

Monopoly

An appendix to *Pass-through as an Economic Tool*

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I. Mark-up contraction

Proof of Theorem 1. The monopolist's first-order condition, again, is

$$\mu(p) - p + c = 0$$

If this is decreasing in p then the first-order condition is sufficient for optimization. Clearly MUC guarantees this. Furthermore if a floor (ceiling) is imposed above (below) the unconstrained optimum, the monopolist will always charge this price as for every price above (below) p^* , the unconstrained optimum, the derivative of profits with respect to price is negative (positive).

Conversely suppose that demand does not satisfy weak MUC. Note that for some c (possibly negative), any positive value of $p - c$ can be achieved for any chosen $p \in \mathbb{R}_+$. Thus for any point on $\mu(p)$ there exists some cost c such that $p - c$ intersects that point. Taking the second derivative of the monopolist's profit function with respect to price yields

$$2Q'(p) + Q''(p)(p - c) \tag{12}$$

Suppose one is at a solution to equation (1) where $\mu' > 1$, which from above can happen if there is even a single point with $\mu' > 1$. Then expression (12) becomes

$$2Q'(p) + Q''(p)\mu(p)$$

This expression is strictly positive if

$$\frac{Q''(p)Q(p)}{(Q'(p))^2} > 2$$

But

$$\mu' = \frac{Q''(p)Q(p)}{(Q'(p))^2} - 1$$

Thus the second derivative of the monopolist's objective function is strictly positive if $\mu' > 1$ at this point, implying that this point is a local minimum and therefore clearly not an optimum. Let this point of intersection be \tilde{p} and let the monopolist's unconstrained optimal price be p^* assuming by hypothesis this exists. Either $\tilde{p} < p^*$ or $\tilde{p} > p^*$. Assume for the moment the first case. Then if a price ceiling is put on the monopolist at \tilde{p} then the monopolist will always charge a price strictly below \tilde{p} as she is prohibited from charging above it and \tilde{p} is a local minimum of profits, so she can only do better by charging below it. The result in the case that $\tilde{p} > p^*$ follows by the same logic.

II. Pass-through as an elasticity

To derive equation (3) from the text express profits in terms of mark-up by letting $m \equiv p - c$. Then $\pi(m + c) = mQ(m + c)$ and

$$-\frac{d^2\pi}{dm^2} \frac{m^2}{\pi} = (2Q' + mQ'') \left(-\frac{m}{Q} \right)$$

At optimal prices $m = -\frac{Q}{Q'}$ so this becomes

$$2 - \frac{QQ''}{(Q')^2} = 1 - \mu' = \frac{1}{\rho}$$

Note that this continues to hold if $\tilde{q} \neq 0$, one must simply redefine $\tilde{Q} = Q - \tilde{q}$.

III. Pass-through, surplus and tails

As steps towards proving Theorem 3, we first show a lemma which holds under the assumptions of the theorem.

Lemma 1. *Maintain the assumptions of Theorem 3. Then*

1. V is well-defined and finite.
2. $\lim_{p \rightarrow \bar{p}_-} \mu(p)Q(p) = 0$.

Proof. We first establish the result in the case that $\bar{p} \in \mathbb{R}$ and then in the case when $\bar{p} = \infty$.

$\bar{p} < \infty$

In this case 1) is trivial, as on the interval $[p, \bar{p}]$, Q is bounded above by $Q(p)$ and below by 0 and so its integral is finite. To prove 2) note that for any $p < \bar{p}$

$$\lim_{q \rightarrow \bar{p}_-} \mu(q) = \mu(p) + \lim_{q \rightarrow \bar{p}_-} \int_p^q \mu'(r) dr < \mu(p) + \bar{p} - p$$

The integral exists as μ' is bounded above by 1 and the integral is bounded below by $-\mu(p)$ as for all $q < \bar{p}$, $Q(q) > 0 > Q'(q)$ and thus $\mu(q) > 0$. Thus

$$\lim_{q \rightarrow \bar{p}_-} \mu(q)Q(q) < \lim_{q \rightarrow \bar{p}_-} (m[p] + \bar{p} - p)Q(q) = 0$$

as $Q(\bar{p}) = 0$.

$\bar{p} = \infty$

The assumption

$$\lim_{p \rightarrow \infty} \mu'(p) < 1$$

implies that there will be numbers p_0 and $a \in (0, 1)$ such that for all $p > p_0$ $m'(p) < a$. Using

the definition of $\mu(p)$,

$$\left(-\frac{Q(p)}{Q'(p)} \right)' < a,$$

$$\frac{Q(p)Q''(p) - Q'^2(p)}{Q'^2(p)} < a,$$

$$\frac{Q(p)Q''(p) - (1+a)Q'^2(p)}{Q'^2(p)} < 0,$$

$$\frac{Q(p)^{1+a}Q''(p) - (1+a)Q'^2(p)Q(p)^a}{Q^{2+2a}(p)} < 0,$$

$$\left(\frac{Q'(p)}{Q(p)^{1+a}} \right)' < 0.$$

By integration,

$$\frac{Q'(p)}{Q(p)^{1+a}} < \frac{Q'(p_0)}{Q(p_0)^{1+a}}.$$

For later purposes, rewrite this as

$$\mu(p)Q(p)^a < \mu(p_0)Q(p_0)^a. \quad (13)$$

At this point, however, the following form of the inequality will be more useful.

