Multiple Products

An appendix to *Pass-through as an Economic Tool*

E. Glen Weyl

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The approach taken in proving the results in this appendix is somewhat more involved than those in other appendices so I briefly outline it before going into the details. This appendix is concerned with the properties of pass-through rates with multiple products. Because multiproduct equilibrium is an N firm fixed-point problem, its comparative statics are solutions to a matrix algebra equation representing linear approximations around the equilibrium. By the assumption that firms are symmetric, all on-diagonal entries of these matrices are the same, as are all off-diagonal entries. Solving for pass-through rates therefore involves finding the inverse of such a matrix, though I defer the formal presentation of this reasoning to the proof of Lemma 4 in Section II below. Once these formulae are obtained, the results follow from simple substitution and standard algebra.

I. Preliminaries

Therefore all of my results on multiple products therefore depend on a simple formula for the entries of the inverse of a matrix with identical on-diagonal and off-diagonal entries. I am certain this trivial formula is well-known, but could not find a proof of it, so I supply one here.

Lemma 3. *Let $\mathbf{A}_N$ be the NxN matrix whose on-diagonal entries are all a and off diagonal entries are all b. The determinant of $\mathbf{A}_N$ is $(a-b)^{N-1}(a+[N-1]b)$ and its off-diagonal co-factor is $-(a-b)^{N-2}b$.*

Proof. I establish the result by induction. For the base case consider $N=2$ for the determinant and $N=3$ for the co-factor. The determinant is clearly $a^2-b^2=(a-b)^1(a+1\cdot b)$ and therefore clearly satisfies the formula. Its off-diagonal co-factor is $-b=-(a-b)^0b$

and thus again satisfies the formula. The off-diagonal co-factor of $\mathbf{A}_3$ is the negative of the determinant of $\begin{bmatrix} b & b \\ b & a \end{bmatrix}$ which is $b(b-a) = -b(a-b)$ confirming the formula.

Now suppose the formula for determinants is correct for all matrices of size $N \leq M$ and for co-factors for all matrices of size no greater than $M+1$. I want to show that $\mathbf{A}_{M+1}$ satisfies the formula for determinants and $\mathbf{A}_{M+2}$ satisfies the formula for co-factors. By Cramer's rule $|\mathbf{A}_{M+1}| = a|\mathbf{A}_M| + Mbc\left(\mathbf{A}_{M+1}\right)$ where $c(\mathbf{M})$ is the off-diagonal co-factor of $\mathbf{M}$. Thus by the inductive hypothesis

$$|\mathbf{A}_{M+1}| = a(a-b)^{M-1}\left(a + [M-1]b\right) - Mb^2(a-b)^{M-1} =$$

$$(a-b)^{M-1}\left(a^2 + [M-1]ab - Mb^2\right) = (a-b)^{M-1}(a-b)(a+Mb) = (a-b)^M(a+Mb)$$

which establishes the inductive step for determinants.

Again by Cramer's rule $c(\mathbf{A}_{M+2}) = -\left(b|\mathbf{A}_M| + Mbc[\mathbf{A}_{M+1}]\right)$. By the inductive hypothesis therefore

$$-c(\mathbf{A}_{M+2}) = b(a-b)^{M-1}\left(a + [M-1]b\right) - Mb(a-b)^{M-1}b = b(a-b)^{M-1}(a-b) = b(a-b)^M$$

so

$$c(\mathbf{A}_{M+2}) = -b(a-b)^M$$

which establishes the lemma.

II. Bertrand horizontal demand systems

I now use Lemma 3 to derive explicit formulae for Lop, Lxp and industry pass-through as a function of primitives (Sop, strength of interaction, the number of firms and the derivative of slope of industry demand).

Lemma 4. *If goods are substitutes and there is a SHDS then*

$$\rho_i^{eq} = \frac{\rho_i\left(N - 1 + [N-2][\rho_i - 1]s\right)}{\left(1 + [\rho_i - 1]s\right)\left(N - 1 - s[\rho_i - 1]\right)}$$

$$\rho_{ij}^{eq} = -\frac{\rho_i s(\rho_i - 1)}{\left(1 + [\rho_i - 1]s\right)\left(N - 1 - s[\rho_i - 1]\right)}$$

$$\rho_I = \frac{\rho_i}{1 + (\rho_i - 1)s}$$

If goods are complements, the same formulae are valid substituting $-s$ for s.

