Problem Set #2

1. Let $F_n, n \geq 1$ be a random sequence of distribution functions on the real line, and let F be a nonrandom distribution function on the real line. Suppose $F_n(x) \xrightarrow{P} F(x)$ for all continuity points x of F. Suppose further that F is continuous at $F^{-1}(1-\alpha) = \inf\{x \in \mathbf{R} : F(x) \geq 1-\alpha\}$. Show that $F_n^{-1}(1-\alpha) = \inf\{x \in \mathbf{R} : F_n(x) \geq 1-\alpha\} \xrightarrow{P} F^{-1}(1-\alpha)$. (Hint: Generalize the proof of Lemma 2.1 in the lecture notes on the bootstrap.)

2. Complete the proof of part (ii) of Theorem 1.1. in the lecture notes on subsampling. (Hint: Use the above exercise.)

3. Let $X_i, i = 1, \ldots, n$ be i.i.d. with distribution F on the real line.

   (a) Consider testing the null hypothesis $H_0 : F$ is symmetric about θ. Find a test ϕ of this null hypothesis that satisfies $E_F[\phi] = \alpha$ whenever the null hypothesis is true.

   (b) Consider testing the null hypothesis $H_0 : \text{Med}(F) = \theta$ and F is continuous at θ. Find a test ϕ of this null hypothesis that satisfies $E_F[\phi] = \alpha$ whenever the null hypothesis is true. (Hint: Consider the "data" $I\{X_i - \theta > 0\}, i = 1, \ldots, n$.)

4. Suppose one observes data $X \sim P \in \Omega$ and wishes to test null hypotheses $H_i : P \in \omega_i, i = 1, \ldots, s$. Let $\hat{p}_i, i = 1, \ldots, s$ be mutually independent p-values for testing $H_i, i = 1, \ldots, s$. In particular, they satisfy

$$\Pr_P\{\hat{p}_i \leq u\} \leq u \text{ for all } u \in (0,1) \text{ and } P \in \omega_i \ .$$

   (a) Show that the single-step testing method with cutoff $c = c(\alpha, s) = 1 - (1-\alpha)^{1/s}$ controls the familywise error rate at level α.

   (b) Is it possible to improve upon the choice of c above?

(c) Compare c from (a) with the Bonferonni cutoff $c = \alpha/s$. In particular, show that

$$\lim_{s\to\infty} \frac{c(\alpha, s)}{\alpha/s} = \frac{-\log(1-\alpha)}{\alpha} \; .$$

(Hint: Use L'hopital's Rule.)

(d) For $\alpha = .05$, approximately how much larger is $c(\alpha, s)$ than α/s for large s? Is the improvement substantial?