M, R, P, Q, W

M has to be constant in equilibrium. Since we know that δM firms die each period, such amount of firms must enter every period for M to be constant. We need something on the other side of this equation.

The other side of this equation is the birth process, which is $M_e 1 - G(\varphi^*)$, where M_e is the number of new entrants that has to enter each period so that after the new firms find out their productivity, the residual new firms will exactly make up the number of firms who die. So we have the equation

$$M_e(1 - G(\varphi^*)) = \delta M$$

Timing matters here. When new firms enter, they have not hired labor for production yet. The only resources they have taken up in the economy is L_e, the amount of labor devoted to building up the fixed cost. At that snapshot of time then, labor market clearing requires that

$$\text{(Labor market clearing condition)} \qquad L = L_e + L_p$$

where L_p is the labor used by the currently producing firms. Another condition we know is that the production worker cost must be the residual of aggregate revenue minus profit,

$$\text{(Producing firm accounting condition)} \qquad R - \Pi = L_p$$

New firms account must also balance,

$$\text{(New firm accounting condition)} \qquad L_e = M_e f_e$$

Combining three plus the definition of $\bar{\pi}$, gives an expression for L_e,

$$L_e = M_e f_e = \frac{\delta M f_e}{(1 - G(\varphi^*))} = \bar{\pi} M = \Pi$$

Thus we have shown that $L_e = \Pi$. Then we can conclude the following about the aggregate variable R,

$$R = \Pi + L_p = L_e + L_p = L$$

Since L is exogenously fixed, this means that R is exogenously fixed as well. The bigger the country the more revenues it earn.

Next, we can say something about M, using definition of $R = M\bar{r}$ and $\pi = \frac{r}{\sigma} - f$,

$$M = \frac{R}{\bar{r}} = \frac{L}{\sigma(\bar{\pi} + f)}$$

To solve for P, recall the following:

$$P = M^{\frac{1}{1-\sigma}} p(\tilde{\varphi}) = \frac{M^{\frac{1}{1-\sigma}}}{\rho\tilde{\varphi}}$$

Since M is solved for, we have also solved for P.

- Now we have solved for M, R, and P in terms of exogenous variables.

Summary of closed economy results
The following is a list of the equilibrium variables we have solved for:

$$\begin{aligned}
\text{ZCP (downward slope):} &\qquad \bar{\pi} = fk(\varphi*) \\
\text{FE (upwward slope):} &\qquad \bar{\pi} = \frac{f\delta}{1 - G(\varphi^*)} \\
\text{Revenue:} &\qquad R = L \\
\text{Mass of firms:} &\qquad M = \frac{L}{\sigma(\bar{\pi} + f)} \\
\text{Price Index:} &\qquad P = \frac{M^{\frac{1}{1-\sigma}}}{\rho\tilde{\varphi}} \\
\text{Welfare:} &\qquad W = P^{-1} = \rho\tilde{\varphi}M^{\frac{1}{\sigma-1}}
\end{aligned}$$

Interpretations of the equilibirum
From the previous results we see the following:
Welfare is higher in larger country because of more varieties.

Open economy equilibrium
For this equilibrium, we are interested in how opening to trade affect the endogenous pair $\{\bar{\pi}, \varphi^*\}$, and the other aggregate variables
Assumptions:

- exporters have to bear a fixed cost, f_x. But the knew their productivity before they have to decide to export. (thus fixed export cost is irrelevant to export decision, and hereby model as a per-period amortized cost, $f_{ex} = \delta f_x$)
- exporters have to incur an iceberg cost, $\tau > 1$ for every unit export, and this cost is same for all countries. (thus if a firm export to one country, it must export to all countries)
- n symmetric countries trade (size difference will induce wage difference across countries. By assuming symmetry, wages same)
- wage is normalized to one

Modification of variables

- Iceberg cost implies that the unit cost of exporting a good is:

$$mc_{ex} = \frac{w\tau}{\varphi}$$

- Thus the price the firm charge for selling its good abroad, with the same markup, is:

$$p_{ex} = \frac{\tau}{\rho\varphi}$$

- Revenue from exports then are, from the implied CES ratio of revenue (from the FOC):

$$r_x(\varphi) = r_d(\varphi)\,\tau^{1-\sigma}$$

- Firm level revenue can be differentiated between those that export and those that don't:

$$\begin{aligned} r(\varphi) &= r_d(\varphi) \quad \text{if the firm export,} \\ r(\varphi) &= r_d(\varphi) + nr_x(\varphi) = r_d(\varphi)\left(1 + n\tau^{1-\sigma}\right) \end{aligned}$$

