Scaling limits of Abelian sandpiles

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Self-organized criticality

The laws of physics can explain how an apple falls but not why Newton, a part of a complex world, was watching the apple

—Per Bak

- how can complex behavior arise from simple rules?
  - **self-organized criticality**: nature perpetually self-organizes itself into a critical state, in which microscopic fluctuations can lead to macroscopic, complex changes

- in 1987, physicists Bak, Tang, Wiesenfeld, invented the **Abelian sandpile** as a simple prototype of self-organized criticality
What is a sandpile?
What is an Abelian sandpile?

- collection of indistinguishable grains distributed among the vertices of a graph
Sandpile dynamics

- one rule
  - a vertex is *unstable* if it has at least as many grains as its degree
  - an unstable vertex can *topple* sending one grain to each neighboring vertex
Sandpile dynamics
Sandpile dynamics
Sandpile dynamics
Sandpile dynamics
Sandpile dynamics
Sandpile dynamics
Sandpile dynamics

[Diagram of a network with nodes labeled 0, 2, 2, 0, 1, 1, 1, 1]
Stabilizing sandpiles

order of topples doesn’t change final sandpile - model is Abelian!
Do Abelian sandpiles always stabilize?
Do Abelian sandpiles always stabilize?
Do Abelian sandpiles always stabilize?
Do Abelian sandpiles always stabilize?
Do Abelian sandpiles always stabilize?
Do Abelian sandpiles always stabilize – no!
One way to ensure stabilization

add a ‘sink’ vertex which absorbs grains
One way to ensure stabilization
One way to ensure stabilization

1 2 1
s
One way to ensure stabilization
One way to ensure stabilization
One way to ensure stabilization
One way to ensure stabilization
One way to ensure stabilization
One way to ensure stabilization
One way to ensure stabilization

Abelian sandpiles on finite, connected graphs with a sink vertex always stabilize.
by adding grains to the sandpile and stabilizing, you eventually will enter the set of *recurrent* sandpiles.
The sandpile group

to any finite connected graph with a sink, we can associate an Abelian group called *the sandpile group* which consists of recurrent sandpiles.
The sandpile group

- the set of stable sandpile configurations forms a commutative monoid under the operation $\bigoplus$ of adding pointwise and then stabilizing
The sandpile group

addition in the sandpile group is not linear!
The sandpile group

- the set of stable sandpile configurations forms a commutative monoid under the operation of adding pointwise and then stabilizing
- the minimal ideal of this commutative monoid is an Abelian group which we call the sandpile group
  - recall that an ideal of a monoid \((M, \oplus)\) is a subset \(J \subset M\) satisfying \(\sigma \oplus J \subset J\) for all \(\sigma \in M\)
  - the minimal ideal is the intersection over all nonempty ideals
  - it is a general fact that the minimal ideal of a finite commutative monoid is an Abelian group
What is the identity of the sandpile group?
What is the identity of the sandpile group?

![Diagram showing a sandpile group with nodes 0, 0, 0, and s connected in a triangle]

**Figure:** not the identity

![Diagram showing a sandpile group with nodes 1, 0, 1, and s connected in a triangle]

**Figure:** identity sandpile

- the identity element of a group constructed in this way is not easy to guess
- for the sandpile group, it is generally *not* the all 0 configuration
How do you find the identity sandpile?

- The group is finite: can enumerate all stable sandpile configurations and check
  - This is unfeasible for large graphs
  - Also, the cardinality of the sandpile group is exponential in the size of the graph
Dhar’s burning algorithm

- an algorithm introduced by statistical physicist and sandpile pioneer Deepak Dhar in 1987
- can be used to find the identity of the sandpile group
- roughly: push in sand through the sink until every vertex topples
Example of burning algorithm

\begin{center}
\begin{tikzpicture}
    \node (s) at (0,0) {$s$};
    \node (0) at (1.5,0) {0};
    \node (00) at (3,0) {0};
    \node (000) at (4.5,0) {0};
    \draw (s) -- (0);
    \draw (s) -- (00);
    \draw (s) -- (000);
\end{tikzpicture}
\end{center}
Example of burning algorithm
Example of burning algorithm
Example of burning algorithm
Example of burning algorithm
Example of burning algorithm
Example of burning algorithm

identity - line graph with 3 vertices
Identity on a line graph

what about a longer line graph?
Identity on a line graph

identity - line graph with 4 vertices
Identity on a line graph

identity - line graph with 5 vertices
can show by induction that the identity for a line graph with $n$ vertices is 1 everywhere, except for a 0 at the center if $n$ is odd.
Identity on a square graph