$$\begin{aligned} Q'(p)Q(p)^{-1-a} &< Q'(p_0)Q(p_0)^{-1-a}, \\ -\frac{1}{a}(Q(p)^{-a})' &< Q'(p_0)Q(p_0)^{-1-a}. \end{aligned}$$

Integrating this inequality further,

$$\begin{aligned} Q(p)^{-a} &> Q(p_0)^{-a} - aQ'(p_0)Q(p_0)^{-1-a}(p - p_0), \\ 0 < Q(p) &< \frac{1}{[Q(p_0)^{-a} - aQ'(p_0)Q(p_0)^{-1-a}(p - p_0)]^{\frac{1}{a}}}. \end{aligned}$$

The fact that

$$\lim_{p \rightarrow \infty} \frac{1}{[Q(p_0)^{-a} - aQ'(p_0)Q(p_0)^{-1-a}(p - p_0)]^{\frac{1}{a}}} = 0$$

implies that

$$\lim_{p \rightarrow \infty} Q(p)$$

exists and is equal to zero. Moreover, since

$$\int_{p_0}^{\infty} \frac{1}{[Q(p_0)^{-a} - aQ'(p_0)Q(p_0)^{-1-a}(p - p_0)]^{\frac{1}{a}}} dp$$

is finite, we see that

$$V(p) = \int_p^{\infty} Q(q) dq$$

is well-defined, finite, and positive establishing 1).

Now return to inequality (13). It implies that

$$0 < \mu(p)Q(p) < \mu(p_0)Q(p_0)^a Q(p)^{1-a},$$

for all $p > p_0$. Because

$$\lim_{p \rightarrow \infty} \mu(p_0)Q(p_0)^a Q(p)^{1-a} = \mu(p_0)Q(p_0)^a \lim_{p \rightarrow \infty} Q(p)^{1-a} = 0,$$

we can conclude that

$$\lim_{p \rightarrow \infty} \mu(p)Q(p)$$

exists and is equal to zero. This proves 2).

Given this lemma, the proof of Theorem 3 is straightforward.

Proof of Theorem 3. The result states that

$$\int_p^{\bar{p}} Q(q) dq = \frac{\mu(p)Q(p) \int_p^{\bar{p}} Q(q) dq}{\int_p^{\bar{p}} \frac{Q(q)}{\rho(q)} dq}$$

and so we must show that for $p < \bar{p}$

$$\mu(p)Q(p) = \int_p^{\bar{p}} \frac{Q(q)}{\rho(q)} dq$$

Substituting into the right hand side expression and invoking Lemma 1

$$\begin{aligned} \int_p^{\bar{p}} \frac{Q(q)}{\rho(q)} dq &= \int_p^{\bar{p}} Q(q) (1 - \mu'(q)) dq = \int_p^{\bar{p}} \left(1 - \left(-\frac{Q(q)}{Q'(q)} \right)' \right) Q(q) dq = \\ &= \int_p^{\bar{p}} \left(1 + \frac{Q'^2(q) - Q(q)Q''(q)}{Q'^2(q)} \right) Q(q) dq = \int_p^{\bar{p}} \frac{2Q(q)Q'^2(q) - Q^2(q)Q''(q)}{Q'^2(q)} dq = \\ &= \int_p^{\bar{p}} \left(\frac{Q^2(q)}{Q'(q)} \right)' dq = \int_{\bar{p}}^p (\mu(q)Q(q))' dq = \mu(p)Q(p) - \lim_{q \rightarrow \bar{p}^-} \mu(p)Q(p) = \mu(p)Q(p) \end{aligned}$$

The second part of Theorem 3 follows directly by duality with the first part.

It may appear a bit mysterious that the size of tails are determined by an average of any property of demand. Intuitively the reason is that log-curvature of demand, which the pass-through rate is a measure of, determines how quickly it is “forced” towards 0. The following theorem formalizes this claim, which is also made in the text.

Theorem 7. *Suppose that $\lim_{p \rightarrow \bar{p}} \rho(p)$ exists and is equal to l . Then if $l < 1$, $\bar{p} < \infty$ and if $l > 1$, $\bar{p} = \infty$.*

Proof. First suppose $l < 1$. Then there $\exists \tilde{p} < \bar{p}, l < l' < 1 : \forall p > \tilde{p}$ we have $\rho(p) < l'$. Thus $\forall p > \tilde{p}$, $\mu'(p) < 1 - \frac{1}{l'}$ so $\mu(p) < \mu(\tilde{p}) + (1 - \frac{1}{l'}) (p - \tilde{p})$. Let $\hat{p} \equiv \tilde{p} + \frac{\mu(\tilde{p})l'}{1-l'}$. It must be that

$\bar{p} < \hat{p} < \infty$. Otherwise, $\mu(\hat{p}) < 0$ which contradicts the definition of Q being derived from a survival function.

Now suppose that $l > 1$ and that, contrary to the proof, $\bar{p} < \infty$. Then $\exists \tilde{p} < \bar{p}, 1 < l' < l : \forall p > \tilde{p}$ we have $\rho(p) > l'$. Thus, by the reasoning above, $\forall p > \tilde{p}, \mu(p) > \mu(\tilde{p}) > 0$. But because $\bar{p} < \infty$

$$Q(\bar{p}) = Q(\tilde{p})e^{-\int_{\tilde{p}}^{\bar{p}} \frac{1}{\mu(q)} dq} \geq Q(\tilde{p})e^{-\frac{\bar{p}-\tilde{p}}{\mu(\bar{p})}} > 0$$

But this contradicts the definition of \bar{p} , namely that $Q(\bar{p}) = 0$, completing the proof.