Taxonomy of Functional Forms

An appendix to *Pass-through as an Economic Tool*

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My categorizations of demand functions into cost-absorbing and cost-amplifying come from the literature on log-concavity and log-convexity; see, for example, Bagnoli and Bergstrom (2005). The exception to this is AIDS; I am not aware of any previous classification of its log-curvature. The categorizations regarding violations of MUC and the slope of pass-through are, as far as I know, novel and are therefore stated and proved, along with the characterization of AIDS, in the following proposition.

The single-product version of AIDS can be written (over a particular range of prices as discussed below) as

$$D(p) = \frac{a + b\log(p)}{p} \tag{22}$$

The range of prices over which this formula can be viewed as valid depends on whether b is positive or negative. With $b > 0$, demand behaves very strangely, sloping upwards for low enough prices. We therefore only consider the (more commonly used) case when $b \leq 0$. If $b = 0$ this is just constant elasticity demand with an elasticity of 1, which violates (strict) MUC as discussed below. With $b < 0$, formula (22) is valid only for $p \leq e^{-\frac{a}{b}}$; for prices above this, demand is 0. It is this demand function that is considered in the table below as AIDS.

Proposition 2. *For any shape parameter $\alpha < 1$ and for any non-degenerate value of other parameters, there exists a value in the support of the following probability distributions such that the demand function generated by these distributions violates MUC at the corresponding price:*

1. *Type II Extreme Value (Fréchet) distribution with shape α*

2. *Weibull distribution with shape α*

3. *Gamma distribution with shape* α

The following distributions give rise to demand functions exhibiting increasing pass-through:

1. *Type I Extreme Value (Gumbel) distribution*

2. *Normal (Gaussian) distribution*

3. *Logisitic distribution*

4. *Laplace distribution*

5. *Type II Extreme Value (Fréchet) distribution with shape* $\alpha > 1$

6. *Type III Extreme Value (Reverse Weibull) distribution*

7. *Weibull distribution with shape* $\alpha > 1$

8. *Gamma distribution with shape* $\alpha > 1$

For any shape $\alpha > 1$*, the Type II Extreme Value (Fréchet) distribution exhibits cost-absorbption at some prices and cost-amplifcation at others. For the single-product, constant-expenditure AIDS demand function of equation (22) above with* $b < 0$*, there are always some prices (yielding positive demand) at which the demand is cost-amplifying and others at which it is cost-absorbing. AIDS always exhibits decreasing pass-through.*

Proof. Before considering individual probability distributions, note that for any probability distribution of the form $\tilde{F}(x; m, \sigma) = F\left(\frac{x-m}{\sigma}\right)$, $\tilde{F}(\cdot; m, \sigma)$ will exhibit globally increasing (decreasing) pass-through for any $\sigma > 0$ and any real m if and only if F exhibits globally increasing (decreasing) pass-through. To see this note that the pass-through rate for a given price for $\tilde{F}$ is

$$\tilde{\rho}(x; m, \sigma) = \frac{1}{2 + \frac{\left(1-\tilde{F}[x;m,\sigma]\right)\tilde{f}'(x;m,\sigma)}{\left(\tilde{f}[x;m,\sigma]\right)^2}} = \frac{1}{2 + \frac{\left(1-F[z]\right)\frac{f'(z)}{\sigma^2}}{\frac{\left(f[z]\right)^2}{\sigma^2}}} = \frac{1}{2 + \frac{\left(1-F[z]\right)f'(z)}{\left(f[z]\right)^2}} = \rho(z)$$

where $\tilde{f}$ and f are the density functions of $\tilde{F}$ and F respectively, $z \equiv \frac{x-m}{\sigma}$ and ρ is the pass-through rate of F. Thus as z is clearly a positive monotone transformation of x it is order-preserving and $\tilde{\rho}$ is globally increasing (decreasing) for any $m \in \mathbb{R}$ and $\sigma > 0$ if and only if ρ is globally increasing (decreasing). This is also obviously true of other properties of pass-through and of the slope of market power. This is useful as many of the probability distributions I consider below have scale and position parameters that this fact allows me to neglect.

