Online Appendix to
Materialistic Genius and Market Power

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Note bene: Given the length requisite to provide a detailed proof of all of the results in this appendix, many of the proofs are closer to proof sketches than full proofs. While we hope these are sufficient to persuade the reader of the truth of the results, more details of any proof are available on request. This is particularly true of Section 2, at the beginning of which we supply another note similar to this one further explaining the reason for the light touch there.

1 Re-statement of Results to be Proven

Lemma 1: *Under an arbitrary differentiable pricing policy* $a(\cdot,\cdot)$*, incentive compatibility requires that* T *be weakly monotone in both its arguments and that rewards be constant along any curve* $\tilde{\sigma}(\tilde{m})$ *obeying for all* $\tilde{m}$

$$\tilde{m}'(\tilde{\sigma};\sigma,m) = -\frac{\tilde{m}(\tilde{\sigma})}{\tilde{\sigma}\epsilon\left(a\left(\tilde{\sigma},\tilde{m}\left(\tilde{\sigma}\right)\right)\right)}$$

and passing through a point (σ,m)*, except that, on a countable set of such curves, this may fail. However, changing* T *along such a set has no effect on the social planner's value function and thus* an *optimum for the social planner subject to incentive compatibility obeys this constraint. Every point* (σ,m) *is assigned to a unique such curve, whose value may be defined by the point at which it intersects the* $45^\circ, \sigma = m$ *line.*

Corollary 1: *Under proportional pricing, incentive compatibility requires* T *being constant along curves of the form* $k = \sigma^{\frac{1}{1+\epsilon(a)}} m^{\frac{\epsilon(a)}{1+\epsilon(a)}}$ *and monotonically increase in* k*.*

Lemma 2: $W(a)$ *is differentiable for all* $a \in (0,1)$ *and its derivative may be evaluated by the envelope theorem, holding* $T^\star$ *fixed. Formally:*

$$W'(\hat{a}) = \frac{\partial}{\partial a}\left[\int_\sigma\int_m\int_{c=0}^{T^\star\left(\sigma^{\frac{1}{1+\epsilon(a)}} m^{\frac{\epsilon(a)}{1+\epsilon(a)}};\hat{a}\right)}\left[\sigma m S(a) - c\right] f(c,\sigma,m)\,dc\,dm\,d\sigma\right]\Bigg|_{a=\hat{a}}$$

Lemma 3: *Suppose that* (q,p) *is a strict MOPSD of* $\mathbb{R}^2_+$ *with* $q(\sigma,m) = \sigma Q\left(\frac{p(\sigma,m)}{m}\right)$*. Then the conditions in Lemma 1 are necessary and sufficient for incentive compatibility. Almost conversely if* (q,p) *is differentiable then any incentive compatible* T *implementing* (q,p) *is constant over any neighborhood where* (q,p) *fails to be MOPSD.*

Corollary 2: *Under proportional pricing, the incentive compatibility constraint is equivalent to the relaxed constraints of Corollary 1.*

Proposition 1: *The derivative of social welfare* W^λ *maximized over transfers (with maximizer* T^λ*) with respect to a is proportional to expression (4) if*

$$(1-\lambda)\frac{\epsilon'}{(1+\epsilon)^2}E_{k,\tilde{f}}\left[k^4\frac{T^{\lambda\prime}}{k}E\left(x^{\frac{1-\epsilon}{1+\epsilon}}|c=T^\lambda\right)\left[E\left(\log(x)|c=T^\lambda\right)-E\left(\log(x)|c<T^\lambda\right)\right]\right]$$

is also added to it and that η *is replaced by* $1-\lambda+\eta$ *.*

Proposition 2: *Let the social surplus generated by an innovation be* $[\gamma+S(a)]\,\sigma m$*. Then Corollary 3 applies except that* $T^\star(k)=[\gamma+S(a)]\,k^2E_{x,\tilde{f}}\left[x^{\frac{1-\epsilon(a)}{1+\epsilon(a)}}\middle|\,k,c=T^\star(k)\right]$ *rather than*

$S(a)k^2E_{x,\tilde{f}}\left[x^{\frac{1-\epsilon(a)}{1+\epsilon(a)}}\middle|\,k,c=T^\star(k)\right]$.

Proposition 3: *Let the social surplus generated by an innovation be* $(1+\gamma)S(a)\sigma m$*. Then Corollary 3 applies except that* $T^\star(k)=(1+\gamma)S(a)k^2E_{x,\tilde{f}}\left[x^{\frac{1-\epsilon(a)}{1+\epsilon(a)}}\middle|\,k,c=T^\star(k)\right]$ *rather than*

$S(a)k^2E_{x,\tilde{f}}\left[x^{\frac{1-\epsilon(a)}{1+\epsilon(a)}}\middle|\,k,c=T^\star(k)\right]$ *and the second, ex-post distortion term is scaled up (or down) by* $1+\gamma$*.*

Proposition 4: *The first-order net benefits of increasing a in the mutually-exclusive innovations model are proportional to*

$$\underbrace{(1-\epsilon_h)\,\epsilon_\epsilon}_{\text{incentivizing high quality}}\quad-\quad\underbrace{\epsilon_{\epsilon_{\epsilon_h}}\epsilon_S}_{\text{ex-post distortion}}$$

where ϵ_f *is the elasticity of the function* f*.*

Proposition 5: *In the model with ex-ante uncertain demand, suppose* $T^\star$ *is differentiable at a and that conditional on the fact that an innovation ends up on isoreward curve k and is marginal, the average distribution of* $\hat{k}$ *and* $\hat{x}$ *that the social planner infers the innovator anticipated are independent and the distribution of x does not depend on k. Assume also a similar independence condition on each s. Then* $W'(a)\propto$

$$E_k\left[k^4\left[(1-\epsilon)\frac{E_s\left[E_{\hat{k}}\left[\hat{k}^2\middle|s\right]\middle|k\right]}{k^2}\frac{2\epsilon'}{(1+\epsilon)^2}\frac{\eta\left(T^\star|k\right)Cov_s\left(E_x[\log(x)|s],E_x\left[x^{\frac{1-\epsilon}{1+\epsilon}}|s\right]\middle|k,c=T^\star(k)\right)}{(1-\epsilon)}-\epsilon QE_x\left[x^{\frac{1-\epsilon}{1+\epsilon}}|k,c\leq T^\star(k)\right]x^{\widetilde{\frac{1-\epsilon}{1+\epsilon}}}\right]\right]$$

Proposition 6: *Holding fixed quantity, a monopolist will always have too little incentive to supply m, but, so long as demand is log-concave (linear-cost pass-through is less than 1), a monopolist will have excessive incentive to supply* σ*.*

Proposition 7: *If* $T^\star$ *is differentiable in k, the first-order net benefit of increasing a at* $\left(\hat{k},\hat{x}\right)$ *beginning from a strict MOPSD pricing policy* $a(\cdot,\cdot)$ *is, if* $x\geq1$*, proportional to*

$$\frac{T^{\star\prime}}{\hat{k}}\frac{\epsilon'\left(\hat{k},\hat{x}\right)}{\hat{x}\left[1+\epsilon\left(\hat{k},\hat{x}\right)\right]^2}\left[1+\log(\hat{x})\overline{\frac{\epsilon'\left(\epsilon_{a_m}-\epsilon_{a_\sigma}\right)}{(1+\epsilon)^2}}\left(\hat{k},\hat{x}\right)\right]\left(\frac{E_{\tilde{f},x>\hat{x}}\left[Sx^{\frac{1-\tilde{\epsilon}}{1+\tilde{\epsilon}}}\right]}{E_{\tilde{f},x}\left[Sx^{\frac{1-\tilde{\epsilon}}{1+\tilde{\epsilon}}}\right]}-1\right)\eta\left(T^\star|\hat{k}\right)-Q\epsilon\hat{x}^{\frac{1-\tilde{\epsilon}}{1+\tilde{\epsilon}}}H\left(\hat{x}|\hat{k},T^\star\right)\frac{E\left[\tilde{f}\left(\hat{x}|\hat{k},c\right)|c<T^\star,\hat{k}\right]}{\tilde{f}\left(x|c=T^\star,\hat{k}\right)}$$

where H *is the (conditional) hazard rate of* x *under* $\tilde{f}$ *and if* $x < 1$*, proportional to*

$$\frac{T^{\star\prime}}{\hat{k}} \frac{\epsilon'(\hat{k},\hat{x})}{\hat{x}\left[1+\epsilon(\hat{k},\hat{x})\right]^2} \left[1+\log(\hat{x}) \overline{\frac{\epsilon'(\epsilon_{a_m}-\epsilon_{a_\sigma})}{(1+\epsilon)^2}}(\hat{k},\hat{x})\right] \left(\frac{E_{\tilde{f},x>\hat{x}}\left[Sx^{\frac{1-\tilde{\epsilon}}{1+\tilde{\epsilon}}}\right]}{E_{\tilde{f},x}\left[Sx^{\frac{1-\tilde{\epsilon}}{1+\tilde{\epsilon}}}\right]} - 1 \right) \eta\left(T^\star|\hat{k}\right) - Q\epsilon\hat{x}^{\frac{1-\tilde{\epsilon}}{1+\tilde{\epsilon}}} H\left(\hat{x}|\hat{k},T^\star\right) \frac{E\left[\tilde{f}(\hat{x}|\hat{k},c)|c<T^\star,\hat{k}\right]}{\tilde{f}(x|c=T^\star,\hat{k})}$$

where R *is the reversed hazard rate. A necessary condition for a strict MOPSD (no-bunching) solution is that these equal zero at every point in the plane.*

Proposition 8: *Starting from any strict MOPSD policy* a *with* $a < 1$ *and assuming* $T^\star$ *is differentiable at this policy, the first variation of* W *in the direction* $a + \frac{(1+\epsilon)^2}{\epsilon'}$ *(uniform decrease in* $\frac{1}{1+\epsilon}$*) is proportional to*

$$E_{k,\tilde{f}}\left[k^4\left(\frac{\eta T^{\star\prime}}{k} Cov_{x,\tilde{f}}\left[\frac{\epsilon'}{(1+\epsilon)^2}\left(1+\overline{\log\left(\frac{x}{z}\right)(\epsilon_{a_m}-\epsilon_{a_\sigma})}\right)\log(x), Sx^{\frac{1-\tilde{\epsilon}}{1+\tilde{\epsilon}}}\right]\right) - E_{\tilde{f},x}\left[Q\epsilon x^{\frac{1-\tilde{\epsilon}}{1+\tilde{\epsilon}}}\middle| k, c<T^\star(k)\right] E_{\tilde{f},x}\left[Sx^{\frac{1-\tilde{\epsilon}}{1+\tilde{\epsilon}}}\right]\right]$$

Proposition 9: *Suppose that for all* k, c *and a fixed* a*,*

$$Cov_{x,\tilde{f}}\left[x^{\frac{1-\epsilon(a)}{1+\epsilon(a)}}, \frac{\partial \log(f)}{\partial c}\middle| k, c\right] \leq \frac{1}{k^2 S(a)} \tag{1}$$

and

$$2E_{x,\tilde{f}}\left[x^{\frac{1-\epsilon(a)}{1+\epsilon(a)}}|k,c\right] \geq -k\, Cov_{x,\tilde{f}}\left[x^{\frac{1-\epsilon(a)}{1+\epsilon(a)}}, \frac{\partial \log(\tilde{f})}{\partial k}\middle| k, c\right] \tag{2}$$

Then the optimal reward function $T^\star(k;a)$*, given* a*, is defined for each* k *by the unique value at which* $D(\cdot;k,a)$ *and* $S(\cdot;k,a)$ *intersect if* $D(0;k,a) > S(0;k,a), D(1;k,a) < S(1;k,a)$ *for all* k, a*. If* $D(0;k,a) \leq S(0;k,a)$ *then the optimal reward is* 0 *and if* $D(1;k,a) \geq S(1;k,a)$ *then the optimal reward is anything exceeding the maximal possible cost* $\overline{c}$ *given* a *and* k*, or infinite if no such cost exists.*

Proposition 10: *Suppose that at least one of the conditions of Proposition 9 is obeyed. Then if the expectation is taken over all* c*'s other than the (at most) countable set where* ${T^\star}^{-1}$ *is not well-defined* $W'(a) \propto$

$$\frac{S\epsilon'}{[1+\epsilon]^2} E_{c<\overline{c},\tilde{f}}\left[\left[{T^\star}^{-1}(c;a)\right]^3 \eta\, Cov_{\tilde{f},x}\left[\log(x), x^{\frac{1-\epsilon}{1+\epsilon}}\right] - Q\epsilon E_{x,k,\tilde{f}}\left[k^2 x^{\frac{1-\epsilon}{1+\epsilon}}\middle| k \geq {T^\star}^{-1}(c;a), c\right] E_{x,k,\tilde{f}}\left[k^2 x^{\frac{1-\epsilon}{1+\epsilon}}\right]\right] \tag{3}$$

where, again, if not otherwise stated, expectations are taken over the marginal set for which $T^\star(k;a) = c$*,* η^k *is the elasticity of innovation supply and* $\overline{c} \equiv \lim_{x\to\infty} T^\star(x)$ *(typically* ∞*).*