- Key question we have to ask with this modification is: what type of firms would export, given the iceberg cost and fixed cost? They must earn a positive profit after incurring the sunk fixed cost and iceberg cost. Thus we can define a cutoff φ^*_{ex} where the profit generated from sales abroad is given by,

$$\pi_{ex}(\varphi^*_{ex}) = 0$$

- and $\pi_d(\varphi^*_{ex})$ and $\pi_{ex}(\varphi^*_{ex})$ are defined as,

$$\begin{aligned} \pi_d(\varphi) &= \frac{r_d(\varphi)}{\sigma} - f \\ \pi_{ex}(\varphi) &= \frac{r_x(\varphi)}{\sigma} - f_{ex} \end{aligned}$$

- We also want to refine the definition of M with the following,

$$\begin{aligned} M &= \text{equilibrium mass of incumbent firms in any country} \\ M_x &= p_x M = \text{the mass of exporting firms} \\ p_x &= \frac{1 - G(\varphi^*_{ex})}{1 - G(\varphi^*)} = \text{probability of exporting in addition to surviving entry} \\ M_t &= M + nM_x = \text{ total mass of available varieties to consumers in any country} \end{aligned}$$

- We need to redefine an average productivity that can be used as a representative agent in this integrated economy (the old one will not suffice because it did not take into account the iceberg cost. But I don't know how that comes into the old expression. Anyway....)

$$\begin{aligned} \text{(all firms, same as old)} \quad \tilde{\varphi} &= \tilde{\varphi}(\varphi^*) \\ \text{(export firms only)} \quad \tilde{\varphi}_x &= \tilde{\varphi}(\varphi^*_x) \\ \text{(the representative avg in the intergrated economy)} \quad \tilde{\varphi}_t &= \left\{ \frac{1}{M_t}\left[M\tilde{\varphi}^{\sigma-1} + nM_x(\tau^{-1}\tilde{\varphi})^{\sigma-1}\right] \right\}^{\frac{1}{\sigma-1}} \end{aligned}$$

- Like in the closed economy case, write the aggregate variables in terms of the representative productivity

$$\begin{aligned} P &= M_t^{\frac{1}{1-\sigma}} p(\tilde{\varphi}_t) = M_t^{\frac{1}{1-\sigma}} \frac{1}{\rho\tilde{\varphi}_t} \\ R &= M_t^{\frac{1}{1-\sigma}} p(\tilde{\varphi}_t) \\ W &= \frac{R}{L} M_t^{\frac{1}{1-\sigma}} \rho\tilde{\varphi}_t \end{aligned}$$

Solving for Equilibrium
We want to again characterize the distribution of the productivities of the firms and the aggregate variables, and match empirical facts

- In order to get only a subset of firms to export (as empirical facts suggest), we need $\varphi_{ex}^* > \varphi^*$. To see the condition for this, solve for the φ_{ex}^* from the zero profit condition of φ_{ex}^*,

$$\begin{aligned}
\pi(\varphi_{ex}*) &= \frac{r_{ex}\left(\varphi_{ex}^*\right)}{\sigma} - f_{ex} = 0 \\
&\Updownarrow \\
f_{ex} &= \frac{r_{ex}\left(\varphi_{ex}^*\right)}{\sigma} \\
&\Updownarrow \quad \text{(Plug the above into the equation into the following)} \\
\pi_{ex}(\tilde{\varphi}_{ex}) &= \frac{r_{ex}\left(\tilde{\varphi}_{ex}\right)}{\sigma} - f_{ex} = f_{ex}k(\varphi_{ex}^*) \\
\text{where again, } k(\varphi_{ex}^*) &= \left(\frac{\tilde{\varphi}(\varphi_{ex}^*)}{\varphi_{ex}^*}\right)^{\sigma-1} - 1
\end{aligned}$$