Proof. Each firm maximizes $(p_i - c_i)Q^i(p_i, \mathbf{p}_{-i})$ taking all $\mathbf{p}_{-i}$ as given. Conditions for equilibrium are therefore

$$\mathbf{p}^\star - \mathbf{c} = \mu(\mathbf{p}^\star)$$

where μ is a vector with ith entry $\mu^i = -\frac{Q^i}{Q^i_i}$ and Q^i_i is the own-price derivative of firm i's demand. By the implicit function theorem in matrix form

$$\left[\frac{\partial p_i^\star}{\partial c_j}\right] = (\mathbf{I} - \nabla\mu)^{-1}$$

At the unique symmetric equilibrium all diagonal entries of $\mathbf{I} - \nabla\mu$ are identical, as are all off-diagonal entries. The on-diagonal entry is, in the language of equation (9) in the text Subsection III.A $1 - \tilde{\mu}'$. One oligopolist, holding other firms' prices constant, is equivalent to a monopolist so by the basic monopoly reasoning $\rho_i = \frac{1}{1-\tilde{\mu}'}$. Thus the on-diagonal entry of $\mathbf{I} - \nabla\mu$ is $\frac{1}{\rho_i}$. Its off-diagonal entries are

$$-\frac{\partial g}{\partial p_j}\tilde{\mu}' = \frac{s(\rho_i - 1)}{\rho_i(N-1)}$$

by symmetry and the definitions in the text for the case of substitutes and its negative for the case of complements. By Lemma 3 I have that the determinant of $\mathbf{I} - \nabla\mu$ is , in the substitutes case,

$$\left[\frac{(N-1) - s(\rho_i - 1)}{\rho_i(N-1)}\right]^{N-1}\left[\frac{1 + s(\rho_i - 1)}{\rho_i}\right]$$

and any principal minor is

$$\left[\frac{(N-1) - s(\rho_i - 1)}{\rho_i(N-1)}\right]^{N-2}\left[\frac{N - 1 + s(N-2)(\rho_i - 1)}{\rho_i(N-1)}\right]$$

Note that the (any) diagonal element of $(\mathbf{I} - \nabla\mu)^{-1}$ is ρ_i^{eq} and by Cramer's rule a diagonal element of $(\mathbf{I} - \nabla\mu)^{-1}$ is the ratio of the a principal minor to the determinant of the entire matrix. In other words

$$\rho_i^{eq} = \left[\frac{\rho_i(N-1)}{(N-1) - s(\rho_i - 1)}\right]\left[\frac{N - 1 + s(N-2)(\rho_i - 1)}{(1 + s[\rho_i - 1])\,(N-1)}\right]$$

which immediately simplifies to the formula I wanted to show.

On the other hand the off-diagonal co-factor of $\mathbf{I} - \nabla\mu$ is

$$-\left[\frac{(N-1) - s(\rho_i - 1)}{\rho_i(N-1)}\right]^{N-2}\left[\frac{s(\rho_i - 1)}{\rho_i(N-1)}\right]$$

So again by Cramer's rule the off-diagonal entry of $(\mathbf{I} - \nabla\mu)^{-1}$ is

$$-\rho_{ij}^{eq} = \left[\frac{\rho_i(N-1)}{(N-1) - s(\rho_i - 1)}\right]\left[\frac{s(\rho_i - 1)}{(N-1)\left(1 + s[\rho_1 - 1]\right)}\right]$$

Establishing the formula for cross pass-through. Industry pass-through is just

$$\rho_I = (N-1)\rho_{ij}^{eq} + \rho_i^{eq} = \frac{\rho_i\left(N - 1 + [N-2][\rho_i - 1]s\right) - (N-1)\rho_i s(\rho_i - 1)}{\left(1 + [\rho_i - 1]s\right)\left(N - 1 - s[\rho_i - 1]\right)} =$$

$$\frac{\rho_i\left(N - 1 - s[\rho_i - 1]\right)}{\left(1 + [\rho_i - 1]s\right)\left(N - 1 - s[\rho_i - 1]\right)} = \frac{\rho_i}{1 + [\rho_i - 1]s}$$

which shows the last formula. Identical reasoning shows the result for the case of complements.