Corollary 3: *Assuming* $T^\star$ *is differentiable in* k *at* a*,* $W'(a) \propto$

$$E_{k,\tilde{f}}\left[k^4\left[\underbrace{(1-\epsilon)\frac{T^{\star\prime}}{k}\frac{\epsilon'}{(1+\epsilon)^2}\frac{\eta\, Cov_{x,\tilde{f}}\left(\log(x), x^{\frac{1-\epsilon}{1+\epsilon}}\right)}{1-\epsilon}}_{sorting} - \underbrace{\epsilon Q E_{x,\tilde{f}}\left(x^{\frac{1-\epsilon}{1+\epsilon}}\right) E_{x,\tilde{f}}\left(x^{\frac{1-\epsilon}{1+\epsilon}}\middle| c < T^\star\right)}_{ex\text{-}post\ distortion}\right]\right] \tag{4}$$

where η *is the elasticity of innovations with respect to reward and all quantities inside the expectation are evaluated conditional on* a*,* k *and* $c = T^\star(k;a)$ *where not explicitly stated. As usual, a necessary condition for the optimal choice of* a *is that this equal* 0*.*

Proposition 11: *Let*

$$C(a) \equiv E_{k,\tilde{f}} \left[k^3 T^{\star\prime}(k;a) \frac{Cov_{x,\tilde{f}}\left(\log(x), x^{\frac{1-\epsilon(a)}{1+\epsilon(a)}}\right)}{1-\epsilon(a)} \right]$$

and

$$M(a) \equiv \frac{1}{E_{k,\tilde{f}} \left[\frac{E_{x,\tilde{f}}\left(x^{\frac{1-\epsilon(a)}{1+\epsilon(a)}} | c < T^\star(k;a), k \right) E_{x,\tilde{f}}\left(x^{\frac{1-\epsilon(a)}{1+\epsilon(a)}} | c = T^\star(k;a), k \right)}{\eta(T^\star(k;a); k, a)} \right]}.$$

W is quasi-concave if for all $a \in (0,1)$

$$\frac{d \log(CM)}{da} < \frac{\epsilon(a)}{a} + \epsilon'(a) \left(\frac{1 + 3\epsilon(a) - 2\epsilon^2(a)}{\epsilon(a)\left[1 - \epsilon^2(a)\right]} \right) - \frac{\epsilon''(a)}{\epsilon'(a)}$$

This condition always holds for a sufficiently close to either 0 *or* 1*.*

Theorem 1: *Either optimal rewards are constant everywhere and* $a = 0$ *or the optimal value of a is strictly between* 0 *and* 1*.*

Theorem 2 (Friedman's Conjecture)**:** *Let* π *be the monopoly profit associated with an innovation and* V_1 *the* value of materialistic genius near monopoly *be*

$$\frac{E_{\pi,\tilde{f}} \left[\pi^2 \, Var_{x,\tilde{f}} \left(\log(x) \middle| \, k = \sqrt{\frac{\pi}{Q(1)}}, c = \frac{S(1)}{Q(1)}\pi \right) \eta \left(\frac{S(1)}{Q(1)}\pi \middle| \, k = \sqrt{\frac{\pi}{Q(1)}}, a = 1 \right) \right]}{E_{\pi,\tilde{f}}\left[\pi^2\right]}.$$

Then, within a class of distributions for which W is quasi-concave, those with sufficiently high values of V_1 *have* $a^\star$ *arbitrarily close to* 1*. That is, as the value of materialistic genius near monopoly grows large, monopoly pricing becomes optimal.*

Theorem 3 (Partial converse of Friedman's Conjecture)**:** *Let* V_0*,* the value of materialistic genius near ex-post efficiency *be*

$$\frac{E_{\sigma,f} \left[\sigma^4 \frac{\log(T^\star(\sigma))' \, Cov_{m,f}(\log(m), m | \sigma, c = T^\star(\sigma)) \eta(T^\star(\sigma) | \sigma, a = 0)}{\sigma E(m | \sigma, c < T^\star)} \right]}{E_{\sigma,f}(\sigma^4)}$$

Then, within a class of distributions for which W is quasi-concave, those with sufficiently low values of V_1 *have* $a^\star$ *arbitrarily close to* 0*. That is, as the value of materialistic genius near ex-post efficiency grows small, ex-post efficiency becomes optimal.*

Theorem 4: *At global ex-post efficient pricing there is a local incentive at all points to raise prices at any* (σ, m) *for which* $T^\star(\sigma)$ *is not constant in the neighborhood of* σ*. At global monopoly pricing there is a local incentive at all points to lower prices.*

Theorem 5: *Beginning from any a sufficiently, uniformly close to uniform monopoly pricing (*$a = 1$ *everywhere) but with* $a < 1$ *everywhere, if* V_1 *(of Theorem 2) is sufficiently large there are first-order benefits from moving a small amount (uniformly) towards uniform monopoly pricing. Beginning from any strict MOPSD a sufficiently, log-uniformly close to ex-post efficiency (*$a = 0$ *everywhere) but with* $a > 0$ *everywhere, if* V_0 *(of Theorem 3) is sufficiently small there are first-order benefits from moving a small amount (uniformly) towards ex-post efficiency.*

Corollary 4: *If $T^\star$ is differentiable in k, the first-order net benefit of increasing a at $\left(\hat{k},\hat{x}\right)$ given beginning from a proportional pricing policy a is, if $x \geq 1$, proportional to*

$$\frac{\epsilon' T^{\star\prime}\left(\hat{k}\right)}{(1+\epsilon)^2 \hat{k}\hat{x}^{\frac{2}{1+\epsilon}}}\left(\frac{E_{\tilde{f},x>\hat{x}}\left[x^{\frac{1-\epsilon}{1+\epsilon}}\middle|\hat{k},T^\star\right]}{E_{\tilde{f},x}\left[x^{\frac{1-\epsilon}{1+\epsilon}}\middle|\hat{k},T^\star\right]}-1\right)\eta\left(T^\star|\hat{k}\right)-Q\epsilon H\left(\hat{x}|\hat{k},T^\star\right)\frac{E\left[\tilde{f}\left(\hat{x}\middle|\hat{k},c\right)\middle|c<T^\star,\hat{k}\right]}{\tilde{f}\left(x\middle|T^\star,\hat{k}\right)}$$

and if $x < 1$

$$\frac{\epsilon' T^{\star\prime}\left(\hat{k}\right)}{(1+\epsilon)^2 \hat{k}\hat{x}^{\frac{2}{1+\epsilon}}}\left(1-\frac{E_{\tilde{f},x<\hat{x}}\left[x^{\frac{1-\epsilon}{1+\epsilon}}\middle|\hat{k},T^\star\right]}{E_{\tilde{f},x}\left[x^{\frac{1-\epsilon}{1+\epsilon}}\middle|\hat{k},T^\star\right]}\right)\eta\left(T^\star|\hat{k}\right)-Q\epsilon R\left(\hat{x}|\hat{k},T^\star\right)\frac{E\left[\tilde{f}\left(\hat{x}\middle|\hat{k},c\right)\middle|c<T^\star,\hat{k}\right]}{\tilde{f}\left(x\middle|T^\star,\hat{k}\right)}$$

2 Necessary Conditions for Incentive Compatibilty

Nota bene: the proof in this section was written jointly by Weyl and Michal Fabinger. We are very grateful to Michal and these proofs will be spun off as a separate paper (Fabinger and Weyl, 2011) which this paper will then simply cite at some point in the near future. For this reason, this subappendix provides more of a sketch of a proof of Lemma 1 than a fully detailed proof.

Our strategy for establishing Lemma 1, using a series of sublemmata, consists of five steps. In describing them, we use the terminology "conjectured isoreward curve" to refer to curves defined by the differential equations in the lemma statement and "actual isoreward curve" to describe a curve along which T must in fact be constant. We also repeatedly rely on the fact that conjectured isoreward curves pass through each point in the (σ, m) plane, smoothly deform, do not intersect,and intersect the 45° line at exactly one point. These are classical results on the solutions to classes of first-order ordinary differential equations of the form $\tilde{m}'(\tilde{\sigma};\sigma,m) = -\frac{\tilde{m}(\tilde{\sigma})}{\tilde{\sigma}\epsilon(a(\tilde{\sigma},\tilde{m}(\tilde{\sigma})))}$ where $\epsilon \geq 0$ and smooth and the curve passes through (σ, m).

1. We extend the classic theorem of Young and Young (1924) to show that the set of discontinuities of a monotone function of several variables can be placed along a countable set of non-increasing curves, which we call *extended discontinuity curves.*
2. We argue that any curve of discontinuities must lie entirely along a conjectured isoreward curve, since if it were to "cut through" an isoreward curve, it would offer a profitable opportunity for deviation.
3. Because each extended discontinuity curve can contain at most a countable number of curves (or almost-curves, curves from which a set of measure zero has been removed) of discontinuity, these lie along at most a countable number of conjectured isoreward curves.
4. We show that any conjectured isoreward curve not including discontinuities must be an actual isoreward curve.
5. Finally, we conclude that the set of conjectured isoreward curves failing to be actual isoreward curves is at most countable and, as such, can be disregarded.

We use a concept that, while in some sense common in economics, does not have a standard name we are aware of and thus we introduce the terminology that follow.

Definition 1: *A* non-increasing curve *is a curve in* $\mathbb{R}^2$ *with the property that if* (x, y) *belongs to the curve and* $x' > x, y' > y$ *then* (x', y') *do not belong to the curve. A* non-increasing almost-curve *is a non-increasing curve from which a set of one-dimensional Lebesgue measure zero has been removed.*

Note that a demand curve is really a non-increasing curve, not a non-increasing function, if it may be perfectly inelastic over some range. For verbal economy we will refer to a point at which T fails to be continuous in (σ, m) (and not merely those in which it is discontinuous in some direction) as points of discontinuity of T. Furthermore by countable we mean any set of cardinality less than or equal to that of the integers.

Sublemma 1: *The points of discontinuity of* T *are a subset of a countable set of non-increasing curves, which we refer to as* extended discontinuity curves*, through the* (σ, m) *plane.*

Proof. The proof of this sublemma is fairly involved so we begin with a brief outline:

1. We begin by considering a finite box on the real plane and investigate the maximal number of non-increasing curves needed to accommodate all of its points of discontinuity of a size larger than a fixed amount.
2. Along any non-increasing curve through this box there must be no more than a finite number of discontinuities larger than some size in any direction. In the step that forms the heart of the proof, we show this implies that the number of non-increasing curves needed to accommodate all points of sufficiently large discontinuity is finite.
3. We then take the limit as the size of discontinuities grows small, obtaining a countable number of non-increasing curves for each box.
4. Finally we take the limit as the box grows to encompass the full plane, obtaining a countable union of countable sets of non-increasing curves.

Note that by the monotonicity of T in σ and m, any discontinuity of T in a direction from the $(-, -)$ quadrant to the $(+, +)$ quadrant of a point must be a jump discontinuity and therefore have a (supremal) size s (in some such direction). We refer to such discontinuities as *monotone discontinuities of size* s. Let us refer to the set of all monotone discontinuities of size great than δ contained in closed box $[0, N] \times [0, N]$ as $D(N, \delta)$ and to its closure as $\overline{D}(N, \delta)$.

We will construct a set of no more than $\left\lfloor \frac{T(N,N)}{\delta} \right\rfloor$ (where $\lfloor x \rfloor$ represents the greatest integer function) non-increasing curves, the union of which contains $\overline{D}(N, \delta)$. We do so inductively and thus define the base inductive case $\overline{D}_0(N, \delta) \equiv \overline{D}(N, \delta)$.

Let the *weakly undominating set* of $\overline{D}_i(N, \delta)$,

$$U\left(\overline{D}_i\left(N, \delta\right)\right) \equiv \left\{(\sigma, m) \in \overline{D}_i(N, \delta) : \nexists(\sigma', m') \in \overline{D}_i(N, \delta) s.t. (\sigma', m') \ll (\sigma, m)\right\}$$

Note that $U\left(\overline{D}_i\left(N, \delta\right)\right)$ is a closed subset of a non-increasing curve:

1. $U\left(\overline{D}_i\left(N, \delta\right)\right)$ is closed because it is defined by the failure of strict inequalities among elements of a closed set.
2. It is (at most) one-dimensional as any two-dimensional closed subset of $\mathbb{R}^2$ contains a interior point and thus dominates another point.

