The Planning Solution in a Textbook Model of Search and Matching: Discrete and Continuous Time

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1 Introduction

This note uses recursive methods to provide a simple characterization of the planner's solution of the continuous time and discrete time version of the simplest Pissarides (2000) model. I show that the solutions are virtually identical, characterized by a constant vacancy-unemployment ratio θ^*. In the continuous time model, employment adjusts monotonically to steady state, while it may oscillate in the discrete time model if the length of the time period is particularly long.

These results appear to contradict the findings of a recent note by Bhattacharya and Bunzel (2003), which concludes that the planner's solution in the discrete time model may exhibit explosive oscillatory cycles. Although the two papers take different approaches to solving the same problem, both approaches are correct. The difference in results apparently reflects Bhattacharya and Bunzel's (2003) failure to impose a necessary transversality condition in their characterization of the planner's solution.

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2 Discrete Time

Following Bhattacharya and Bunzel (2003), consider a social planner who in each period chooses a sequence of vacancies levels $v_t > 0$ and employment levels $n_t \in [0, 1]$ in order to maximize the present discounted value of output net of vacancy costs,

$$\max_{\{v_t, n_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t \big(y n_t + z(1 - n_t) - c v_t \big), \tag{1}$$

where $y > 0$ is output per employed worker, $z < y$ is output per unemployed worker, and $c > 0$ is the cost of a vacancy. Employment evolves according to

$$n_{t+1} = (1 - s) n_t + \mu(\theta_t)(1 - n_t), \tag{2}$$

where $s \in (0, 1)$ is the separation rate, μ is the rate at which unemployed workers match, and $\theta_t \equiv v_t/(1 - n_t)$ is the vacancy-unemployment ratio. I assume for simplicity that μ is strictly concave. Since μ is a probability, I also impose $\mu : \mathbb{R} \to [0, 1]$. In addition, the planner takes the initial level of employment n_0 as given.

Rather than formulate this problem as a Lagrangian, I express the problem recursively. In addition, I think of the planner choosing the vacancy-unemployment ratio θ rather than the vacancy level v; since n is predetermined, these choices are isomorphic. Let $V(n)$ be the expected present value of output as a function of the beginning-of-period employment level n. This must solve

$$V(n) = \max_{\theta} \Big(yn + z(1 - n) - c\theta(1 - n) + \beta V\big((1 - s)n + \mu(\theta)(1 - n)\big)\Big). \tag{3}$$

A standard argument based on Blackwell's theorem ensures that there is a unique solution V to this functional equation. I conjecture that in fact V is a linear function, say $V(n) = a_0 + a_1 n$, and solve for the constants a_0 and a_1. Given this conjecture, the first order condition with respect to θ is

$$-c + \beta a_1 \mu'(\theta) = 0. \tag{4}$$

Moreover, this condition is both necessary and sufficient, since μ is concave. In addition, the envelope theorem implies

$$a_1 = y - z + c\theta + \beta a_1 \big(1 - s - \mu(\theta)\big) \tag{5}$$

Eliminating a_1 between these equations gives an implicit definition of θ:

$$1 - \beta\big(1 - s - \mu(\theta) + \theta\mu'(\theta)\big) = \beta\mu'(\theta)\frac{y - z}{c} \tag{6}$$

Concavity of μ implies that the left hand side is increasing in θ and the right hand side is decreasing in θ. Thus there is at most one solution to this equation, the equilibrium vacancy-unemployment ratio θ^*.[1] Given this, equation (4) or (5) determines the value of a_1. Finally, go back to the Bellman equation to determine a_0. Matching constant coefficients, I get

$$a_0 = z - c\theta + \beta\big(a_0 + a_1\mu(\theta)\big),$$

which pins down a_0 and verifies the functional form of V.

Given the constant value of $\theta = \theta^*$, the remainder of the characterization of the planner's solution is simple: employment evolves so as to satisfy equation (2). This is a linear first order difference equation. The steady state employment level is

$$n^* = \frac{\mu(\theta^*)}{s + \mu(\theta^*)}.$$

If employment starts off away from steady state, it converges towards steady state. The speed of convergence is governed by the eigenvalue of this difference equation, $1 - s - \mu(\theta^*) \in (-1, 1)$. If this is positive, convergence is monotone. If it is negative, convergence is oscillatory, with employment jumping from above to below its steady state value in alternating periods. Obviously whether convergence is monotone depends on the period length; for short period lengths, both the separation and matching rates are small, and so the eigenvalue is positive. But if one takes the discrete time structure of this model seriously, it is possible to have non-monotone convergence.

3 Continuous Time

In continuous time, the planner solves

$$\max_{\{v(t),n(t)\}_{t=0}^{\infty}} \int_0^{\infty} e^{-rt}\big(yn(t) + z(1 - n(t)) - cv(t)\big)dt \tag{7}$$

[1]Standard conditions are needed to ensure existence of an interior solution. See Bhattacharya and Bunzel (2003) for details.

subject to

$$\dot{n}(t) = \mu(\theta(t))(1 - n(t)) - sn(t). \tag{8}$$

Now the discount rate is $r > 0$. I find it simplest to express this problem recursively again, using the state variable n:

$$rW(n) = \max_{\theta} \Big(yn + z(1-n) - c\theta(1-n) + W'(n)\big(\mu(\theta)(1-n) - sn\big) \Big) \tag{9}$$

Standard arguments again imply uniqueness of the solution to this functional equation. Again, I conjecture that the Bellman operator is linear, $W(n) = b_0 + b_1 n$. Given this conjecture, the first order and envelope conditions are

$$-c + b_1\mu'(\theta) = 0$$
$$rb_1 = y - z + c\theta - b_1\big(\mu(\theta) + s\big)$$

Eliminating b_1 between these equations gives

$$r + s + \mu(\theta) - \theta\mu'(\theta) = \mu'(\theta)\frac{y - z}{c}. \tag{10}$$

Again, the left hand side is increasing in θ and the right hand side is decreasing, so there is at most one equilibrium vacancy-unemployment ratio θ^*. I again ignore existence issues. As before, I can also calculate the exact values of b_0 and b_1, verifying the functional form conjecture.

In the continuous time model, the employment rate satisfies a linear first order differential equation, with steady state

$$n^* = \frac{\mu(\theta^*)}{s + \mu(\theta^*)}.$$

I can verify that the eigenvalue of the linear differential equation, $-\mu(\theta^*) - s$, is negative, and so the unemployment rate adjusts monotonically to its steady state value. Since vacancies are proportional to $1 - n$, vacancies also adjust monotonically to steady state.

References

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Pissarides, Christopher (2000): *Equilibrium Unemployment Theory.* MIT Press, Cambridge, MA, second edn.