Problem Set #1

1. Consider the binary choice model

$$Y = \mathbf{1}\{\beta_1 + \beta_2 X_2 - \epsilon \geq 0\} \ .$$

   Suppose $\text{supp}(X_2) = \mathbf{R}$ and $\text{Med}(\epsilon|X) = 0$. Show that $\text{sign}(\beta_2)$ is identified; that is, determine from the distribution of the observed data whether $\beta_2 > 0$, $\beta_2 = 0$, or $\beta_2 < 0$. (Hint: Since β_2 may be equal to zero, this result does not follow immediately from Theorem 4.2 in the lecture notes on identification. Instead, use Lemma 4.1, which was used in the proof of Theorem 4.2.)

2. For each $\epsilon > 0$, let $A_n(\epsilon)$ be a sequence of numbers such that $A_n(\epsilon) \to 0$. Show that there exists a sequence $\epsilon_n \to 0$ such that $A_n(\epsilon_n) \to 0$. (Hint: First argue that for each k there exists n_k such that for $n > n_k$ we have $|A_n(\frac{1}{k})| < \frac{1}{k}$. Argue next that we may assume w.l.o.g. that $n_1 < n_2 < \cdots$. Finally, show that the sequence $\epsilon_n = \frac{1}{k}$ where $n_k \leq n \leq n_{k+1}$ works.)

3. Consider the problem of testing $H_0 : \theta = \theta_0$ versus $H_1 : \theta = \theta_1 \neq \theta_0$. Suppose that for each $\alpha \in (0, 1)$ there exists a test of this null hypothesis that is asymptotically of level α and that is consistent (i.e., has power tending to one). Show that there exists a sequence of tests in which the probability of both Type 1 and Type 2 error go to zero simultaneously!

4. Let $X_i, i = 1, \ldots, n$ be an i.i.d. sequence of random variables with distribtution $N(\theta, 1)$ where $\theta \in \mathbf{R}$. Let $0 < \alpha_n \to 0$, but such that $\sqrt{n}\alpha_n \to \infty$. Let S_n be Hodges' estimator; that is,

$$S_n = \begin{cases} \bar{X}_n & \text{if } |\bar{X}_n| > \alpha_n \\ 0 & \text{otherwise} \end{cases} .$$

   Suppose θ_n is such that $\sqrt{n}\theta_n \to \infty$, but $\theta_n/\alpha_n \to 0$. Show that $\sqrt{n}(S_n - \theta_n)$ converges in probability to $-\infty$ under θ_n.

5. Consider the setup of the previous exercise. The Convolution Theorem says that superefficiency can happen for at most a set of θ values with Lebesgue measure 0. This suggests that superefficiency could possibly happen for values of θ in a countably infinite set. Can you find an estimator that is superefficient for all values of θ in a countably infinite set?

6. Let F_n and F be nonrandom distribution functions on $\mathbf{R}$. Suppose F_n converges in distribution to F and that F is continuous. Show that

$$\sup_{x \in \mathbf{R}} |F_n(x) - F(x)| \to 0 \ .$$

(This result holds more generally on $\mathbf{R}^k$ with $k \geq 1$. It is sometimes referred to as Polya's Theorem.)

7. Let $X_i, i = 1, \ldots, n$ be an i.i.d. sequence of random variables with distribution P on $\mathbf{R}$ with finite, nonzero variance $\sigma^2(P)$. Let $J_n(x, P)$ denote the distribution of $\sqrt{n}(\bar{X}_n - \theta(P))$. Let $J(x, P) = \Phi(x/\sigma(P))$. Show that

$$\sup_{x \in \mathbf{R}} |J_n(x, \hat{P}_n) - J(x, P)| \to 0$$

a.s., where $\hat{P}_n$ is the empirical distribution of the $X_i, i = 1, \ldots, n$. (Hint: Use the above exercise.)

8. Show by example that $X_n \xrightarrow{d} X$ does not imply that $E[X_n] \to E[X]$.

9. Suppose $X_i, i = 1, \ldots, n$ is an i.i.d. sequence of random variables with distribution Bernoulli(p). Construct a set $C_n = C_n(X_1, \ldots, X_n)$ (not equal to $[0, 1]$) such that

$$\text{Pr}_p\{p \in C_n\} \geq 1 - \alpha$$

for all n and p. (Hint: Consider first the problem of testing the null hypothesis $H_0 : p = p_0$ for some prespecified value of p_0. Figure out way to do this in a way that controls the probability of a Type 1 error at level α for all n. Next, consider doing this for all values p_0. One of these values is bound to be the correct value of p.)

10. Let $X_i, i = 1, \ldots, n$ be an i.i.d. sequence of random variables with continuous distribution F on $\mathbf{R}$. Define

$$F^{-1}(u) = \inf\{x \in \mathbf{R} : F(x) \geq u\}$$

and let $\hat{F}_n(x)$ be the empirical distribution function of the $X_i, i = 1, \ldots, n$. The following exercise steps you through proving that

$$\sup_{x \in \mathbf{R}} \sqrt{n} |\hat{F}_n(x) - F(x)| \tag{1}$$

is a pivot; that is, its distribution does not depend on F.

(a) Show that $F^{-1}(u) \leq x$ if and only if $u \leq F(x)$.

(b) Let $U \sim \text{Unif}(0, 1)$. Use (a) to show that $F^{-1}(U) \sim F$. (This useful trick is known as the quantile transformation.)

(c) Use (a) and (b) to show that the distribution of (1) is the same as the distribution of

$$\sup_{x \in \mathbf{R}} \sqrt{n} |\frac{1}{n} \sum_{1 \leq i \leq n} \mathbf{1}\{U_i \leq F(x)\} - F(x)| \ . \tag{2}$$

(d) Complete the proof by using the fact that F is continuous to show that the distribution of (2) is the same as the distribution of

$$\sup_{u \in [0,1]} \sqrt{n} |\frac{1}{n} \sum_{1 \leq i \leq n} \mathbf{1}\{U_i \leq u\} - u| \ . \tag{3}$$

(e) We only used continuity of F in part (d). How would the distributions of (1) and (3) be related if F were not continuous?