3. It may be chosen to be non-increasing because it is undominating.

Thus $U\left(\overline{D}_i\left(N,\delta\right)\right)$ is either connected, and thus forms a non-increasing curve itself, or there is a non-increasing curve, of which it is a subset, which connects it. While there may be many such curves, choose one and note that because the curve is non-increasing and every point in $U\left(\overline{D}_i\left(N,\delta\right)\right)$ is undominating, this curve may also be chosen to be undominating. Call this *extended* curve $\mathfrak{c}_{i+1}(N,\delta)$, define $\overline{D}_{i+1}(N,\delta)$ as the closure of $\overline{D}_i(N,\delta)\backslash\mathfrak{c}_{i+1}(N,\delta)$ and repeat this process unless $\overline{D}_{i+1}(N,\delta)=\emptyset$.

$\overline{D}_{\left\lfloor\frac{T(N,N)}{\delta}\right\rfloor}(N,\delta)$ must be empty. To see this, suppose that this were not the case. Then there would be some point $(\sigma,m)\in\overline{D}_{\left\lfloor\frac{T(N,N)}{\delta}\right\rfloor}(N,\delta)$ which is actually in $D(N,\delta)$, not merely its closure, as only closed sets are removed at each step and thus mere limit points will always be removed. Therefore $(\sigma,m)\notin U\left(\overline{D}_{\left\lfloor\frac{T(N,N)}{\delta}\right\rfloor-1}(N,\delta)\right)$ and thus strictly dominates some point in $\overline{D}_{\left\lfloor\frac{T(N,N)}{\delta}\right\rfloor-1}(N,\delta)$. This point again cannot be merely a limit point but must actually be an element of $D(N,\delta)$ as dominance is strict. But this point in turn must strictly dominate a point of $\overline{D}_{\left\lfloor\frac{T(N,N)}{\delta}\right\rfloor-2}(N,\delta)\cap D(N,\delta)$ and so on. Thus, by the transitivity of strict dominance, (σ,m) lies atop a hierarchy of (at least) $\left\lfloor\frac{T(N,N)}{\delta}\right\rfloor$ strict dominance relations among points in the box.

However, because the dominance is strict, it is possible to draw a non-decreasing curve (analogous to a non-increasing curve) between these points hitting each at from any direction from the $(-,-)$ to the $(+,+)$ quadrant. Thus, by the monotonicity of T, $T(N,N)\geq\left(\left\lfloor\frac{T(N,N)}{\delta}\right\rfloor+1\right)\delta>T(N,N)$ which is a contradiction. Thus $\overline{D}_{\left\lfloor\frac{T(N,N)}{\delta}\right\rfloor}(N,\delta)$ is in fact empty.

Thus $\overline{D}(N,\delta)\subset\bigcup_{i=1}^{\left\lfloor\frac{T(N,N)}{\delta}\right\rfloor}\mathfrak{c}_i(N,\delta)$ as desired. Thus clearly the set of all monotone discontinuities T (of any size) is a subset of $\lim_{N\to\infty}\bigcup_{j=1}^{N}\bigcup_{i=1}^{\lfloor NT(N,N)\rfloor}\mathfrak{c}_i\left(N,\frac{1}{N}\right)$. But by Theorem 4 of Young and Young (1924), any point at which a monotone function of two variables is continuous in all directions from the $(-,-)$ to $(+,+)$ quadrants it is continuous from all directions. Thus the set of discontinuities of T is a subset of $\lim_{N\to\infty}\bigcup_{j=1}^{N}\bigcup_{i=1}^{\lfloor NT(N,N)\rfloor}\mathfrak{c}_i\left(N,\frac{1}{N}\right)$, a countable union of non-increasing extended discontinuity curves. □

Sublemma 2: *Every non-increasing curve or almost-curve of discontinuities of T is a subset of some conjectured isoreward curve.*

Proof. If the discontinuity curve consists of a point, it clearly lies along a conjectured isoreward curve. For non-point curves and almost-curves, our proof strategy for this sublemma is again quite intricate so we again outline it before diving in:

1. We begin by investigating discontinuity curves (rather than almost-curves) in a finite box consisting entirely of discontinuities of size at least δ. In particular we focus on a single curve lying along the highest extended discontinuity curve, to avoid any interference by other curves of discontinuity.

2. We use an inductive argument to show that no curve of discontinuity of at least size δ may "cut through" a conjectured isoreward curve as a series of local dominance relationships of the smooth imitation frontiers might then be established which would force T to take an infinite value in this finite region.

3. Because the discontinuity curve may not cut through, it must either be locally differentiable or kinked (in a Dini derivative sense). But if kinked a local value can be found which will cut through a sufficiently close isoreward curve, contradicting the prior step and establishing the differentiability of the discontinuity curve.

4. Any differentiable discontinuity curve which cannot cut through conjectured isoreward curves must be everywhere tangent to any such curve it intersects, determining a differential equation "pinning" the discontinuity curves to the conjectured isoreward curves.

5. The argument extends without any trouble to almost-curves, then to other extended discontinuity curves in the same finite region and then over all such curves as the region grows large and increment size small.

First we define the notion of tangency we are interested in, using the standard notation for Dini derivatives: D_-, D^-, D_+, D^+ represent respectively the lower left, upper left, lower right and upper right Dini derivatives. For a treatment of Dini derivatives, which are not used commonly in economics, see for example Royden (1988). We say that a non-increasing curve is *tangent* to a conjectured isoreward curve at point (σ, m) if

1. The discontinuity curve is a function $\check{m}(\check{\sigma})$ in a neighborhood about (σ, m).

2. The discontinuity curve does not cut the conjectured isoreward curve from above at (σ, m): $D_-\check{m}(\sigma) < \tilde{m}'(\sigma; \sigma, m) \implies D_+\check{m}(\sigma) \geq \tilde{m}'(\sigma; \sigma, m)$.

3. The discontinuity curve does not cut the conjectured isoreward curve from below at (σ, m): $D^-\check{m}(\sigma) > \tilde{m}'(\sigma; \sigma, m) \implies D^+\check{m}(\sigma) \leq \tilde{m}'(\sigma; \sigma, m)$.

Consider $D\left(N, \frac{1}{N}\right)$ as defined above and its highest extended discontinuity curve and any non-point curve in this extended discontinuity curve, if such exists. Choose any interior point of the curve; this is some (σ, m) corresponding to an isoreward and imitation frontier. Suppose that the discontinuity curve fails to be tangent to this conjectured isoreward curve at (σ, m). There are two possibilities. Either the curve is locally a correspondence or it is a continuous function whose upper and lower Dini derivatives from each side exist, but they fail to obey the specified bounds. We focus on the second case to begin with, as the argument in the first case is essentially a special case of the second. Furthermore, we focus on showing the impossibility of a cut from above in the case when T is continuous at (σ, m) from the $(+, +)$ quadrant rather than the $(-, -)$ quadrant (it must be from one by monotonicity), as the argument for a contradiction in all other cases is perfectly analogous. Thus we assume that $D_+\check{m}(\sigma) < \tilde{m}'(\sigma; \sigma, m)$ and that $D_-\check{m}(\sigma) < \tilde{m}'(\sigma; \sigma, m)$, seeking a contradiction.

By differentiability and tangency to the conjectured isoreward curve of the imitation frontier anchored at (σ, m), if $D_+\check{m}(\sigma) < \tilde{m}'(\sigma; \sigma, m)$, there exists a region to the southeast of (σ, m) above $\check{m}$ and below the imitation frontier $\hat{m}(\sigma, m)$. By the arguments from the proof of Sublemma 1, $T \geq \frac{1}{N}$ in this region. But, by continuity of the imitation frontier in its parameters for any (σ', m') along the conjectured isoreward curve $\check{m}$ sufficiently close to (σ, m), the imitation frontier $\hat{m}(\sigma', m')$ contain points dominating points in his region. Thus by incentive compatibility, $T(\sigma', m') \geq \frac{1}{N}$.

However, again by continuity and monotonicity of the imitation frontiers in the (σ, m) at which they are anchored, if we chose any $(\sigma'', m'') > (\sigma, m)$ interior to $[0, N] \times [0, N]$, which is possible because (σ, m) was constructed as an interior point, there is some point of the form (σ', m') described in the preceding paragraph dominated by a neighborhood of (σ'', m''). Furthermore because

$D_{-}\check{m}(\sigma) < \tilde{m}'(\sigma;\sigma,m)$, by the exact argument of the preceding paragraph, some such point lies below $\check{m}$ and that there is some region, in the dominating neighborhood of (σ'', m'') lying above $\check{m}$ and below $\tilde{m}(\sigma'', m'')$. By the argument above, T in this region is at least $\frac{1}{N}$ greater than $T(\sigma', m')$ which it dominates through $\check{m}$. Thus T must be at least $\frac{2}{N}$ in this region. Thus we can iterate the argument to show that there is some point, interior to $[0, N] \times [0, N]$ on which T takes arbitrarily high values, a contradiction.

To prove that $\check{m}$ may not be vertical at (σ, m) follows precisely the same logic and to prove it cannot cut below follows the same logic in reverse (starting from the left, then moving to the right, moving down rather than up across conjectured isoreward curves).

Therefore $\check{m}$ must be tangent to any isoreward curve it intersects. Thus either $D_{-}\check{m}(\sigma) \geq \tilde{m}'(\sigma;\sigma,m)$ or $D_{+}\check{m}(\sigma) \geq \tilde{m}'(\sigma;\sigma,m)$ and either $D^{-}\check{m}(\sigma) \leq \tilde{m}'(\sigma;\sigma,m)$ or $D^{+}\check{m}(\sigma) \leq \tilde{m}'(\sigma;\sigma,m)$. We now wish to establish that at all σ, $\check{m}$ is differentiable and $\check{m}'(\sigma) = \tilde{m}'(\sigma;\sigma,m)$. This is equivalent to

$$D_{-}\check{m}(\sigma) = D^{-}\check{m}(\sigma) = D_{+}\check{m}(\sigma) = D^{+}\check{m}(\sigma) = \tilde{m}'(\sigma;\sigma,m)$$

This may fail, for example, if $D_{-}\check{m}(\sigma) < \tilde{m}'(\sigma;\sigma,m)$. If so, $\exists \delta > 0 : D_{-}\check{m}(\sigma) < \tilde{m}'(\sigma;\sigma,m) - \delta$. A well-known property of Dini derivatives[1] is their *potential continuity*: $\forall \eta > 0, \exists \sigma^{\star} \in [\sigma - \eta, \sigma] : D_{-}\check{m}\left(\sigma^{\star}\right), D_{+}\check{m}\left(\sigma^{\star}\right) < D_{-}\check{m}(\sigma) + \eta$. But by the continuous differentiability of ϵ and a and continuity of $\check{m}$, for η sufficiently small $\tilde{m}'\left(\sigma^{\star};\sigma^{\star},\check{m}\left(\sigma^{\star}\right)\right)$ is arbitrarily close to $\tilde{m}'(\sigma;\sigma,m)$. Thus $D_{-}\check{m}\left(\sigma^{\star}\right), D_{+}\check{m}\left(\sigma^{\star}\right) < \tilde{m}'\left(\sigma^{\star};\sigma^{\star},\check{m}\left(\sigma^{\star}\right)\right)$ and $\check{m}$ is not tangent to the isoreward curve passing through $\left(\sigma^{\star},\check{m}\left(\sigma^{\star}\right)\right)$. But this contradicts our above reasoning. All other ways in which $\check{m}$ may fail to be differentiable can be ruled out similarly. Thus $\check{m}(\sigma)$ is differentiable at all σ in its support and

$$\check{m}'(\sigma) = \tilde{m}'(\sigma;\sigma,m)$$

an ordinary differential equation with a unique solution passing through any point, corresponding to the conjectured isoreward curve passing through that point. Thus $\check{m}$ lies entirely along a single conjectured isoreward curve.

This argument may be repeated first for all other curves lying along the extended discontinuity curve in consideration, showing each of these lies along a conjectured isoreward curve. Furthermore it is straightforward to demonstrate that the same argument applies to almost-curves as to curves: removing a set of measure zero does not interfere with any of the steps as proper closure was applied in all of our arguments above.

Moving to the next extended discontinuity curve down, we may now ignore all points in $D\left(N, \frac{1}{N}\right)$, as these dominate all points in this curve and we can thus consider the argument over a sufficiently small box so as to exclude these points. The same argument can then be repeated to show all curves along this, and inductively all other, extended discontinuity curves containing $D\left(N, \frac{1}{N}\right)$ lie entirely along conjectured isoreward curves.

But of course this argument hold for any N and thus, by the same limiting argument as in the proof of Sublemma 1, establishes that all curves and almost-curves of discontinuities lie each entirely along a single conjectured isoreward curve.

$\square$

Sublemma 3: *At most a countable number of conjectured isoreward curves contain discontinuities of* T.

---

[1] See, for example, Hagood and Thomson (2006).