These formulae make Theorems 4 and 5 easy to establish.

Proof of Theorem 4.

$$1 > \frac{(\rho_i - 1)s}{N-1}$$

because this is proportional to a 2x2 principal minor of $\mathbf{I} - \nabla\mu$, which must be positive by my assumption that the matrix of cross-partials of firm profits is Hicksian. For the same reason, $1 > -(\rho_i - 1)s$. Thus the denominator of all the expressions in Lemma 4 are positive, and by the same reasoning this holds for the case of complements. Therefore the sign of ρ_{ij}^{eq} is clearly determined by the sign of $1 - \rho_i$ for substitutes and $\rho_i - 1$ for complements as claimed.

On the other hand

$$\rho_i^{eq} - 1 = (\rho_i - 1)\frac{N - 1 + (\rho_i - 1)s\left(N + s - 2\right)}{\left(1 + [\rho_i - 1]s\right)\left(N - 1 - s[\rho_i - 1]\right)}$$

in the case of substitutes. The numerator of the second factor of this expression is positive. To see this note that if $\rho_i > 1$ then the expression is at least $N - 1 - (\rho_i - 1)s$ which is positive as shown above. On the other hand if $\rho_i < 1$ then the expression is at least $N - 1 - (1 - \rho_i)s(N + s - 2) > (N-1)\left(1 + [\rho_i - 1]s\right)$ which again is positive as shown above. For the case of complements the numerator is $N - 1 - (\rho_i - 1)s\left(N - s - 2\right)$. This can be shown to be positive as well by the same reasoning. Therefore the sign of $\rho_i^{eq} - 1$ in either case is the same as the sign of $\rho_i - 1$ which is what I wanted to show.

Finally in the case of substitutes

$$\rho_I - 1 = \frac{(\rho_i - 1)(1 - s)}{1 + (\rho_i - 1)s}$$

which clearly has the same sign as $\rho_i - 1$ as $s < 1$. In the case of complements

$$\rho_I - 1 = \frac{(\rho_i - 1)(1 + s)}{1 - (\rho_i - 1)s}$$

which again clearly has the same sign as $\rho_i - 1$. Note that the entry of a new firm is equivalent to one firm reducing its prices. Because this has a symmetric effect on all pre-entry firm prices, it must move them in the direction of the direct strategic effect, which is given by the sign of the strategic effect. Similarly because the two merging firms are symmetric, the merger provides both a local incentive to raise prices and the equilibrium is stable, both firms must raise their prices in equilibrium. Again the effect of this on the other (symmetric) firm prices must follow the strategic effect, completing the proof. A more detailed, mechanical proof of these claims available on request, but omitted here given its similarity to the above results on the GCS model.

Proof of Theorem 5. First consider the easiest case of industry pass-through.

$$\frac{\partial \rho_I}{\partial \rho_i} = \frac{1 - s}{\left(1 + [\rho_i - 1]s\right)^2} > 0$$

The comparative statics with respect to s are immediately clear.

Now consider Lop.

$$\frac{\partial \rho_i^{eq}}{\partial s} = \frac{(\rho_i - 1)^2 \rho_i s \left(2[N-1] + [N-2][\rho_i - 1]s\right)}{\left(1 + [\rho_i - 1]s\right)^2 \left(N - 1 - s[\rho_i - 1]\right)^2} > 0$$

The result can easily be seen to also hold for the case of complements.

$$\frac{\partial \rho_i^{eq}}{\partial N} = -\frac{(\rho_i - 1)^2 \rho_i s^2}{\left(1 + [\rho_i - 1]s\right) \left(N - 1 - s[\rho_i - 1]\right)^2} < 0$$

as stated. The expression for the derivative with respect to ρ_i is a bit more complicated and requires more involved algebra to show is positive so I omit it here. The analysis is available on request.

