# 1 A Result of Bahadur and Savage (1956)

I thought I would end the class with the following simple, but somewhat striking result. Consider the following problem. Let $X_i, i = 1, \ldots, n$ be a sequence of i.i.d. random variables with distribution $P \in \mathbf{P} = \{P \text{ on } \mathbf{R} : 0 < \sigma^2(P) < \infty\}$. Suppose one wishes to test the null hypothesis $H_0 : \mu(P) = 0$ versus $> 0$. Earlier, we showed that the $t$-test, i.e. $\phi_n = I\{\sqrt{n}\bar{X}_n > \hat{\sigma}_n z_{1-\alpha}\}$ where $z_{1-\alpha}$ is the $1 - \alpha$ quantile of the standard normal distribution, had size one if $\mathbf{P}$ were sufficiently large. In particular, we showed that

$$\sup_{P \in \mathbf{P} : \mu(P) = 0} E_P[\phi_n] = 1 \ .$$

This result was perhaps a bit shocking, but it is possible that it is unique to the $t$-test – perhaps there are other tests of the same null hypothesis that would behave more reasonably. Unfortunately, we can show that this is not the case, provided that $\mathbf{P}$ is "'sufficiently rich". Formally, we have the following result:

**Theorem 1.1** Let $\mathbf{P}$ be a class of distributions on $\mathbf{R}$ such that

(i) For every $P \in \mathbf{P}$, $\mu(P)$ exists and is finite;

(ii) For every $m \in \mathbf{R}$, there is $P \in \mathbf{P}$ such that $\mu(P) = m$;

(iii) $\mathbf{P}$ is convex in the sense that if $P_1$ and $P_2$ are in $\mathbf{P}$, then $\gamma P_1 + (1-\gamma)P_2$ is in $\mathbf{P}$ for $\gamma \in [0, 1]$.

Let $X_i, i = 1, \ldots, n$ be i.i.d. with distribution $P \in \mathbf{P}$. Let $\phi_n$ be any test of the null hypothesis $H_0 : \mu(P) = 0$. Then,

(a) Any test of $H_0$ which has size $\alpha$ for $\mathbf{P}$ has power $\leq \alpha$ for any alternative $P \in \mathbf{P}$.

(b) Any test of $H_0$ which has power $\beta$ against some alternative $P \in \mathbf{P}$ has size $\geq \beta$.

The proof of this result will follow from the following lemma:

**Lemma 1.1** Let $X_i, i = 1, \ldots, n$ be i.i.d. with distribution $P \in \mathbf{P}$, where a $\mathbf{P}$ is the class of distributions on $\mathbf{R}$ satisfying (i) - (iii) in Theorem 1.1. Let $\phi_n$ be any test function. Define

$$\mathbf{P}_m = \{P \in \mathbf{P} : \mu(P) = m\} \ .$$

Then,

$$\inf_{P \in \mathbf{P}_m} E_P[\phi_n] \text{ and } \sup_{P \in \mathbf{P}_m} E_P[\phi_n]$$

are independent of $m$.

Proof: We show first that $\sup_{P \in \mathbf{P}_m} E_P[\phi_n]$ does not depend on $m$. Let $m$ be given and choose $m' \neq m$. We wish to show that

$$\sup_{P \in \mathbf{P}_{m'}} E_P[\phi_n] = \sup_{P \in \mathbf{P}_m} E_P[\phi_n] \ .$$

To this end, choose $P_j, j \geq 1$ so that

$$\lim_{j \to \infty} E_{P_j}[\phi_n] = \sup_{P \in \mathbf{P}_m} E_P[\phi_n] \ .$$

Let $h_j$ be defined so that

$$m' = (1 - \frac{1}{j})m + \frac{1}{j}h_j \ .$$

Choose $H_j$ so that $\mu(H_j) = h_j$. Define

$$G_j = (1 - \frac{1}{j})P_j + \frac{1}{j}H_j \ .$$

Thus, $G_j \in \mathbf{P}_{m'}$. Note that with probability $(1 - \frac{1}{j})^n$, a sample of size $n$ from $G_j$ is simply a sample of size $n$ from $P_j$. Therefore,

$$\sup_{P \in \mathbf{P}_{m'}} E_P[\phi_n] \geq E_{G_j}[\phi_n] \geq (1 - \frac{1}{j})^n E_{P_j}[\phi_n] \ .$$

But $(1 - \frac{1}{j})^n \to 1$ and $E_{P_j}[\phi_n] \to \sup_{P \in \mathbf{P}_m} E_P[\phi_n]$ as $j \to \infty$. Therefore,

$$\sup_{P \in \mathbf{P}_{m'}} E_P[\phi_n] \geq \sup_{P \in \mathbf{P}_m} E_P[\phi_n] \ .$$

Interchanging the roles of $m$ and $m'$, we can establish the reverse inequality

$$\sup_{P \in \mathbf{P}_{m'}} E_P[\phi_n] \leq \sup_{P \in \mathbf{P}_m} E_P[\phi_n] \ .$$

We could replace $\phi_n$ with $1 - \phi_n$ to establish that $\inf_{P \in \mathbf{P}_m} E_P[\phi_n]$ does not depend on $m$. ■

Proof of Theorem 1.1: (a) Let $\phi_n$ be a test of size $\alpha$ for $\mathbf{P}$. Let $P'$ be any alternative. Define $m = \mu(P')$. Then,

$$E_{P'}[\phi_n] \leq \sup_{P \in \mathbf{P}_m} E_P[\phi_n] = \sup_{P \in \mathbf{P}_0} E_P[\phi_n] = \alpha \ .$$

The proof of (b) is similar. ■

The class of distributions with finite second moment satisfies the requirements of the theorem, as does the class of distributions with infinitely many moments. Thus, the failure of the $t$-test is not special to the $t$-test; in this setting, there simply exist no "reasonable" tests.