Proof. It is a classic real analysis result that every subset of the real line is dense in a countable union of closed intervals. Thus, because any non-increasing curve is uni-dimensional and therefore isomorphic to (a subset of) the real line, the points of discontinuity along every extended discontinuity curve from Sublemma (1) is a countable union of non-increasing curves or almost-curves. But by Sublemma 2 each of these intersects at most a single isoreward curve. Thus the set of isoreward curves containing discontinuities has the cardinality of non-increasing curves of discontinuity, which are a countable union of countable sets and therefore countable. □

Sublemma 4: *Suppose that T is continuous in (σ, m) at every point along a conjectured isoreward curve. Then this is an actual isoreward curve.*

Proof. Our proof proceeds in three steps:

1. We argue that along any conjectured isoreward curve where T is continuous, the upper Dini derivative from the direction below and perpendicular to the conjectured isoreward curve is continuous along that curve.
2. We show that if this Dini derivative is bounded from above at a point, then the conjectured isoreward curve is constant in the neighborhood of that point.
3. We show that arbitrarily close to any conjectured isoreward curve from below there must exist another conjectured isoreward curve along which all points have finite upper Dini derivatives and, therefore, that this other conjectured isoreward curve is an actual isoreward curve.
4. Finally, by continuity, we conclude that the conjectured isoreward curve is, in fact, an isoreward curve.

We begin by establishing the continuity of Dini derivatives along conjectured isoreward curves at which T is continuous. Consider a point (σ, m). We would like to show that the upper Dini derivative of T at (σ, m) from below in the direction $\left(1, -\frac{1}{\tilde{m}'(\sigma;\sigma,m)}\right)$ is continuous along the conjectured isoreward curve. We will denote this derivative by $\tilde{D}^{-}$. To do this we begin by establishing bounds on these Dini derivatives.

To place a lower bound on $D^{-}T\left(\sigma', \tilde{m}\left(\sigma'; \sigma, m\right)\right)$ we must, for any distance we are challenged with, find a point closer than that distance to $\left(\sigma', \tilde{m}\left(\sigma'; \sigma, m\right)\right)$ in the lower perpendicular direction to the isoreward curve which is sufficiently lower than $T\left(\sigma', \tilde{m}\left(\sigma'; \sigma, m\right)\right)$. To see that this is possible, note that points very close in the negative perpendicular direction to (σ, m) will be dominated by, and therefore with a lower value than, $\left(\sigma', \tilde{m}\left(\sigma'; \sigma, m\right)\right)$; by continuity, such a point, jointly with (σ', m') may be chosen so that the value of T is arbitrarily close to $T(\sigma, m)$. Furthermore the definition of the lower perpendicular Dini derivative at (σ, m) implies that a close point to on the lower perpendicular of (σ, m) may be chosen to be dominated by a slightly lower point on the lower perpendicular of $(\sigma', \tilde{m}(\sigma'; \sigma, m))$ and still have a sufficiently low value of T. This ensures that the desired point on the lower perpendicular of $(\sigma', \tilde{m}(\sigma'; \sigma, m))$ does in fact exist and establishes the lower bound.

By repeating the same argument, but reversing the choice of point dominance we obtain the opposite inequality: $\forall \delta > 0, \exists \eta : \forall \sigma' \in [\sigma, \sigma + \eta], \tilde{D}^{-}T(\sigma, m) - \delta \geq \tilde{D}^{-}T\left(\sigma', \tilde{m}\left(\sigma'; \sigma, m\right)\right)$. Combining these two implies the continuity of the lower Dini derivatives along the conjectured isoreward curve.

Next we argue that Dini derivatives being bounded above at a point from below along the local perpendicular implies the conjectured isoreward curve being constant in that neighborhood. Note that this hypothesis is equivalent, by the continuity of the Dini derivatives shown above to the Dini derivative being bounded in a neighborhood about this point.

Suppose this were false. Then there must be two points, (σ, m) and (σ', m') lying on the same conjectured isoreward curve in a neighborhood of bounded upper Dini derivative; take this upper bound to be M. Suppose we consider another conjectured isoreward curve that is sufficiently close (δ^2 along the local perpendicular) to the original one in question. Then by differentiability of the conjectured isoreward curves and tangency of the imitation frontier, the local imitation frontier will intersect the slightly lower conjectured isoreward curve at a distance of order δ along the isoreward curve (if a local perpendicular is drawn). The value of T at this point must be no greater than $T(\sigma, m)$ by incentive compatibility. Furthermore, the value at the point along the original conjectured isoreward curve must not exceed $T(\sigma, m)$ by more than $M\delta^2$, by the upper bound on the upper Dini derivative. Iterating this argument one finds that if the distance between (σ, m) and (σ', m') is less than d, then $T(\sigma', m') < T(\sigma, m) + d\delta$. Because δ maybe chosen arbitrarily small we obtain that $T(\sigma', m') \leq T(\sigma, m)$.

The argument may be repeated in the opposite direction to show that $T(\sigma', m') \geq T(\sigma, m)$ and thus it must be that $T(\sigma', m') = T(\sigma, m)$. Thus any neighborhood of bounded upper Dini derivatives from below along perpendicular must have a locally constant T value along the conjectured isoreward curve.

Now consider the actual value of these upper Dini derivatives from below along the local perpendicular. Note that for the perpendicular emanating from any point along the conjectured isoreward curve, the upper Dini derivative from below along this curve cannot be infinite, in any neighborhood of the curve, over any set of positive measure; otherwise the value of the curve would, by monotonicity, be infinite at some finite point. Thus along each perpendicular at most a set of measure zero has points where the upper Dini derivative from below is infinite.

Choose a countable, dense set of points in a neighborhood about (σ, m) and consider the perpendiculars emanating from these points. The set of conjectured isoreward curves passing through as point of infinite Dini derivative from below along any of these perpendiculars must be of measure zero, as a countable union of sets of measure zero is of measure zero. So must be the set of these conjectured isoreward curves that contain discontinuities of T by Sublemma 3. Therefore we can always find a "normative" conjectured isoreward curve avoiding both of these sins and arbitrarily close to the original conjectured isoreward curve of interest.

This normative curve must have uniformly bounded upper Dini derivatives along the diagonal in the neighborhood of each point in the countable dense set. It must therefore be constant over these sets. Furthermore because it may be chosen arbitrarily close to the original conjectured isoreward curve, at which T is continuous, this original curve must have constant T over this neighborhood as well.

Putting this all together, we have established that each conjectured isoreward curve, if it contains no discontinuities of T, must be constant in the neighborhood of each point along the curve. But clearly this establishes constancy over the whole curve. □

Proof of Lemma 1. By Sublemma 3 and Sublemma 4, the set of conjectured isoreward curves which fail to be actual isoreward curves is countable. Now suppose some $T^{\star\star}$ was incentive compatible, but violated the isoreward property on some countable set of pathological conjectured isoreward curves. Then consider another $T^{\star}$ which matches the values of $T^{\star\star}$ on the non-pathological conjectured isoreward curves but assigned to all point of the pathological isoreward curves any value taken by $T^{\star\star}$ along these pathological isoreward curves. Because f is continuous and has all finite moments social welfare is exactly the same under $T^{\star\star}$ as under $T^{\star}$ as changing the value of a function along a countable set does not alter its Riemann-Stieljes or Lebesgue integral. And clearly $T^{\star}$ satisfies the isoreward property along all conjectured isoreward curves. Thus *some* optimal T always satisfies the isoreward property of the Lemma. □

3 Main results

This appendix provides proofs and more detailed developments relevant to the derivation of optimal transfers and pricing policies, given the broad isoreward-monotonicity approach.

3.1 Conditions for the a "simple solution"

Proof of Proposition 9. The first-order condition for the monotonicity-unconstrained version of program

$$\max_{\tilde{T}(\cdot)} \int_k \int_{c=0}^{\tilde{T}(k)} \int_x \left(k^2 x^{\frac{1-\epsilon(a)}{1+\epsilon(a)}} S(a) a - c \right) \tilde{f}(k, x, c; a) dx dc dk \tag{5}$$

is exactly that $\overline{V}(T; k, a) = T$ or equivalently that supply equal demand. This condition is sufficient for the monotonicity-unconstrained optimum if the first-order derivative with respect to T, $\overline{V} - T$, is monotone decreasing in T (Guesnerie and Laffont, 1984). Because these expressions are clearly differentiable, given the smoothness assumed, this is equivalent to

$$\frac{\partial \overline{V}}{\partial c} \leq 1 \iff k^2 S(a) \frac{\partial E_{x,\tilde{f}} \left[x^{\frac{1-\epsilon}{1+\epsilon}} \middle| k, c \right]}{\partial c} \leq 1 \iff \text{Cov}_{x,\tilde{f}} \left[x^{\frac{1-\epsilon}{1+\epsilon}}, \frac{\partial \log \left(\tilde{f} \right)}{\partial c} \right] \leq \frac{1}{k^2 S(a)}$$

But note that because $\tilde{f}$ differs from f as a function of c only by a multiplicative factor, we can replace $\tilde{f}$ with f and obtain inequality (1) in the proposition. Thus, so long as that inequality is satisfied, the solution from the supply and demand (first-order condition) approach yields the optimal monotonicity-relaxed transfers $T^{\star\star}$.

Further more an upward shift in $\overline{V}$ must then increase this optimal transfer (Milgrom and Shannon, 1994). Thus we are guaranteed that $T^{\star\star}$ will be monotone so long as $\overline{V}$ increases in k. Again by differentiability this is equivalent to (taking logs)

$$\frac{\partial \overline{V}}{\partial k} \geq 0 \iff \frac{2}{k} + \frac{\text{Cov}_{x,\tilde{f}} \left[x^{\frac{1-\epsilon}{1+\epsilon}}, \frac{\partial \log(\tilde{f})}{\partial k} \middle| k, c \right]}{E_{x,\tilde{f}} \left[x^{\frac{1-\epsilon}{1+\epsilon}} \middle| k, c \right]} \geq 0$$

which simplifies to condition (2). Thus if this condition also holds the $T^{\star\star}$ derived from supply and demand is in fact $T^{\star}$.

Existence conditions are entirely classical for a solution and thus require no explanation. □

3.2 Envelope Theorem

The application of the general Milgrom and Segal (2002) envelope theorem at $\hat{a}$ requires two conditions: equidifferentiability *across feasible reward functions* of social welfare with respect to a, holding fixed the reward function, and continuity of the derivative *given the (locally) optimal choice of the reward function* , both at $\hat{a}$. This subsection therefore establishes four results:

1. First, we establish equidifferentiability in Sublemma 5. We begin by deriving a general expression for the first-order derivative of social welfare holding fixed the reward function that is valid even when $T^\star$ is not differentiable. This expression, for any reward function, can be broke into two pieces, one along the boundary and one on the interior. The second always converges "quickly" because the derivative is proportional to S, which is constant across T. The first does as well, because the mass along the boundary is bounded above by monotonicity and the rate of movement across the boundary by uniform bounds we establish on the covariance of $x^{\frac{1-\epsilon}{1+\epsilon}}$ and $\log(x)$.

2. Second, we establish continuity of the derivative in Sublemma 6. This follows from the continuity of the optimal reward boundary; this does not require that the optimal rewards, as a function of k, be continuous but rather that the curve dividing innovations that are created and those that are not continuously deform as a changes. Combined with the smoothness of $\tilde{f}$ both in its arguments and in a suffices to establish continuity of the derivative.

3. Given these two results, we invoke Milgrom and Segal (2002)'s Theorem 3 to establish differentiability and the validity of Lemma 2.