By definition and Lemma 4, for the case of substitutes,

$$\sigma_{ij} = \frac{(1-\rho_i)s}{\rho_i\left(N-1+[N-2][\rho_i-1]s\right)}$$

so

$$\frac{\partial \sigma_{ij}}{\partial \rho_i} = -\frac{s\left(N-1-[N-2][\rho_i-1]^2 s\right)}{\rho_i^2\left(N-1+[N-2][\rho_i-1]s\right)^2} > 0$$

if $\rho_i < 2$ as this implies $\rho_i - 1 < 1$.

$$\frac{\partial \sigma_{ij}}{\partial N} = \frac{(\rho_i-1)s\left(1+[\rho_i-1]s\right)}{\rho_i\left(N-1+[N-2][\rho_i-1]s\right)^2}$$

which is clearly positive (negative) when σ_{ij} is negative (positive) and thus increasing N reduces the absolute value of σ_{ij}.

$$\frac{\partial \sigma_{ij}}{\partial s} = \frac{(1-\rho_i)(N-1)}{\rho_i\left(N-1+[N-2][\rho_i-1]s\right)^2}$$

which has the same sign as σ_{ij} and thus increasing s increases the absolute value of σ_{ij}.

Now I turn to the limit results.

$$\lim_{N\to\infty} \rho_i^{eq} = \frac{\rho_i\left(1+s[\rho_i-1]\right)}{1+(\rho_i-1)s} = \rho_i$$

σ_{ij} clearly approaches 0 as claimed.

$$\lim_{s\to 1} \rho_i^{eq} = \frac{N-1+(N-2)(\rho_i-1)}{(N-\rho_i)} \equiv \tilde{\rho}_i^{eq}$$

Simple, but tedious, algebra shows $\rho_i^{eq} > \rho_i$. Being on the same side of 1 follows from Theorem 4. The σ_{ij} limit in s is trivial, as is the ρ_I result. The complements result for the ρ_i^{eq} result is proved identically

III. Cournot duality

In this appendix I state the corollaries to Theorems 4 and 5 that apply by the duality of generalized Cournot and Bertrand oligopoly. I do not provide any proof of these results, as they follow immediately by duality.

Let

1. ρ_i^Q be the *Short-run own quantity* (Soq) pass-through, the amount an increase in exogenous quantity is passed-through to total quantity when other firms' production are held fixed

2. $\rho_i^{Q,eq}$ be the *Long-run own quantity* (Loq) pass-through, the amount an increase in exogenous quantity is passed-through to the equilibrium total production of a good

3. ρ_{ij}^Q be the *Long-run cross quantity* (Lxq) pass-through, the amount an increase in exogenous quantity of one good is passed-through to the equilibrium production of another good

4. ρ_I^Q be the *Industry quantity* pass-through, the amount that an increase by the same small amount in the exogenous quantity of all goods is passed-through to equilibrium production of each good

These are the natural analogs of the notions of pass-through under Bertrand oligopoly that are introduced in Subsection III.A; see these for more detailed definitions (in the Bertrand context). The following corollaries maintain the Cournot analogs of the assumptions made for Bertrand oligopoly in Subsection III.A: production contraction for each firm and Hicksian $\nabla\kappa - \mathbf{I}$. It also assumes that firms are symmetric (considering perturbations around the symmetric quantities) and inverse demand is horizontal: $P_i(q_i, \mathbf{q}_{-i}) = \tilde{P}\left(q_i - g[\mathbf{q}_{-i}]\right)$ for some decreasing $\tilde{P}$.

Corollary 1. *$\rho_i^Q - 1, \rho_i^{Q,eq} - 1$ and $\rho_I^Q - 1$ all have the same signs and these are the same (opposite) as the sign of $\rho_{ij}^{Q,eq}$ if the goods are substitutes (complements). Therefore the entry of a new firm into the industry will reduce (increase) production of other firms if there is production absorption and the new good is a substitute (complement) for the existing goods or production amplification and the new good is a complement (substitute) for the existing goods. A merger (with no efficiencies) between two firms producing substitutes (complements) will lower (raise) the merging firm production and raise the production of other firms under production absorption, but lower their production under production amplification.*

Let the strength of interaction $s \equiv \left|\sum_{j\neq i} \frac{\partial g}{\partial p_j}\right|$. In the case of complements $s \leq 1$: the increase in consumer willingness to pay for a good from the production of all other goods increase must be fully offset of if the production of the good increases by a corresponding amount. Goods cannot be more than perfect complements in aggregate. In the case of substitutes $s \leq N - 1$: a good cannot be better than a perfect substitute for another.