I begin by considering the first part of the proof, that for any shape parameter $\alpha < 1$ the Fréchet, Weibull and Gamma distributions with shape α violate MUC at some price. I show this for each distribution in turn:

1. Type II Extreme Value (Fréchet) distribution: *Up to scale and position* (USP,) this distribution is $F(x) = e^{-x^{-\alpha}}$ with domain $x > 0$. Algebra shows that

$$\mu'(x) = \frac{(e^{x^{-\alpha}} - 1)x^{\alpha}(1+\alpha) - e^{x^{-\alpha}}\alpha}{\alpha}$$

As $x \to \infty$ and therefore $x^{-\alpha} \to 0$ (as shape is always positive), $e^{x^{-\alpha}}$ is well-approximated by its first-order approximation about 0, $1 + x^{-\alpha}$. Therefore the limit of the above expression is the same as that of

$$\frac{x^{-\alpha}x^{\alpha}(1+\alpha) - e^{x^{-\alpha}}\alpha}{\alpha} = \frac{1 + \alpha - e^{x^{-\alpha}}\alpha}{\alpha} \to \frac{1}{\alpha}$$

as $x \to \infty$. Clearly this is greater than 1 for $0 < \alpha < 1$ so that for sufficiently large x MUC is violated.

2. Weibull distribution: USP, this distribution is $F(x) = 1 - e^{-x^{\alpha}}$. Again algebra yields:

$$\mu'(x) = \frac{1-\alpha}{\alpha x^{\alpha}}$$

Clearly for any $\alpha < 1$ as $x \to 0$ this expression goes to infinity, so that for sufficiently small x MUC is violated.

3. Gamma distribution: USP, this distribution is $F(x) = \frac{\gamma(\alpha,x)}{\Gamma(\alpha)}$ where $\gamma(\cdot,\cdot)$ is the lower incomplete Gamma function, $\Gamma(\cdot,\cdot)$ is the upper incomplete Gamma function and $\Gamma(\cdot)$ is the complete Gamma function:

$$\mu'(x) = \frac{e^{x}(1-\alpha+x)\Gamma(\alpha,x)}{x^{\alpha}} - 1$$

By definition, $\lim_{x\to 0}\Gamma(\alpha,x) = \Gamma(\alpha) > 0$ so

$$\lim_{x\to 0}\mu'(x) = +\infty$$

as $1-\alpha > 0$ for $\alpha < 1$. Thus clearly for small enough x, the Gamma distribution with shape $\alpha < 1$ violates MUC.

I now turn to the categorization of demand functions as having increasing or decreasing pass-through. As price always increases in cost, this can be viewed as either pass-through as a function of price or pass-through as a function of cost.

1. Normal (Gaussian) distribution: USP, this distribution is given by $F(x) = \Phi(x)$, where Φ is the cumulative normal distribution function. Algebraic computations show that for this distribution

$$\rho'(x) = \frac{\big(1-\Phi[x]\big)(1+x^2)\sqrt{2\pi}e^{\frac{x^2}{2}} - x}{\left(e^{\frac{x^2}{2}}\sqrt{2\pi}x\big[1-\Phi(x)\big]-2\right)^2} \tag{23}$$

This has the same sign as

$$\big(1-\Phi[x]\big)(1+x^2)\sqrt{2\pi}e^{\frac{x^2}{2}} - x \tag{24}$$

This is positive if and only if

$$\chi(x) \equiv 1-\Phi(x) - \frac{x}{\sqrt{2\pi}(1+x^2)e^{\frac{x^2}{2}}} > 0$$

Note that $\lim_{x\to\infty}\chi(x) = 0$. Therefore if $\chi'(x) < 0$ for all x, as I show in a minute[1], the $\chi > 0$ for any finite x which establishes the result. I now show that $\chi' < 0$.