4. We then simplify the formula from the first step to establish the general first-order derivative of Proposition 10.

5. Finally, we show how this can be transformed into an expectation over k rather than c to derive Corollary 3.

In what follows we will use the (slightly abusive of our earlier) notation $W(a; T(\cdot))$ to denote the social welfare when pricing policy is a and the transfers assigned to each k (given a) are $T(k)$:

$$W(a; T(\cdot)) \equiv \int_\sigma \int_m \int_{c=0}^{T\left(\sigma^{\frac{1}{1+\epsilon(a)}} m^{\frac{\epsilon(a)}{1+\epsilon(a)}}\right)} (\sigma m S(a) - c) f(\sigma, m, c) dc dm d\sigma$$

Sublemma 5: *The class of functions $W(a; T(\cdot))$ are equidifferentiable in a across all monotone increasing $T(\cdot)$ at each $a \in [0,1]$.*

Proof.

$$W'(a, T(\cdot)) = \left. \frac{\partial \int_\sigma \int_m \int_{c=0}^{T\left(\sigma^{\frac{1}{1+\epsilon(a)}} m^{\frac{\epsilon(a)}{1+\epsilon(a)}}\right)} (\sigma m S(a) - c) f(\sigma, m, c) dc dm d\sigma}{\partial a} \right|_{a=\hat{a}} =$$

$$\lim_{\delta\to 0}\frac{\int_\sigma\int_m\left(\int_{c=0}^{T\left(\sigma^{\frac{1}{1+\epsilon(a+\delta)}}m^{\frac{\epsilon(a+\delta)}{1+\epsilon(a+\delta)}}\right)_{(\sigma mS(a+\delta)-c)f(\sigma,m,c)dc-}}\int_{c=0}^{T\left(\sigma^{\frac{1}{1+\epsilon(a)}}m^{\frac{\epsilon(a)}{1+\epsilon(a)}}\right)_{(\sigma mS(a)-c)f(\sigma,m,c)dc}}\right)dmd\sigma}{\delta}=$$

letting $\epsilon_\delta\equiv\epsilon(a+\delta)$ and $\epsilon\equiv\epsilon(a)$ and correspondingly for S,

$$\lim_{\delta\to 0}\frac{\int_x\int_k\left(\int_{c=0}^{T\left(kx^{\frac{\epsilon_\delta-\epsilon}{(1+\epsilon_\delta)(1+\epsilon)}}\right)}\left(k^2x^{\frac{1-\epsilon}{1+\epsilon}}S_\delta-c\right)\tilde{f}(k,x,c)dc-\int_{c=0}^{T(k)}\left(k^2x^{\frac{1-\epsilon}{1+\epsilon}}S-c\right)\tilde{f}(k,x,c)dc\right)dkdx}{\delta}=$$

dropping the arguments of $\tilde{f}$ and letting $x_\delta\equiv x^{\frac{\epsilon_\delta-\epsilon}{(1+\epsilon_\delta)(1+\epsilon)}}$,

$$\lim_{\delta\to 0}\frac{\int_x\int_k\left(\int_{c=T(k)}^{T(kx_\delta)}\left(k^2x^{\frac{1-\epsilon}{1+\epsilon}}S_\delta-c\right)\tilde{f}dc+\int_{c=0}^{T(k)}\left(k^2x^{\frac{1-\epsilon}{1+\epsilon}}\left[S_\delta-S\right]-c\right)\tilde{f}dc\right)dkdx}{\delta}=$$

$$\lim_{\delta\to 0}\frac{\int_x\int_k\int_{c=T(k)}^{T(kx_\delta)}\left(k^2x^{\frac{1-\epsilon}{1+\epsilon}}S_\delta-c\right)\tilde{f}dcdkdx}{\delta}-Q(a)\epsilon(a)\int_k k^2\int_x\int_{c=0}^{T(k)}x^{\frac{1-\epsilon}{1+\epsilon}}\tilde{f}dcdkdx \qquad (6)$$

by differentiability of S. The second term is as we want it, so we focus on the first. We can break it into the sum of three terms: an integral over the region where $x>1$, the point where $x=1$ and the integral over the region where $x<1$. The second of these is identically zero as the integral over c always runs over a degenerate region. The third can be transformed in a manner exactly parallel to that of the first, so we focus on the first in most of the remaining proof; thus the quantity we will analyze is

$$\lim_{\delta\to 0}\frac{\int_{x=1_+}^{\infty}\int_k\int_{c=T(k)}^{T(kx_\delta)}\left(k^2x^{\frac{1-\epsilon}{1+\epsilon}}S_\delta-c\right)\tilde{f}dcdkdx}{\delta} \qquad (7)$$

The integral over k, which runs from 0 to ∞ is just the limit of an integral running from bounds $\underline{k}(\delta,z)$ to $\overline{k}(\delta,z)$ as $z\to\infty$ so long as $\lim_{z\to\infty}\underline{k}(\delta,z)=0$ and $\lim_{z\to\infty}\overline{k}(\delta,z)=\infty$. In particular let $\underline{k}(\delta,z)\equiv\frac{1}{\overline{k}(\delta,z)}$ and

$$\overline{k}(\delta,z)\equiv x_\delta^{\left\lceil\frac{\log(z)}{\log(x_\delta)}\right\rceil}$$

where $\lceil x\rceil$ is the smallest integer greater than x. Clearly $\lim_{z\to\infty}\overline{k}(\delta,z)=\infty$ and thus $\lim_{z\to\infty}\underline{k}(\delta,z)=0$. Then we can re-write expression (7) as

$$\lim_{\delta\to 0}\lim_{z\to\infty}\frac{\int_{x=1_+}^{\infty}\int_{k=\underline{k}(\delta,z)}^{k=\overline{k}(\delta,z)}\int_{c=T(k)}^{T(kx_\delta)}\left(k^2x^{\frac{1-\epsilon}{1+\epsilon}}S_\delta-c\right)\tilde{f}dcdkdx}{\delta} \qquad (8)$$

Furthermore, we may approximate the integral over k by a Riemann sum, the intervals of which are given by

$$\left(\underline{k}(\delta,z),\underline{k}(\delta,z)x_\delta\right),\left(\underline{k}(\delta,z)x_\delta,\underline{k}(\delta,z)x_\delta^2\right),\ldots,\left(\overline{k}(\delta,z)x_\delta^{-1},\overline{k}(\delta,z)\right)$$

Let us number these intervals starting from 1 and moving outwards, both down and up: "up" interval 1 is $(1, x_\delta)$ and "down" interval 1 is $\left(x_\delta^{-1}, 1\right)$. There are then $\left\lceil \frac{\log(z)}{\log(x_\delta)} \right\rceil$ of each up and down intervals and the length of the ith up interval is $x_\delta^{i-1}(x_\delta - 1)$ while of the ith down interval is $x_\delta^{-i}(x_\delta - 1)$. Thus note that the upper bound on the length any interval, $\overline{k}(\delta, z)(x_\delta - 1) \to 0$ as $\delta \to 0$, so that any choice of a point within the interval at which to evaluate the Riemann sum will lead to a sum converging to the integral as $\delta \to 0$. Thus, if we evaluate the Riemann sum at the bottom of the interval, the expression (8) becomes the limit as δ becomes small and z becomes large of

$$\frac{(x_\delta - 1)\displaystyle\int_{x=1_+}^{\infty} \left(\Sigma_{N=1}^{\left\lceil \frac{\log(z)}{\log(x_\delta)} \right\rceil} x_\delta^{-N} \int_{c=T\left(x_\delta^{-N}\right)}^{T\left(x_\delta^{-N+1}\right)} \left(x_\delta^{-2N} x^{\frac{1-\epsilon}{1+\epsilon}} S_\delta - c \right) \tilde{f} dc + x_\delta^{N} \int_{c=T\left(x_\delta^{N-1}\right)}^{T\left(x_\delta^{N}\right)} \left(x_\delta^{2(N-1)} x^{\frac{1-\epsilon}{1+\epsilon}} S_\delta - c \right) \tilde{f} dc \right) dx}{\delta} =$$

$$\lim_{\delta \to 0} \lim_{z \to \infty} \int_{x=1_+}^{\infty} \frac{x_\delta - 1}{\delta} \int_{c=T(\underline{k}(\delta,z))}^{T\left(\overline{k}(\delta,z)\right)} T^{-1} \left(x_\delta^{\left\lfloor \frac{\log(c)}{\log(x_\delta)} \right\rfloor} \right) \left(T^{-1} \left(x_\delta^{\left\lfloor \frac{\log(c)}{\log(x_\delta)} \right\rfloor} \right)^2 x^{\frac{1-\epsilon}{1+\epsilon}} S_\delta - c \right) \tilde{f} dc dx =$$

where, in the cases where T^{-1} is not single valued, it is defined to pick out the single value to make this equality correct, and where c is not in the range of T because of a discontinuity, $T^{-1}(c)$ is taken to be the unique value of k such that $T(k_-) \le c \le T(k_+)$. Thus if $\lim_{c \to 0} T(c) = \underline{c} > 0$ then for all $c < \underline{c}$, $T(c) = 0$ as clearly $T(0) = 0$.

$$\lim_{\delta \to 0} \int_{x=1_+}^{\infty} \frac{x_\delta - 1}{\delta} \int_{c=0}^{\overline{c}} T^{-1} \left(x_\delta^{\left\lfloor \frac{\log(c)}{\log(x_\delta)} \right\rfloor} \right) \left(T^{-1} \left(x_\delta^{\left\lfloor \frac{\log(c)}{\log(x_\delta)} \right\rfloor} \right)^2 x^{\frac{1-\epsilon}{1+\epsilon}} S_\delta - c \right) \tilde{f} dc dx \tag{9}$$

The equidifferentiability of the second term of (6) follows from the finiteness of the moments of f and $\tilde{f}$; by the same argument we can, for the purposes of establishing equidifferentiability reduce equation (9) to

$$\lim_{\delta \to 0} \int_{c=0}^{\overline{c}} T^{-1} \left(x_\delta^{\left\lfloor \frac{\log(c)}{\log(x_\delta)} \right\rfloor} \right) \left(T^{-1} \left(x_\delta^{\left\lfloor \frac{\log(c)}{\log(x_\delta)} \right\rfloor} \right)^2 x^{\frac{1-\epsilon}{1+\epsilon}} - c \right) \tilde{f} dc$$

We must show this converges uniformly across T. Clearly $\left| x_\delta^{\left\lfloor \frac{\log(c)}{\log(x_\delta)} \right\rfloor} - c \right| \le x_\delta c$ and thus

$$\left| \int_{c=0}^{\overline{c}} T^{-1} \left(x_\delta^{\left\lfloor \frac{\log(c)}{\log(x_\delta)} \right\rfloor} \right) \left(T^{-1} \left(x_\delta^{\left\lfloor \frac{\log(c)}{\log(x_\delta)} \right\rfloor} \right)^2 x^{\frac{1-\epsilon}{1+\epsilon}} - c \right) \tilde{f} dc - \int_{c=0}^{\overline{c}} T^{-1}(c) \left(T^{-1}(c)^2 x^{\frac{1-\epsilon}{1+\epsilon}} - c \right) \tilde{f} dc \right| <$$

$$\max \left\{ \left| x^{\frac{1-\epsilon}{1+\epsilon}} \int_{c=0}^{\overline{c}} \left[T^{-1}(c x_\delta)^3 - T^{-1}(c)^3 \right] \tilde{f} dc \right|, \left| \int_{c=0}^{\overline{c}} c \left[T^{-1}(c x_\delta) - T^{-1}(c) \right] \tilde{f} dc \right| \right\}$$

For brevity's sake, we only show that the second of these must converge uniformly, as the argument that the first does follows by the same logic.

$$\left| \int_{c=0}^{\overline{c}} c \left[T^{-1}(c x_\delta) - T^{-1}(c) \right] \tilde{f} dc \right| = \left| \int_{c=0}^{\overline{c}} c T^{-1}(c x_\delta) \tilde{f} dc - \int_{c=0}^{\overline{c}} c T^{-1}(c) \tilde{f} dc \right| =$$

by the change of variables $\tilde{c} = cx_\delta$ on the first integral

$$\left|\int_{\tilde{c}=0}^{x_\delta \overline{c}} \tilde{c}T^{-1}\left(\tilde{c}\right)\tilde{f}\left(\cdot,\cdot,\frac{\tilde{c}}{x_\delta}\right)d\tilde{c} - \int_{c=0}^{\overline{c}} cT^{-1}\left(c\right)\tilde{f}(\cdot,\cdot,c)dc\right| =$$

$$\int_{c=\frac{\overline{c}}{x_\delta}}^{\overline{c}} cT^{-1}(c)\tilde{f}dc + \int_{c=0}^{\frac{\overline{c}}{x_\delta}} cT^{-1}(c)\left(\tilde{f}\left(\cdot,\cdot,\frac{c}{x_\delta}\right) - \tilde{f}\left(\cdot,\cdot,c\right)\right)dc$$

But both of these clearly converge rapidly and uniformly across T as first term is just some upper tail of the kc moment, which is finite, and the second term is, in the limit, just the (bounded by smoothness) partial slope of $\log(\tilde{f})$-weighted value of the kc moment along the T^{-1} curve given that finite moments imply finite moments along any one-dimensional curve. ☐

Sublemma 6: *$W_1\left(a; T^\star(\cdot;a)\right)$ is continuous on $[0,1]$ as a function of a.*

Proof. We wish to show that

$$\lim_{\delta\to 0} W_1\left(a+\delta, T^\star\left(\cdot, a+\delta\right)\right) = \lim_{\delta\to 0} W_1\left(a-\delta, T^\star\left(\cdot, a-\delta\right)\right)$$

It is clearly from the reasoning in the proof of Sublemma 5 that $W_1\left(a, T(\cdot)\right)$ is continuous in a for any $T(\cdot)$ so it suffices to show that

$$\lim_{\delta\to 0} W_1\left(a, T^\star\left(\cdot, a+\delta\right)\right) = \lim_{\delta\to 0} W_1\left(a, T^\star\left(\cdot, a-\delta\right)\right)$$

Now, abbreviating $W\left(x, T^\star(\cdot, y)\right)$ to $W(x,y)$, this is equivalent to:

$$\lim_{\delta\to 0}\left|\lim_{\eta\to 0}\frac{W(a+\eta, a+\delta) - W(a, a+\delta)}{\eta} - \frac{W(a+\eta, a-\delta) - W(a, a-\delta)}{\eta}\right| = 0$$

Interchanging the limits and rearranging it suffices to show that

$$\lim_{\eta\to 0}\left|\lim_{\delta\to 0}\frac{W(a+\eta, a+\delta) - W(a+\eta, a-\delta)}{\eta}\right| = 0$$

and

$$\lim_{\eta\to 0}\left|\lim_{\delta\to 0}\frac{W(a, a+\delta) - W(a, a-\delta)}{\eta}\right| = 0$$