Corollary 2. *$\rho_i^{Q,eq}$ and ρ_I^Q are increasing in ρ_i^Q. $\rho_i^{Q,eq}$ is increasing in s and decreasing in N. ρ_I is constant in N and when goods are complements (substitutes) is increasing (decreasing)* *in s under production absorption and decreasing (increasing) in s under production amplification. σ_{ij} is decreasing (increasing) in ρ_i^Q when goods are complements (substitutes), so long as $\rho_i^Q < 2$, and when goods are complements (substitutes) is increasing in absolute value in s and decreasing in absolute value in N.*

As N becomes large, $\rho_i^{Q,eq}$ approaches ρ_i^Q and σ_{ij} approaches 0. Under complements as s approaches 1, ρ_I^Q approaches 1.

While we rarely observe changes to exogenous quantities, the logic of appendix *Generalized Cournot-Stackelberg Models* Section II applies here as well: shocks to cost substitute for shocks to exogenous quantities. For example, assuming $\tilde{q}_i = 0$,

$$\rho_i^{Q,eq} \equiv \frac{dq_i^\star}{d\tilde{q}_i} = -\frac{m_i^\star}{q_i^\star}\frac{dq_i^\star}{dc_i}$$

This is true for any demand system (not just horizontal ones) and regardless of symmetry. However, translating these quantity effects into price effects requires the full matrix of first derivatives of demand, just as moving from Bertrand effects on prices to quantities requires the same matrix.

IV. Discrete choice

Gabaix et al. (2009) consider a model where consumer i's utility from good j is

$$u_{ij} = v - \beta_i p_j + \epsilon_{ij}$$

and consumer receive 0 utility from consuming no good. v is drawn from some distribution, the β_i's are drawn from some distribution i.i.d. across consumers and ϵ_{ij}'s are drawn from some distribution with p.d.f. f i.i.d. across consumers and goods. The number of goods are assumed to be large and the number of consumers continuously infinite. In an unpublished result Gabaix et al. (2009) also show the same results cited below hold for the non-parametric version of the original BLP with logarithmic rather than quasi-linear utility; however, for expositional simplicity they, and therefore we, stick with the modified version of the model developed by Petrin (2002). This is a close approximation to the original model because income effects are small over the range typically considered in industrial organization applications, as shown formally by Gabaix et al. (2009).

Proof of Theorem 6. The only property of horizontal demand systems I used in the proofs of Theorems 4 and 5 above were that

$$\frac{\frac{\partial Q_i}{\partial p_j}}{\frac{\partial Q_i}{\partial p_i}} = \frac{\frac{\partial \mu_i}{\partial p_j}}{\frac{\partial \mu_i}{\partial p_i}} \tag{19}$$

Furthermore, I only used the fact that this equation holds *at the symmetric equilibrium* not away from it. Thus all I need show is that this equation holds at equilibrium in the symmetric, non-parametric BLP model with many firms considered by Gabaix et al. (2009).

By the quotient rule

$$\frac{\partial \mu_i}{\partial p_j} = \frac{\frac{\partial^2 Q_i}{\partial p_i \partial p_j} Q_i - \frac{\partial Q_i}{\partial p_i}\frac{\partial Q_i}{\partial p_j}}{\left(\frac{\partial Q_i}{\partial p_i}\right)^2}$$

and

$$\frac{\partial \mu_i}{\partial p_i} = \frac{\frac{\partial^2 Q_i}{\partial p_i^2} Q_i - \left(\frac{\partial Q_i}{\partial p_i}\right)^2}{\left(\frac{\partial Q_i}{\partial p_i}\right)^2}$$

Thus to establish equation (19) it is sufficient to show that

$$\frac{\frac{\partial^2 Q_i}{\partial p_i \partial p_j}}{\frac{\partial^2 Q_i}{\partial p_i^2}} = \frac{\frac{\partial Q_i}{\partial p_j}}{\frac{\partial Q_i}{\partial p_i}} \tag{20}$$

at equilibrium prices.