$$\chi'(x) = -\Phi'(x) - \left(\frac{x}{\sqrt{2\pi}(1+x^2)e^{x^2/2}}\right)' =$$

$$-\frac{1}{\sqrt{2\pi}e^{x^2/2}} - \frac{1}{2\pi}\frac{(1+x^2)e^{x^2/2} - (x.2xe^{x^2/2} + x(1+x^2)e^{x^2/2}x)}{(1+x^2)^2e^{x^2}} =$$

$$-\frac{1}{\sqrt{2\pi}e^{x^2/2}} - \frac{1}{2\pi}\frac{(1+x^2) - (2x^2 + (1+x^2)x^2)}{(1+x^2)^2e^{x^2/2}} =$$

$$-\frac{1}{\sqrt{2\pi}e^{x^2/2}}\left(1+\frac{(1+x^2)-2x^2-x^2-x^4)}{(1+x^2)^2}\right) = -\frac{1}{\sqrt{2\pi}e^{x^2/2}}\frac{2}{(1+x^2)^2} < 0$$

[1]The demonstration that $\chi' < 0$ is due to my research assistant Rosen Kralev. Thanks again to him.

2. Logistic distribution: USP, this distribution is $F(x) = \frac{e^x}{1+e^x}$. Again algebra yields

$$\rho'(x) = \frac{e^x}{(1+e^x)^2} > 0$$

Thus the logistic distribution exhibits increasing pass-through.

3. Type I Extreme Value (Gumbel) distribution : USP, this distribution has two forms. For the minimum version it is $F(x) = 1 - e^{-e^x}$. Algebra shows that for this distribution

$$\rho'(x) = \frac{e^x}{(1+e^x)^2}$$

Note that this is the same as for the logistic distribution; in fact the pass-through rates for the Gumbel minimum distribution are identical to the logistic distribution. This is not surprising given the close connection between these distributions (McFadden, 1974).

For the maximum version it is $F(x) = e^{-e^{-x}}$. Again algebra yields

$$\rho'(x) = \frac{e^{-x}\left(e^{e^{-x}}[1+e^x] + e^{2x}[e^{e^{-x}} - 1]\right)}{\left(1 + e^{e^{-x}} + e^x - e^{e^{-x}+x}\right)^2}$$

But clearly $e^{-x} > 0$ so $e^{e^{-x}} > 1$ and therefore the numerator and the entire expression is greater than 0 and the demand function generated by the Gumbel distribution therefore exhibits increasing pass-through.

4. Laplace distribution: USP, this distribution is

$$F(x) = \begin{cases} 1 - \frac{e^{-x}}{2} & x \geq 0 \\ \frac{e^x}{2} & x < 0 \end{cases}$$

For $x > 0$, $\rho = 1$ (so in this range pass-through is not strictly increasing). For $x < 0$

$$\rho'(x) = \frac{2e^x}{(2+e^x)^2} > 0$$

So the Laplace distribution exhibits globally weakly increasing pass-through, strictly increasing for prices below the mode. The pass-through rate for this distribution is $\frac{e^x}{2+e^x}$ as opposed to $\frac{e^x}{1+e^x}$ for Gumbel and Logistic. However these are very similar, again pointing out the similarities among pass-through functions assumed by common demand forms.

5. Type II Extreme Value (Fréchet) distribution with shape $\alpha > 1$: From the formula above it is easy to show that the derivative of the pass-through rate is

$$\rho'(x) = \frac{x^{-(1+\alpha)}\alpha^2\Big([1+\alpha]\big[x^{2\alpha}(e^{x^{-\alpha}}-1)+e^{x^{-\alpha}}x^{\alpha}\big]+\alpha e^{x^{-\alpha}}\Big)}{\big(\alpha[1+e^{x^{-\alpha}}]-[e^{x^{-\alpha}}]-1]x^{\alpha}(1+\alpha)\big)^2} > 0$$

as $x > 0$ in the range of this demand function and $e^x > 1$ for positive x. Thus this distribution, as well, exhibits increasing pass-through.