Using the definition of revenues and the zero profit condition, solve for ratio of the cutoffs,

$$\frac{r_{ex}\left(\varphi_{ex}^*\right)}{r_d\left(\varphi^*\right)} = \tau^{1-\sigma}\left(\frac{\varphi_{ex}^*}{\varphi^*}\right)^{\sigma-1} = \frac{f_x}{f}$$

$$\varphi_{ex}^* = \varphi^* \tau \left(\frac{f_x}{f}\right)^{\frac{1}{\sigma-1}}$$

We can see that if $\tau\left(\frac{f_x}{f}\right)^{\frac{1}{\sigma-1}} < 1$, then only a subset of firms would export.

- The equilibrium variables we want to solve for are $\bar{\pi}$ and the mass of firms. Because of trade, we redefine the average profit

$$\bar{\pi} = \pi_d\left(\tilde{\varphi}\right) + p_x n \pi_x\left(\tilde{\varphi}_{ex}\right)$$

where $p_x = \left[1 - G\left(\varphi_x^*\right)\right] / \left[1 - G\left(\varphi^*\right)\right]$ represents the ex ante conditional probability that entered firms will export, or the ex-post fraction of firms that export.

With the above relation between the φ_{ex}^* and φ^*, and our ability to write $\tilde{\varphi}_{ex}$ in terms of φ_{ex}^* because of $\pi_{ex}(\tilde{\varphi}_{ex}) = f_{ex}k(\varphi_{ex}^*)$

We can write $\bar{\pi}$ in terms of φ^*

$$\begin{aligned}
\bar{\pi} &= \pi_d\left(\tilde{\varphi}\right) + p_x n \pi_x\left(\tilde{\varphi}_x\right) \\
&= f k\left(\varphi^*\right) + p_x n f_x k\left(\varphi_{ex}^*\right)
\end{aligned}$$

- So our new ZCP is

$$\bar{\pi} = f k\left(\varphi^*\right) + p_x n f_x k\left(\varphi_{ex}^*\right) \qquad \text{(ZCP)}$$

- The FE condition is unchanged at $\bar{\pi} = \delta f_e / p_{in}$ since regardless of profit differences across firms, the expected vaue of future profits, in equilibrium, must equal the fixed investment cost

Determination of the Equilibrium, φ^*

After determining φ^*, all the other interested variables are determined: $\{\tilde{\varphi}, \tilde{\varphi}_x, \tilde{\varphi}_t, p_{in}, p_x, M, M_t, P\}$

$$\begin{aligned} \text{FREE ENTRY} \quad p_{in} M_e &= \delta M \\ &\Rightarrow \quad R = L \end{aligned}$$

The free entry conditoin ensures that the aggregate payment to the investment workers L_e equals the aggregate profit level Π. Thus aggregate revenue R remains exogenously fixed by the size of the labor force: R=L

Same as before, average firm revenue is determined by the ZCP and FE conditions, giving us M

$$\begin{aligned} \bar{r} &= r_d(\tilde{\varphi}) + p_x n r_x(\tilde{\varphi}_x) \\ &= \sigma(\bar{\pi} + f + p_x n f_x) \end{aligned}$$

$$M = \frac{R}{\bar{r}} = \frac{L}{\sigma(\bar{\pi} + f + p_x n f_x)}$$

Which in turn pins down M_t, the mass of variety available in every country

$$M_t = (1 + n p_x) M$$

And their price index

$$P = \frac{M_t^{1/(1-\sigma)}}{\rho \tilde{\varphi}_t}$$

$$\begin{aligned} \text{ZCP (downward slope):} & \quad \bar{\pi} = f k(\varphi *) \\ \text{FE (upwward slope):} & \quad \bar{\pi} = \frac{f\delta}{1 - G(\varphi *)} \\ \text{Revenue:} & \quad R = L \\ \text{Mass of firms:} & \quad M = \frac{L}{\sigma(\bar{\pi} + f)} \\ \text{Price Index:} & \quad P = \frac{M^{\frac{1}{1-\sigma}}}{\rho \tilde{\varphi}} \\ \text{Welfare:} & \quad W = P^{-1} = \rho \tilde{\varphi} M^{\frac{1}{\sigma - 1}} \end{aligned}$$