Both of these can be shown in the same manner, so we focus on the second. Note that it suffices to demonstrate that

$$\lim_{\delta\to 0} W(a, a+\delta) - W(a, a-\delta) = 0$$

To see this note that $W(a,\hat{a})$ is continuous in a so that for any $\nu > 0$ we can find a sufficiently small δ such that $W(a, a-\delta) \geq W(a-\delta, a-\delta) - \frac{\nu}{2}$ and $W(a-\delta, a) \geq W(a,a) - \frac{\nu}{2}$. Combining this with revealed preference yields

$$W(a, a-\delta) \geq W(a-\delta, a-\delta) + \frac{\nu}{2} \geq W(a-\delta, a) + \frac{\nu}{2} \geq W(a,a) + \nu \geq W(a, a+\delta) + \nu$$

A similar inequality may be established by the same reasoning in the other direction, establishing the desired limit. Again the same reasoning applies to the other limit and establishes the desired continuity. ☐

Proof of Lemma 2. The choice set for $T(\cdot)$ is the set of all monotone increasing functions. Thus from Milgrom and Segal (2002)'s Theorem 3, given equidifferentiability across this class and continuity on $[0,1]$ from Sublemmata 5 and 6 we have that

$$W'(\hat{a}) = W_1\left(\hat{a}; T^\star(\cdot;\hat{a})\right)$$

□

3.3 First-order derivative

Proof of Proposition 10. Clearly $\lim_{\delta\to 0} x_\delta^{\left\lfloor \frac{\log(c)}{\log(x_\delta)} \right\rfloor} = \log(c)$ and $\lim_{\delta\to 0} S_\delta = S(a)$. Thus we need only analyze

$$\lim_{\delta\to 0} \frac{x_\delta - 1}{\delta} \underbrace{= \frac{\partial x_\delta}{\partial \delta}(0)}_{\text{by L'Hôpital's rule}} = \frac{\log(x)\epsilon'(a)}{\left[1+\epsilon(a)\right]^2}$$

to see that expression (9) becomes

$$\frac{\epsilon'(a)}{\left[1+\epsilon(a)\right]^2} \int_{x=1_+}^{\infty} \log(x) \int_{c=0}^{\overline{c}} T^{-1}(c) \left(T^{-1}(c)^2 x^{\frac{1-\epsilon(a)}{1+\epsilon(a)}} S(\hat{a}) - c\right) \tilde{f} dc dx$$

where the integral over c leaves out the measure-zero set of c's for which T is not invertible (by monotonicity). A perfectly analogous argument for $x < 1$ shows that the full first term of expression (6) is

$$\frac{\epsilon'(a)}{\left[1+\epsilon(a)\right]^2} \int_{x=0}^{\infty} \log(x) \int_{c=0}^{\overline{c}} T^{-1}(c) \left(T^{-1}(c)^2 x^{\frac{1-\epsilon(a)}{1+\epsilon(a)}} S(a) - c\right) \tilde{f} dc dx$$

By the first-order conditions for optimal transfers and the proof of Sublemma 5, $W'\left(a, T^\star(\cdot, a)\right) =$

$$\frac{\epsilon'(a)}{[1+\epsilon(a)]^2} \int_{x=0}^{\infty} \log(x) \int_{c=0}^{\overline{c}} \left[{T^\star}^{-1}(c;a)\right]^3 S(a) \left(x^{\frac{1-\epsilon(a)}{1+\epsilon(a)}} - E_{x,\tilde{f}}\left[x^{\frac{1-\epsilon(a)}{1+\epsilon(a)}}\right]\right) \tilde{f} - Q(a)\epsilon(a) x^{\frac{1-\epsilon}{1+\epsilon}} \int_{k={T^\star}^{-1}(c;a)}^{\infty} k^2 \tilde{f} dk dc dx \propto$$

$$\frac{\epsilon'}{[1+\epsilon]^2} \int_{c=0}^{\overline{c}} \left[{T^\star}^{-1}(c;a)\right]^3 S \operatorname{Cov}_{\tilde{f},x}\left[\log(x), x^{\frac{1-\epsilon}{1+\epsilon}} \middle| k = {T^\star}^{-1}(c;a), c\right] - Q\epsilon E_{x,k,\tilde{f}}\left[k^2 x^{\frac{1-\epsilon}{1+\epsilon}} \middle| k \geq T^{-1}(c;a), c\right] \frac{1-\tilde{F}^{-1}\left({T^\star}^{-1}(c;a);c\right)}{\tilde{f}\left({T^\star}^{-1}(c;a),c\right)} dc \tag{10}$$

As long as we are not in an ironing region, the average reward given to an innovation with $k = {T^\star}^{-1}(c)$ that is created is, by the first-order condition for socially optimal transfers,

$$S E_{x,k,\tilde{f}}\left[k^2 x^{\frac{1-\epsilon}{1+\epsilon}} \middle| k = {T^\star}^{-1}(c;a), c\right]$$

and thus

$$\frac{1-\tilde{F}^{-1}\left({T^\star}^{-1}(c;a);c\right)}{S E_{x,k,\tilde{f}}\left[k^2 x^{\frac{1-\epsilon}{1+\epsilon}} \middle| k = {T^\star}^{-1}(c;a), c\right] \tilde{f}\left({T^\star}^{-1}(c;a),c\right)}$$

is exactly the elasticity of of innovation supply described in the proposition statement. We can thus rewrite expression (10) as

$$\frac{S\epsilon'}{[1+\epsilon]^2} E_{c,\tilde{f}}\left[\left[{T^\star}^{-1}(c;a)\right]^3 \eta \operatorname{Cov}_{\tilde{f},x}\left[\log(x), x^{\frac{1-\epsilon}{1+\epsilon}} \middle| k={T^\star}^{-1}(c;a), c\right] - Q\epsilon E_{x,k,\tilde{f}}\left[k^2 x^{\frac{1-\epsilon}{1+\epsilon}} \middle| k \geq {T^\star}^{-1}(c;a), c\right] E_{x,k,\tilde{f}}\left[k^2 x^{\frac{1-\epsilon}{1+\epsilon}} \middle| k = {T^\star}^{-1}(c;a), c\right] \middle| c < \overline{c}\right]$$

If we *are* in an ironing region, rewards are constant over the ironing region in Guesnerie and Laffont (1984)'s solution, so this may only occur at one of the countable discontinuity points of c, which have no effect on the integral and thus may be ignored.

$\square$

Proof of Theorem 1. By equation (3) we have that $W'(a) \propto$

$$E_{c\leq\overline{c},\tilde{f}}\left[\left({T^{\star}}^{-1}(c)\right)^3\left((1-\epsilon)\frac{\epsilon'}{(1+\epsilon)^2}\frac{\eta\text{Cov}_{x,\tilde{f}}\left(\log(x),x^{\frac{1-\epsilon}{1+\epsilon}}\right)}{1-\epsilon}\right)-\epsilon QE_{x,\tilde{f}}\left(x^{\frac{1-\epsilon}{1+\epsilon}}\right)E_{x,\tilde{f}}\left(x^{\frac{1-\epsilon}{1+\epsilon}}\middle|k\geq{T^{\star}}^{-1}(c)\right)\right]$$

Note that by the smoothness of $\tilde{f}$ the expectation and covariance terms are non-explosive so that as $a \to 0$ the second term approaches 0 as $\lim_{a\to 0}\epsilon(a)Q(a) = 0$ while $\epsilon'(a) > 0$. Thus for a in an open ball about 0 the expression is weakly positive so that, if $a = 0$ is a maximizer, so is some $\epsilon > 0$ and that it is strictly positive if T is non-constant. By (the proof of) Theorem 2 we know that the covariance ratio term approaches the finite limit of the variance of $\log(x)$; thus by an identical argument the first term must approach 0 as $a \to 1$ and thus expression must be strictly negative at $a = 1$; thus this cannot be a maximum.

Finally, to see that any $a > 1$ is dominated, note that when $a = 1$ the solution to the unconstrained program

$$\max_{\{T(\cdot,\cdot),a(\cdot,\cdot)\}}\int_{\{\theta:c<T(\sigma,m)\}}\left[\sigma m S(a(\sigma,m))-c\right]f(\theta)d\theta \tag{11}$$

holding fixed a, $T^{\star}(\sigma,m) = S(1)\sigma m$, satisfies the constraint in $\sigma Q\left(a\left(\hat{\sigma},\widehat{m}\right)\frac{\widehat{m}}{m}\right) \geq \hat{\sigma}Q\left(a\left(\hat{\sigma},\widehat{m}\right)\right) \implies T(\sigma,m) \geq T(\hat{\sigma},\hat{m})$ because

$$\sigma Q\left(\frac{\hat{m}}{m}\right) = \hat{\sigma}Q(1) \implies \hat{\sigma}\hat{m}S(1) = \sigma m\frac{Q\left(\frac{\hat{m}}{m}\right)\hat{m}}{m}\frac{S(1)}{Q(1)}$$

which is clearly maximized at $\hat{m} = m$ as 1 maximizes $Q(a)a$ by construction. Thus if we let $W\left(a;T\left(\cdot\right)\right)$ represent the maximized social value given a but subject to the incentive compatibility constraint implied by $\hat{a}$, $W(\hat{a};1) > W(\hat{a};\hat{a})$ for any $\hat{a} > 1$. But clearly, given that $S'(a) < 0$, $W(\hat{a};1) < W(1;1)$.

$\square$

Proof of Corollary 3.

$$\tfrac{\epsilon'(a)}{[1+\epsilon(a)]^2}\int_{x=0}^{\infty}\log(x)\int_{c=0}^{\overline{c}}\left[{T^{\star}}^{-1}(c;a)\right]^3 S(a)\left(x^{\frac{1-\epsilon(a)}{1+\epsilon(a)}}-E_{x,\tilde{f}}\left[x^{\frac{1-\epsilon(a)}{1+\epsilon(a)}}\right]\right)\tilde{f}-Q(a)\epsilon(a)x^{\frac{1-\epsilon}{1+\epsilon}}\int_{k={T^{\star}}^{-1}(c;a)}^{\infty}k^2\tilde{f}dkdcdx=$$

changing variables according to $c = T^{\star}(k;a)$ and assuming differentiability of $T^{\star}$

$$\tfrac{\epsilon'}{[1+\epsilon]^2}\int_{x=0}^{\infty}\log(x)\int_{k=0}^{\infty}k^3{T^{\star}}'(k;a)S\left(x^{\frac{1-\epsilon}{1+\epsilon}}-E_{x,\tilde{f}}\left[x^{\frac{1-\epsilon}{1+\epsilon}}\right]\right)\tilde{f}-Q\epsilon x^{\frac{1-\epsilon}{1+\epsilon}}k^2\int_{c=0}^{T^{\star}(k;a)}\tilde{f}dcdkdx$$

The same set of transformations as in the proof of Proposition 10 just above yields the desired formula.

$\square$

3.4 Second-order conditions

Proof of Proposition 11. The first-order derivative from expression (4) is exactly

$$\frac{\epsilon'(1-\epsilon)}{(1+\epsilon)^2}C(a) - \frac{\epsilon(a)Q(a)}{M(a)}$$

Given that all terms here are strictly positive for $a \leq 1$, using the standard ratio monotonicity condition, a sufficient condition for quasi-concavity is that

$$\frac{\epsilon'(1-\epsilon)}{(1+\epsilon)^2\epsilon(a)Q(a)}C(a)M(a)$$

or its log is declining in a:

$$\frac{d\log\left(CM\right)}{da} < \frac{Q'}{Q} + \frac{\epsilon'}{\epsilon} + \frac{2\epsilon'}{1+\epsilon} - \frac{\epsilon''}{\epsilon'} + \frac{\epsilon'}{1-\epsilon} = \frac{\epsilon(a)}{a} + \frac{\epsilon'\left(1-\epsilon^2+2\epsilon-2\epsilon^2+\epsilon+\epsilon^2\right)}{\left(1-\epsilon^2\right)\epsilon} - \frac{\epsilon''}{\epsilon'}$$

which gives the desired inequality.

Note that by differentiability and strictly declining marginal revenue

$$\lim_{a\to 0}\epsilon'(a), \lim_{a\to 1}\epsilon'(a) > 0$$

and that $\lim_{a\to 0} 1+3\epsilon(a)-2\epsilon^2(a) = 1$ and $\lim_{a\to 1} 1+3\epsilon(a)-2\epsilon^2(a) = 2$ while $\frac{\epsilon(a)}{a} > 0$. Therefore for a close to 0 the first two terms in the expression approach infinity and for a close to 1 the second does. Because $\epsilon''(a)$ is assumed bounded the third term is bounded near both extremities, as is $\frac{d\log(CM)}{da}$ since these converge smoothly to their limiting quantities as shown in the following subsection. Therefore the inequalities are always satisfied close to $a = 0$ and $a = 1$.