6. Type III Extreme Value (Reverse Weibull) distribution: USP, this distribution is $F(x) = e^{-(-x)^{\alpha}}$ and has support $x < 0$. Algebra shows

$$\rho'(x) = (-x)^{\alpha-1}\alpha^2 \frac{1-\alpha+e^{(-x)^{\alpha}}\Big([1-\alpha]\big[(-x)^{\alpha}-1\big]+[-x]^{2\alpha}\alpha\Big)}{\Big(\alpha-1+[-x]^{\alpha}\alpha+e^{[-x]^{\alpha}}\Big[1+\big([-x]^{\alpha}-1\big)\alpha\Big]\Big)^2}$$

which has the same sign as

$$1-\alpha+e^{(-x)^{\alpha}}\Big([1-\alpha]\big[(-x)^{\alpha}-1\big]+[-x]^{2\alpha}\alpha\Big) \tag{25}$$

Note that the limit of this expression as $x \to 0$ is

$$1-\alpha-(1-\alpha)=0$$

and its derivative is

$$\frac{e^{(-x)^{\alpha}}(-x)^{2\alpha}\alpha\big(1+\alpha+[-x]^{\alpha}\alpha\big)}{x}$$

which is clearly strictly negative for $x < 0$. Thus expression (25) is strictly decreasing and approaches 0 as x approaches 0. It is therefore positive for all negative x, showing that again in this case $\rho' > 0$.

7. Weibull distribution with shape $\alpha > 1$: As with the Fréchet distribution algebra from the earlier formula shows

$$\rho'(x) = \frac{x^{\alpha-1}(\alpha-1)\alpha^2}{(\alpha-1+x^{\alpha}\alpha)^2}$$

which is clearly positive for $\alpha > 1$ as the range of this distribution is positive x. Thus the Weibull distribution with $\alpha > 1$ exhibits increasing pass-through.

8. Gamma distribution with shape $\alpha > 1$: Again using the formula calculated above for

μ', a bit of algebra and a derivative yield:

$$\rho'(x) = \frac{\alpha - 1 - x + \frac{e^x}{x^\alpha}\big(x^2 - 2x[\alpha-1] + [\alpha-1]\alpha\big)\Gamma(\alpha,x)}{x\Big(\frac{e^x}{x^\alpha}[1+x-\alpha]\Gamma[\alpha,x] - 2\Big)^2}$$

Because the Gamma distribution is only defined for positive x, this has the same sign as

$$\alpha - 1 - x + \frac{e^x}{x^\alpha}\big(x^2 + [\alpha - 2x][\alpha-1]\big)\Gamma(\alpha,x) \tag{26}$$

Note that as long as $\alpha > 1$

$$x^2+(\alpha-2x)(\alpha-1) = x^2-2(\alpha-1)x+\alpha(\alpha-1) > x^2-2(\alpha-1)x+(\alpha-1)^2 = (x+1-\alpha)^2 > 0$$

Therefore so long as $x \le \alpha - 1$ this is clearly positive. On the other hand when $x > \alpha - 1$ the proof depends on the following result of Natalini and Palumbo (2000):

Theorem (Natalini and Palumbo, 2000). *Let a be a positive parameter, and let q(x) be a function, differentiable on* $(0,\infty)$ *, such that* $\lim_{x\to\infty} x^\alpha e^{-x} q(x,\alpha) = 0$*. Let*

$$T(x,\alpha) = 1 + (\alpha - x)q(x,\alpha) + x\frac{\partial q}{\partial x}(x,\alpha)$$

If $T(x,\alpha) > 0$ *for all* $x > 0$ *then* $\Gamma(\alpha,x) > x^\alpha e^{-x} q(x,\alpha)$*.*

Letting

$$q(x,\alpha) \equiv \frac{x - (\alpha-1)}{x^2 + (\alpha-2x)(\alpha-1)}$$

$$T(x,\alpha) = \frac{2(\alpha-1)x}{\big(\alpha^2 + x[2+x] - \alpha[1+2x]\big)^2} > 0$$

for $\alpha > 1, x > 0$. So $\Gamma(\alpha,x) > x^\alpha e^{-x} q(x,\alpha)$. Thus expression (26) is strictly greater than

$$\alpha - 1 - x + x - (\alpha - 1) = 0$$

as, again, $x^2 + (\alpha - 2x)(\alpha - 1) > 0$. Thus again $\rho' > 0$.