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Each square is a vertex which has nearest neighbor edges; all boundary squares have edges to an invisible sink.
Identity on a square graph

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push in sand through the boundary
Identity on a square graph

\[
\begin{array}{ccc}
4 & 2 & 4 \\
2 & 0 & 2 \\
4 & 2 & 4 \\
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keep pushing in sand through the boundary and stabilizing until every vertex topples
Identity on a square graph

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Identity on a square graph

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0 & 4 & 0 \\
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Identity on a square graph

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Identity on a square graph

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Identity on a square graph

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\begin{array}{ccc}
2 & 0 & 2 \\
0 & 4 & 0 \\
2 & 0 & 2 \\
\end{array}
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Identity on a square graph

\[
\begin{array}{ccc}
2 & 1 & 2 \\
1 & 0 & 1 \\
2 & 1 & 2 \\
\end{array}
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eyery vertex has now toppled, this is the identity for the $3 \times 3$ square graph
Identity on a square graph

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$5 \times 5$ square graph
Identity on a square graph

\[
\begin{array}{cccccc}
2 & 3 & 2 & 3 & 2 \\
3 & 2 & 1 & 2 & 3 \\
2 & 1 & 0 & 1 & 2 \\
3 & 2 & 1 & 2 & 3 \\
2 & 3 & 2 & 3 & 2 \\
\end{array}
\]

identity - \(5 \times 5\) square graph
Identity on a square graph

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identity - $10 \times 10$ square graph
Identity on a square graph

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identity - $10 \times 10$ square graph
Identity on a square graph

identity - $20 \times 20$ square graph
Identity on a square graph

identity - $100 \times 100$ square graph
Identity on a square graph

identity - 200 × 200 square graph
Identity on a square graph

identity - $500 \times 500$ square graph
Identity on a square graph

identity - 2000 × 2000 square graph
Convergence of the Abelian sandpile

Theorem (Pegden-Smart 2011, Duke J. Math)

The scaling limit of the sandpile identity on a square exists and is the Laplacian of the solution to an elliptic obstacle problem.
Other scaling limits in the sandpile group.

start with any *periodic* initial state of sand and push in sand through the boundary until every vertex topples; the sandpile that remains has a scaling limit by the Pegden-Smart theorem.
Random initial states?

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Random initial state

3 × 3 square
Random initial state

50 × 50 square
Random initial state

100 × 100 square
Random initial state

200 × 200 square
Random initial state

500 × 500 square
Random initial state

1000 × 1000 square
Random initial state

2000 × 2000 square
Random initial state

**Theorem (B. 2019)**

*The scaling limit of the random sandpile exists and is the Laplacian of the solution to an elliptic obstacle problem.*
How do you prove anything about the Abelian Sandpile?
Do not study the patterns

\[
\begin{array}{cccc}
10 & 5 & 5 & 10 \\
5 & 0 & 0 & 5 \\
5 & 0 & 0 & 5 \\
10 & 5 & 5 & 10
\end{array}
\]

\[
\begin{array}{cccc}
2 & 3 & 3 & 2 \\
3 & 2 & 2 & 3 \\
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\end{array}
\]

stabilize
Study the odometer function

\[ \eta_n + \Delta v_n = s_n \]
Odometer Function

\[ \eta_n + \Delta v_n = s_n \]

- \( \eta_n : □_n \rightarrow \mathbb{Z} \) - initial configuration of pushed in sand
- \( s_n : □_n \rightarrow \mathbb{Z}, s_n \leq 3 \) - stable configuration
- \( v_n : □_n \rightarrow \mathbb{N} \) - number of topples per site when stabilizing
- \( \Delta v_n(x) = \sum_{y \sim x} (v_n(y) - v_n(x)) \) - graph Laplacian
Odometer Function
sample a random background $\eta : \mathbb{Z}^2 \to \mathbb{Z}$ from a distribution which is uniformly bounded and stationary, ergodic under spatial translations

first example: $\eta \sim \text{Bernoulli}(0, 1)$

$\eta_n(x) = \eta(x)$ for $x \in \Box_n$
Convergence of the Random Abelian Sandpile

Theorem (B. 2019)