Gabaix et al. (2009) show that at symmetric equilibrium prices with a large number of firms n

$$\frac{\partial Q_i}{\partial p_i} \sim -E\left[f(M_{n-1})\right] E[\beta_i]$$

where M_{n-1} is the maximum of $n-1$ draws from the distribution whose c.d.f. is f. Similarly

$$\frac{\partial Q_i}{\partial p_j} \sim \frac{1}{n-1} E\left[f(M_{n-1})\right] E[\beta_i]$$

$$\frac{\partial^2 Q_i}{\partial p_i^2} \sim -E\left[f'(M_{n-1})\right]$$

$$\frac{\partial^2 Q_i}{\partial p_i \partial p_j} \sim \frac{1}{n-1} E\left[f'(M_{n-1})\right]$$

This clearly (asymptotically) satisfies equation (20) and thus establishes the first part of the result.

Next I establish the connection of pass-through to the log-curvature of f. By the above formulae,

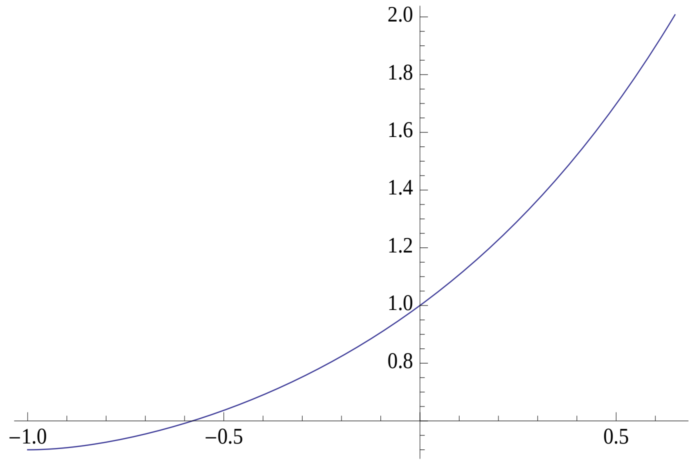

Figure 5: Sop is monotone in the tail of the distribution of idiosyncratic consumer values in the Gabaix et al. (2009) model

$$\rho_i = \frac{1}{2 - \frac{\frac{\partial^2 Q_i}{\partial p_i^2} Q_i}{\left(\frac{\partial Q_i}{\partial p_i}\right)^2}}$$

Thus Sop is monotone increasing in $\frac{\frac{\partial^2 Q_i}{\partial p_i^2} Q_i}{\left(\frac{\partial Q_i}{\partial p_i}\right)^2}$. Gabaix et al. (2009) show that in their context this expression can be rewritten as

$$\frac{\Gamma(2\gamma+3)}{2\Gamma(2+\gamma)^2}$$

where Γ is the Γ function and γ is the tail index of f, so long as $\gamma \geq -1$ ($S(x) \equiv \int_x^\infty f(x)dx$, the survivor function of f, is weakly convex) as Gabaix et al. (2009) assume. The tail index of f is defined as $\gamma \equiv \lim_{x\to\infty}\left(\frac{S(x)}{f(x)}\right)'$. But of course the pass-through rate associated with f in the monopoly case is just

$$\frac{1}{1 - \left(\frac{S(x)}{f(x)}\right)'}$$

as S is demand. Note that if $\gamma > 1$, MUC is violated. Thus I can restrict our attention to cases where $\gamma < 1$; in fact if $\gamma > .65$ it is easy to see that $\frac{\Gamma(2\gamma+3)}{2\Gamma(2+\gamma)^2} > 2$ and thus market-level demand violates MUC. Clearly if one distribution has uniformly higher pass-through than another, γ is also higher. Thus to show that Sop is higher (lower) when f has uniformly higher (lower) pass-through it is sufficient to show that $\frac{\Gamma(2\gamma+3)}{2\Gamma(2+\gamma)^2}$ is monotone increasing in γ for $\gamma \in [-1, .65]$. This is easily shown by graphing the function over this range, as demonstrated in Figure 5. This establishes the results .

References

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