This establishes the second part of the proposition. Turning to my final two claims, algebra shows that the pass-through rate for the Fréchet distribution is

$$\rho(x) = \frac{\alpha}{\alpha + e^{x^{-\alpha}}\big(\alpha - x^{\alpha}[1+\alpha]\big) + x^{\alpha}(1+\alpha)} = \frac{\alpha}{\alpha(1+e^{x^{-\alpha}}) - (e^{x^{-\alpha}}-1)x^{\alpha}(1+\alpha)}$$

Note for any $\alpha > 1$ this is clearly continuous in $x > 0$. Now consider the first version of the expression. Clearly as $x \to 0$, $x^{\alpha} \to 0$ and $e^{x^{-\alpha}} \to \infty$ so the denominator goes to ∞ and the expression goes to 0. So for sufficiently small $x > 0$, $\rho(x) < 1$ and demand is cost-absorbing. On the other consider the second version of the expression. Its denominator is

$$\alpha(1+e^{x^{-\alpha}}) - (e^{x^{-\alpha}}-1)x^{\alpha}(1+\alpha)$$

By the same argument as above with the Fréchet distribution the limit of the above expression as $x \to \infty$ is the same as that of

$$\alpha(1+e^{x^{-\alpha}}) - x^{-\alpha}x^{\alpha}(1+\alpha) = \alpha(1+e^{x^{-\alpha}}) - 1 - \alpha \to \alpha - 1$$

as $x \to \infty$. Thus

$$\lim_{x\to\infty} \rho(x) = \frac{\alpha}{\alpha-1} > 1$$

and thus for sufficiently large x and any $\alpha > 1$, this distribution exhibits cost-amplifcation.

Finally, consider my claim about AIDS. First note that for this demand function

$$\mu'(p) = 1 + \frac{b\big(a - 2b + b\log[p]\big)}{\big(a - b + b\log[p]\big)^2} < 1$$

as $b < 0$ and $p \leq e^{-\frac{a}{b}} < e^{2-\frac{a}{b}}$.

$$\rho(p) = -\left(\frac{a}{b} + \log[p] + \frac{b}{a - 2b + b\log[p]}\right)$$

This is less than 1 iff

$$a^2 + 2ab\big(\log[p] - 2\big) + b^2\Big(1 + \log[p]\big[\log(p) - 2\big]\Big) < b^2\big(2 - \log[p]\big) - ab$$

or

$$\big(a + b\log[p]\big)^2 - b^2\big(\log[p] + 1\big) < 0$$

Clearly as $p \to 0$ the second term is positive; therefore there is always a price at which

$\rho(p) > 1$. On the other hand as $p \to e^{-\frac{a}{b}}$ this expression goes to

$$0 - b^2\left(1 - \frac{a}{b}\right) = b(a-b) < 0$$

Thus there is always a price at which $\rho(p) < 1$.

$$\rho'(p) = \frac{b^2 - \big(a - 2b + b\log[p]\big)^2}{p\big(a - 2b + b\log[p]\big)^2}$$

which has the same sign as

$$b^2 - \big(a - 2b + b\log[p]\big)^2 < b^2 - (2b)^2 = -3b^2 < 0$$

Thus AIDS exhibits decreasing pass-through.

References

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McFadden, Daniel, "Conditional Logit Analysis of Qualitative Choice Behavior," in Paul Zarembka, ed., *Fronteirs in Econometrics*, New York: Academic Press, 1974, pp. 105–142.

Natalini, Pierpaolo and Biagio Palumbo, "Inequalities for the Incomplete Gamma Function," *Mathematical Inequalities and Applications*, 2000, *3* (1), 69–77.