- There exists a unique $\bar{s} : \square_1 \rightarrow [0, 3]$ and $\bar{v} : \square_1 \rightarrow \mathbb{R}^+$ so that almost surely
  \[
  n^{-2} v_n([nx]) \rightarrow \bar{v} \text{ uniformly}
  \]
  \[
  s_n([nx]) \rightarrow \bar{s} \text{ weakly-}\ast
  \]
  and
  \[
  \bar{s}(x) = \Delta \bar{v}(x) + \mathbb{E}(\eta(0)).
  \]
- $\bar{v}$ is the unique viscosity solution to the elliptic obstacle problem
  \[
  \min\{v \in C(\square_1) : v \geq 0, D^2 v \in \bar{\Gamma}_\eta \text{ in } \square_1\},
  \]
  where $\bar{\Gamma}_\eta$ is a unique downwards closed set with Lipschitz boundary.
Convergence of the (Identity) Abelian Sandpile

Theorem (Pegden-Smart 2011, Duke J. Math)

- There exists a unique $\bar{s} : \square_1 \to [0, 3]$ and $\bar{v} : \square_1 \to \mathbb{R}^+$ so that almost surely
  
  $$n^{-2} v_n([nx]) \to \bar{v} \text{ uniformly}$$

  $$s_n([nx]) \to \bar{s} \text{ weakly-*}$$

  and

  $$\bar{s}(x) = \Delta \bar{v}(x).$$

- $\bar{v}$ is the unique viscosity solution to the elliptic obstacle problem

  $$\min\{v \in C(\square_1) : v \geq 0, D^2 v \in \bar{\Gamma}_0\},$$

  where

  $$\bar{\Gamma}_0 = \{ M \in S^2 \text{ so that there exists } u : \mathbb{Z}^2 \to \mathbb{Z} \ \
  \Delta u \leq 3 \text{ and } u(x) \geq \frac{1}{2} x^T M x + o(|x|^2)\}. $$
Proof outline

discrete adaption of the program of Armstrong-Smart (2014) for stochastic homogenization

1. show convergence of $\bar{v}_n$ along subsequences using PDE regularity theory
2. find a subadditive quantity $\mu$
   - show it controls the sandpile
   - show that it is nice
   - implicitly define $\bar{\Gamma}_\eta$ with $\mu$ and the subadditive ergodic theorem
3. conclude that every subsequential limit solves PDE defined by $\bar{\Gamma}_\eta$
the odometer function solves a discrete obstacle problem

\[ v = \inf \{ w : \mathbb{D} \to \mathbb{N} : \Delta w + \eta_n \leq 3 \}, \]

where \( \eta_n : \mathbb{Z}^2 \to \mathbb{N} \) is the initial configuration at step \( n \)

called the least action principle: sandpiles are lazy

equivalent to the Abelian property, the order of topplings doesn’t change the final, stable configuration
What is \( \overline{\Gamma}_0 \)?

\[ \overline{\Gamma}_0 = \{ M \in S^2 \text{ so that there exists } u : \mathbb{Z}^2 \to \mathbb{Z} \]
\[ \Delta u \leq 3 \text{ and } u(x) \geq q_M(x) + o(|x|^2) \} \]

- can look at the boundary \( \partial \overline{\Gamma}_0 \) with a computer algorithm
- parameterize \( M \in S^2 \) by

\[ M(a, b, c) = \frac{1}{2} \begin{bmatrix} c - a & b \\ b & c + a \end{bmatrix} \]

and view \( \partial \overline{\Gamma}_0 \) as a surface in \( \mathbb{R}^3 \)
What is $\Gamma_0$?
What is $\tilde{\Gamma}_0$?

$\partial \tilde{\Gamma}_0$ is an Appolonian circle packing (Levine, Pegden, Smart, Ann. Math 2017)
What is $\bar{\Gamma}_\eta$?

- can also look at the boundary $\partial \bar{\Gamma}_\eta$ with a computer
- will depend on the distribution of $\eta$
What is $\bar{\Gamma}_\eta$?

$\eta \sim \text{Bernoulli}(3, 4)$
What is $\tilde{\Gamma}_\eta$?

$\eta \sim \text{Bernoulli}(3, 5)$
What is $\mathbf{\Gamma}_\eta$?

$\eta \sim \text{Bernoulli}(2, 6)$
Convergence of the Random Abelian Sandpile

Dirichlet problem on square domain $\eta \sim \text{Bernoulli}(3, 4)$
Convergence of the Random Abelian Sandpile

Dirichlet problem on stingray domain $\eta \sim \text{Bernoulli}(3,5)$
Convergence of the Random Abelian Sandpile

free boundary problem with random background $\eta \sim \text{Bernoulli}(0, -1)$
Thank you for listening!