□

3.5 Limit theorems

Proof of Theorem 2. For a close to 1 expression (4) greatly simplifies to

$$E_{k,\tilde{f}}\left[k^4\left((1-\epsilon)\frac{T^{\star\prime}}{k}\frac{\epsilon'(1)}{8}\text{Var}_{x,\tilde{f}}\left(\log(x)\right) - \frac{Q(1)}{\eta}\right)\right] \tag{12}$$

This simplifications follow from taking off all explicit ϵ terms which converge to 1 or $\epsilon'(1)$ (with the exception of $1-\epsilon$, which would approach 0), by noting any expectation of $x^{\frac{1-\epsilon}{1+\epsilon}} \to 1$ and by taking

$$\lim_{a\to 1}\frac{\text{Cov}_{x,\tilde{f}}\left[\log(x), x^{\frac{1-\epsilon}{1+\epsilon}}\right]}{1-\epsilon} = \lim_{a\to 1}\text{Cov}_{x,\tilde{f}}\left[\log(x), \frac{x^{\frac{1-\epsilon}{1+\epsilon}}-1}{1-\epsilon}\right] = \text{Cov}_{x,\tilde{f}}\left[\log(x), \lim_{a\to 1}\frac{x^{\frac{1-\epsilon}{1+\epsilon}}-1}{1-\epsilon}\right]$$

by continuity of $\tilde{f}$. We therefore need to evaluate by L'Hôpital's rule

$$\lim_{a\to 1}\frac{x^{\frac{1-\epsilon}{1+\epsilon}}-1}{1-\epsilon} = \lim_{a\to 1}\frac{2\log(x)}{(1+\epsilon)^2} = \frac{\log(x)}{2}$$

giving expression (12). By the same reasoning, $\lim_{a\to 1}\overline{V} = S(1)k^2$ for all c and thus $\lim_{a\to 1} T^{\star}(k;a) = S(1)k^2$ and $\lim_{a\to 1} T^{\star\prime}(k;a) = 2kS(1)$. Furthermore in the limit $k^2 = \sigma m = \frac{\pi}{Q(1)}$ where π is profits. Re-normalizing and substituting in, expression (4) is proportional to

$$E_{k,\tilde{f}}\left[\pi^2\left[(1-\epsilon)\mathrm{Var}_{x,\tilde{f}}\left(\log(x)\left|k=\sqrt{\tfrac{\pi}{Q(1)}}, c=\tfrac{S(1)}{Q(1)}\pi\right.\right)\eta\left(\tfrac{S(1)}{Q(1)}\pi\right| k=\sqrt{\tfrac{\pi}{Q(1)}}, a=1\right)-\tfrac{4}{\epsilon'(1)}\tfrac{Q(1)}{S(1)}\right]\right]$$

Thus for a near 1, the first-order condition requires

$$(1-\epsilon)\frac{E_{k,\tilde{f}}\left[\pi^2\mathrm{Var}_{x,\tilde{f}}\left(\log(x)\left|k=\sqrt{\tfrac{\pi}{Q(1)}}, c=\tfrac{S(1)}{Q(1)}\pi\right.\right)\eta\left(\tfrac{S[1]}{Q[1]}\pi\right| k=\sqrt{\tfrac{\pi}{Q(1)}}, a=1\right)\right]}{E_{k,\tilde{f}}\left(\pi^2\right)} = \frac{4}{\epsilon'(1)}\frac{Q(1)}{S(1)}$$

Because $\epsilon(a)$ goes to 1 as a goes to 1 and the right hand side of this expression is strictly positive, a solution calls for $a \to 1$ as $V_1 \to 1$. This solution is unique under quasi-concavity, establishing the result.

□

Proof of Theorem 3. By the same process of simplification as above we have when $a \to 0$ the expression (4) becoming

$$E_{k,\tilde{f}}\left[k^4\left(\epsilon'(0)\frac{T^{\star\prime}}{k}Cov_{x,\tilde{f}}\left(\log(x), x\right) - \epsilon E_{x,\tilde{f}}\left(x\right)E_{x,\tilde{f}}\left(x\right|c<T^{\star}\right)\frac{1}{\eta}\right)\right]$$

Also when a goes to 0, $k \to \sigma \to q$ and $x \to m$ so re-normalizing and substituting yields

$$E_{k,\tilde{f}}\left[q^4\left(\epsilon'(0)\frac{T^{\star\prime}(q)}{qE_{m,\tilde{f}}\left(m\left|c=T^{\star}(q)\right.\right)}\frac{\eta\mathrm{Cov}_{m,\tilde{f}}\left(\log(m), m\right)}{qE_{m,\tilde{f}}\left(m\right|c<T^{\star}(q)\right)}-\epsilon\right)\right]$$

Near $a=0$, $\tilde{f}\approx f$ so $T^{\star}\approx qE_{m,\tilde{f}}\left(m\left|c=T^{\star}(q)\right.\right)$ and $\frac{T^{\star\prime}}{qE_{m,\tilde{f}}\left(m|c=T^{\star}(q)\right)}\approx \log(T^{\star})'$. The rest follows by analogous steps to those in the proof of Theorem 2.

□

Assuming elasticity is uncorrelated with the variance of $\log(x)$ under k, the approximation from the proof of Theorem 2 becomes

$$(1-\epsilon)CM \approx \frac{4}{\epsilon'(1)}\frac{Q(1)}{S(1)} \implies \epsilon \approx 1-\frac{4Q(1)}{\epsilon'(1)S(1)CM} \implies a \approx 1-\frac{4Q(1)}{\left(\epsilon'(1)\right)^2 S(1)CM}$$

When demand is linear $\epsilon'(1)=2$, as $\epsilon(a)=\frac{a}{2-a}$ and $Q(1)=\frac{1}{2}$, while $S(1)=\frac{3}{4}$ by standard formulae so $\frac{4Q(1)}{(\epsilon'(1))^2 S(1)} \to \frac{2}{3}$ as in the text.

Note that a similar calculation could be performed near $a=0$ assuming optimal transfers (again). This would allow a (hypothetical) society with a pure prize system (perhaps a socialist society) to calibrate the value of introducing some entrepreneurship.

3.6 Implementation

Under proportional pricing $p = am$ and thus $q = \sigma Q(a)$. Thus optimal rewards must be as stated in the text. This is incentive compatible, as the points along the demand curve at which the innovator may produce receive exactly the rewards corresponding to her imitation frontiers.

4 Extensions and Applications

This appendix describes the mathematics behind some of the extensions and applicatoins discussed briefly in the text

4.1 Rent extraction and distributional concerns

Proof of Proposition 1. To derive this we write out as before

$$\frac{dW^{\lambda}(a)}{da}(\hat{a}) = \frac{d\int_{\sigma}\int_{m}\int_{c=0}^{T^{\lambda}\left(m^{\frac{\epsilon}{1+\epsilon}}\sigma^{\frac{1}{1+\epsilon}};\hat{a}\right)}\left[\sigma m S(a)-\lambda c-(1-\lambda)T^{\lambda}\left(m^{\frac{\epsilon}{1+\epsilon}}\sigma^{\frac{1}{1+\epsilon}};\hat{a}\right)\right]f(c,\sigma,m)dcdmd\sigma}{da}$$

As one might expect from the algebra needed to establish Corollary (3), taking this derivative is quite cumbersome, though eventually it simplifies greatly. This is even more so given the additional terms that have been added. Despite this we think it is worth going through the exercise for completeness and because of the elegance of the simplifications that occur. We begin with the the Leibnitz boundary terms, then consider the corresponding interior terms.

$$\int_{\sigma}\int_{m}\frac{T^{\lambda'}\epsilon'\log\left(\frac{m}{\sigma}\right)m^{\frac{\epsilon}{1+\epsilon}}\sigma^{\frac{1}{1+\epsilon}}}{(1+\epsilon)^2}\left(\sigma m S(a)-T^{\lambda}\right)f(c,\sigma,m)dmd\sigma =$$

$$\int_{k}\int_{x}\frac{T^{\lambda'}\epsilon'\log(x)k^3S(a)}{(1+\epsilon)^2}\left[x^{\frac{1-\epsilon}{1+\epsilon}}-\frac{\eta}{1-\lambda+\eta}E\left(x^{\frac{1-\epsilon}{1+\epsilon}}\right)\right]\tilde{f}(k,x)dxdk =$$

$$\int_{k}\frac{S(a)\epsilon'k^4}{(1+\epsilon)^2}\frac{T^{\lambda'}}{k}\left[\text{Cov}\left(\log(x),x^{\frac{1-\epsilon}{1+\epsilon}}\right)+\frac{1-\lambda}{1-\lambda+\eta}E\left(x^{\frac{1-\epsilon}{1+\epsilon}}\right)E\left(\log(x)\right)\right]\tilde{f}(k)dk$$

Thus we have changed the old boundary term only by adding the second term in the expression above. As for the interior we have:

$$-\int_{\sigma}\int_{m}\int_{c=o}^{T^{\lambda}\left(m^{\frac{a}{2}}\sigma^{1-\frac{a}{2}};\hat{a}\right)}\left[\sigma m\epsilon Q+(1-\lambda)\frac{T^{\lambda'}\epsilon' m^{\frac{\epsilon}{1+\epsilon}}\sigma^{\frac{1}{1+\epsilon}}\log\left(\frac{m}{\sigma}\right)}{(1+\epsilon)^2}\right]f(\sigma,m,c)dcdmd\sigma =$$

$$-\int_{k}k^2\int_{c=0}^{T^{\lambda}(k)}\left[E\left(x^{\frac{1-\epsilon}{1+\epsilon}}\right)\epsilon Q+(1-\lambda)\frac{T^{\lambda'}\epsilon'}{k(1+\epsilon)^2}E\left(\log(x)\right)\right]\tilde{f}(k,c)dcdk =$$

$$-S(a)\int_{k}k^4\frac{E\left(x^{\frac{1-\epsilon}{1+\epsilon}}|c=T^{\lambda}\right)}{1-\lambda+\eta(T^{\lambda})}\left[\epsilon QE\left(x^{\frac{1-\epsilon}{1+\epsilon}}|c<T^{\lambda}\right)+(1-\lambda)\frac{T^{\lambda'}\epsilon'}{k(1+\epsilon)^2}E\left(\log(x)|c<T^{\lambda}\right)\right]\tilde{f}(k,c)dcdk=$$

$$-S(a)\int_{k}k^4\left[\frac{\epsilon QE\left(x^{\frac{1-\epsilon}{1+\epsilon}}|c=T^{\lambda}\right)E\left(x^{\frac{1-\epsilon}{1+\epsilon}}|c<T^{\lambda}\right)}{1-\lambda+\eta(T^{\lambda})}+\frac{1-\lambda}{1-\lambda+\eta(T^{\lambda})}\frac{T^{\lambda'}\epsilon'}{k(1+\epsilon)^2}E\left(x^{\frac{1-\epsilon}{1+\epsilon}}|c=T^{\lambda}\right)E\left(\log(x)|c<T^{\lambda}\right)\right]\tilde{f}(k,c)dcdk$$

Thus note that we have the same ex-post-distortion term as before (except for the replacement of $\frac{1}{\eta}$ with $\frac{1}{1-\lambda+\eta}$ which simply adjusts for rescaling given that it is semi-elasticities that matter). The only additional term is the second, which is simply the second term from the boundary, but where the expectation of $\log(x)$ is evaluated for the average rather than the marginal innovation. Thus the unified expression becomes exactly the adjusted (for the changed elasticity and optimal transfers) form of the first-order condition in Corollary 3 plus

$$(1-\lambda)\frac{S(a)\epsilon'}{(1+\epsilon)^2}\int_{k}k^4\frac{T^{\lambda'}}{k}E\left[x^{\frac{1-\epsilon}{1+\epsilon}}|c=T^{\lambda}\right]\left[E\left(\log(x)|c=T^{\lambda}\right)-E\left(\log(x)|c<T^{\lambda}\right)\right]\tilde{f}(k)dk$$

which is, exactly as we conjectured, the difference between the marginal and infra-marginal innovators of the marginal rewards they receive from raising a.

$\square$

4.2 Externalities

Proof of Proposition 2. Note the equations for $T^\star$ follow directly from our earlier arguments in Subsection 6.2 of the text. In the derivation of Corollary 3, $S(a)$ drops out of the formula because it multiplies both sorting and the ex-post distortion, as this is multiplied by $T^\star$ when transforming the hazard rate into elasticity. Scaling up $T^\star$ therefore simply scales up both terms and can thus be dropped as a constant of proportionality. $\square$

Proof of Proposition 3. In this case we may apply the some arguments as in the proof of Proposition 2, except now the initial ex-post distortion term must be multiplied by $(1+\gamma)$ as $\frac{d(1+\gamma)S(a)}{da} = (1+\gamma)Q(a)\epsilon(a)$ not $Q(a)\epsilon(a)$ as with $S(a)+\gamma$. $\square$

4.3 Residual uncertainty

First we derive optimal transfers. Social welfare is

$$\int_s \int_{c=0}^{E_{\sigma,m}\left[T^\star\left(\sigma^{\frac{1}{1+\epsilon(a)}} m^{\frac{\epsilon(a)}{1+\epsilon(a)}}\right)|s\right]} \left(S(a)E_{\sigma,m}\left[\sigma m|\, s\right] - c\right) g(s,c)dcds$$

changing variables

$$\int_s \int_{c=0}^{E_{k,\tilde{g}}[T^\star(k)|s]} \left(S(a)E_{k,x,\tilde{g}}\left[k^2 x^{\frac{1-\epsilon}{1+\epsilon}}\middle|\, s\right] - c\right) g(s,c)dcds =$$

$$\int_s \int_{c=0}^{\int_k T^\star(k)\tilde{g}(k|s)dk} \left(S(a)E_{\hat{k},\hat{x},\tilde{g}}\left[\hat{k}^2 \hat{x}^{\frac{1-\epsilon}{1+\epsilon}}\middle|\, s\right] - c\right) g(s,c)dcds$$

The standard calculus of variations first-order condition may then be derived using Leibnitz's rule by differentiating with respect to $T^\star$ at each point k:

$$\int_s \left(S(a)E_{\hat{k},\hat{x},\tilde{g}}\left[\hat{k}^2 \hat{x}^{\frac{1-\epsilon}{1+\epsilon}}\middle|\, s\right] - E_{\hat{k},\tilde{g}}\left[T^\star\left(\hat{k}\right)|s\right]\right) \tilde{g}(k|s) g\left(s, E_{\hat{k},\tilde{g}}\left[T^\star\left(\hat{k}\right)|s\right]\right) ds = 0$$

$$S(a)\int_s E_{\hat{k},\hat{x},\tilde{g}}\left[\hat{k}^2 \hat{x}^{\frac{1-\epsilon}{1+\epsilon}}|s\right] \tilde{g}\left(k, s, E_{\hat{k},\tilde{g}}\left[T^\star\left(\hat{k}\right)|s\right]\right) ds = \int_s E_{\hat{k},\tilde{g}}\left[T^\star\left(\hat{k}\right)|s\right] \tilde{g}\left(k, s, E_{\hat{k},\tilde{g}}\left[T^\star\left(\hat{k}\right)|s\right]\right)$$

or

$$S(a)E_s\left[E_{\hat{k},\hat{x},\tilde{g}}\left[\hat{k}^2 \hat{x}^{\frac{1-\epsilon}{1+\epsilon}}\middle|\, s\right]\middle|\, k, c = E_{\hat{k},\tilde{g}}\left[T^\star\left(\hat{k}\right)|s\right]\right] = E_s\left[E_{\hat{k},\tilde{g}}\left[T^\star\left(\hat{k}\right)|s\right]\middle|\, k, c = E_{\hat{k},\tilde{g}}\left[T^\star\left(\hat{k}\right)|s\right]\right] \quad (13)$$

That is, the average reward given to the average (based on Bayesian updating) innovation which is marginal and lands on isoreward curve k must be the average social value of that innovation, in the same sense of averaging. This is a quite complicated expression and can be simplified substantially with some strong[2] but plausible assumption.

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[2] Similar and no more empirically burdensome adjustments could be made to the formulae without these strong independence assumptions here, but these are more complicated to state and thus we omit them.

Of course ironing for either monotonicity or so that this first-order condition actually represents the optimum may be necessary. We ignore these complications here in the interest of brevity and because little changes in addressing these issues from the simpler case.

Now to formalize the assumptions of Proposition 5, for any $\hat{x}, \hat{x}, k, c$ we assume that

$$E_s\left[\tilde{g}\left(\hat{k}, \hat{x}|s, c\right)|k\right] = E_s\left[\tilde{g}\left(\hat{k}|s, c\right)|k\right] E_s\left[\tilde{g}\left(\hat{x}|s, c\right)\right]$$

Then equation (13) becomes

$$E_s\left[E_{\hat{k},\tilde{g}}[T^\star(\hat{k})|s]\Big|k,c=E_{\hat{k},\tilde{g}}[T^\star(\hat{k})|s]\right]=S(a)E_s\left[E_{\hat{k},\tilde{g}}\left[\hat{k}^2\Big|s\right]\Big|k,c=E_{\hat{k},\tilde{g}}[T^\star(\hat{k})|s]\right]E_{x,\tilde{g}}\left[x^{\frac{1-\epsilon}{1+\epsilon}}\Big|c=E_{k,\tilde{g}}[T^\star(k)]\right]$$

which has the trivial (implicit) solution

$$T^\star(k) = k^2 S(a) E_{x,\tilde{g}}\left[x^{\frac{1-\epsilon}{1+\epsilon}}\Big| k, c = E_s\left[E_{\hat{k},\tilde{g}}\left[T^\star\left(\hat{k}\right)|s\right]|k\right]\right] \equiv k^2 S(a)\widetilde{x^{\frac{1-\epsilon}{1+\epsilon}}}$$

Proof of Proposition 5. Social welfare is

$$\int_s \int_{c=0}^{E_{\sigma,m}\left[T^\star\left(\sigma^{\frac{1}{1+\epsilon(a)}} m^{\frac{\epsilon(a)}{1+\epsilon(a)}}\right)|s\right]} \left(S(a)E_{\sigma,m}\left[\sigma m|\, s\right] - c\right) g(s,c)dcds$$

So by the same envelope theorem and Leibnitz rule argument as in Subsection 3.2 above, $W'(a) =$

$$\int_s \frac{\epsilon'}{(1+\epsilon)^2} E_{\sigma,m}\left[\sigma^{\frac{1}{1+\epsilon}} m^{\frac{\epsilon}{1+\epsilon}} \log(\tfrac{m}{\sigma}) T^{\star\prime}\left(\sigma^{\frac{1}{1+\epsilon}} m^{\frac{\epsilon}{1+\epsilon}}\right)\right]\left(S(a)E_{\sigma,m}[\sigma m|s] - E_{\sigma,m}\left[T^\star\left(\sigma^{\frac{1}{1+\epsilon(a)}} m^{\frac{\epsilon(a)}{1+\epsilon(a)}}\right)|s\right]\right) f\left(s, E_{\sigma,m}\left[T^\star\left(\sigma^{\frac{1}{1+\epsilon(a)}} m^{\frac{\epsilon(a)}{1+\epsilon(a)}}\right)|s\right]\right) -$$

$$\int_{c=0}^{E_{\sigma,m}\left[T^\star\left(\sigma^{\frac{1}{1+\epsilon(a)}} m^{\frac{\epsilon(a)}{1+\epsilon(a)}}\right)|s\right]} S'(a) E_{\sigma,m}\left[\sigma m|s\right] g(s,c)dcds$$

As usual, the second term is relatively simple to handle, so we focus on the first. Changing variables to (k, x) this becomes

$$\frac{\epsilon'}{(1+\epsilon)^2}\int_s E_{k,x}\left[k\log(x)T^{\star\prime}(k)\Big|s\right]\left(S(a)E_{k,x}\left[k^2 x^{\frac{1-\epsilon}{1+\epsilon}}|s\right] - E_k[T^\star(k)|s]\right) g(s, E_k[T^\star(k)|s])ds =$$

$$S(a)\frac{\epsilon'}{(1+\epsilon)^2}\int_k\int_s E_{\hat{k},x}\left[\hat{k}\log(x)2\hat{k}\left(\hat{k}\right)|s\right]\left(E_{\hat{k},x}\left[\hat{k}^2 x^{\frac{1-\epsilon}{1+\epsilon}}\Big|s\right] - E_{\hat{k}}\left[\hat{k}^2 x^{\widetilde{\frac{1-\epsilon}{1+\epsilon}}}\Big|s\right]\right)\tilde{g}\left(s, E_k\left[T^\star\left(\hat{k}\right)|s\right]\Big|k\right)\tilde{g}(k)dsdk =$$

$$S(a)\frac{\epsilon'}{(1+\epsilon)^2}\int_k\int_s E_{\hat{k}}2\left[\hat{k}^2\left(\hat{k}\right)|s\right]E_{\hat{k}}\left[\hat{k}^2\left(\hat{k}\right)|s\right]E_x[\log(x)|s]\left(E_{\hat{k},x}\left[x^{\frac{1-\epsilon}{1+\epsilon}}\Big|s\right] - x^{\widetilde{\frac{1-\epsilon}{1+\epsilon}}}\right)\tilde{g}\left(s, E_k\left[T^\star\left(\hat{k}\right)|s\right]\Big|k\right)\tilde{g}(k)dsdk$$

From here the formula obtains through steps similar to the derivation of the first-order conditions above. ☐

To derive the empirical formula in the case of ex-ante uncertainty, we may follow exactly the same argument as in the standard case, with only the anticipated log-variance, except now we have an additional term of

$$\frac{E_s\left[E_{\hat{k}}\left[\hat{k}^2\Big|s\right]|k\right]}{k^2}$$

Recall that in the limit as $a \to 1$, $k^2 \to pq$. Thus the above expression is the ratio of the profits that, on average, an innovator who ends up earning profits π anticipated earning to the profits she actually earned. It seems practical, given all of our other simplifications, to simply ignore this term. However, in principle, incorporating it should not be too difficult; for some discreteization of the profit space one could project the average profits of ex-ante indistinguishable-to-the-innovator projects earning those profits and as what is the average profits earned by projects ex-ante indistinguishable from these. This factor would then scale up or down materialistic creative genius for different values of π that are averaged over.

Finally consider the case of ex-post uncertainty. Now, by exactly the same argument as in the standard case, $\text{Var}\left(\log\left(\frac{\sigma}{m}\right)\middle|\, \sigma m\right)$ is the appropriate measure. However, $\sigma Q(1) \neq q$; rather $\sigma Q(1) = E[q|\sigma]$. Because m is assumed known to the innovator, σ is then just proportional to the innovator's expectation of her sales. This could either be directly empirically recovered if internal estimates are available or estimated based on private information that a number of innovators had using a regression, with fitted values based on available information used to estimate σ.

4.4 Platforms

The text briefly discusses the equivalence between the pricing problem of a platform with many applications and a social planner's problem when $\lambda = 0$ in the distributional concerns model. We briefly discuss this equivalence formally in this subsection.

Let $i \in [0,1]$ denote the set of potential innovations, $\delta_i = 1$ if the innovation is developed and $\delta_i = 0$ otherwise. Consumers are indexed by k, uniform in $[0,1]$. Consumer k's net payoff is

$$\int_0^1 V_i(p_i\,; \sigma_i\,, m_i)\delta_i di - t - \xi_k$$

where ξ_k is distributed according to $H(\cdot)$. Application developer i introduces the innovation if and only if $T_i \geq c_i$. The platform's profit is then:

$$\left[t + \int_0^1 \left[\pi_i(p_i\,; \sigma_i\,, m_i) - T_i\right]\delta_i di\right]\left[1 - H\left(t - \int_0^1 V_i(p_i\,; \sigma_i\,, m_i)\delta_i di\right)\right],$$

or after a change in variables and using $S_i \equiv V_i + \pi_i$,

$$\left[\widehat{t} + \int_0^1 \left[S_i(p_i\,; \sigma_i\,, m_i) - T_i\right]\delta_i di\right]\left[1 - H(\widehat{t})\right].$$

Thus the platform first maximizes

$$\int_0^1 \left[S_i(p_i\,; \sigma_i\,, m_i) - T_i\right]\delta_i di$$

and then chooses the surplus-adjusted price $\widehat{p}$. Put differently, the planner behaves exactly as a social planner with strong redistributive concerns ($\lambda = 0$).

4.5 Stretch parameterization

Proof of Propsoition 6. Social value is $S\left(Q^{-1}\left(\frac{q}{\sigma}\right)\right)\sigma m$ while profits are $qmQ^{-1}\left(\frac{q}{\sigma}\right)$. Thus, the marginal social incentive to supply m is σS while the marginal private incentive is $qQ^{-1} = \sigma\frac{Qp}{m}$. But $\frac{Sm}{Qp}$ is exactly the ratio of average to marginal willingness-to-pay which is clearly above unity.

The social incentive to provide σ holding fixed q is

$$-\frac{S'}{Q'}\frac{q}{\sigma}m + Sm = \frac{Q\epsilon Q}{Q'a}am + Sm = (S - aQ)m$$

while the private incentive is

$$-\frac{q^2 m}{Q'\sigma^2} = -\frac{Q^2 m}{Q'} = \frac{Qam}{\epsilon}$$

At monopoly optimal prices the first simplifies to $(S(1) - Q(1))\, m$ and the second to $Q(1)m$. $\frac{S(1)-Q(1)}{Q(1)}$ is the ratio of consumer to producer surplus at monopoly optimal prices, whose comparison to unity is dictated by the average pass-through rate at prices above the monopoly optimum (Weyl and Fabinger, 2009). At prices other than monopoly optimal a similar result may be shown. $\square$

5 General Pricing

5.1 A price theoretic (demand profile) approach

The price theoretic approach, restricting attention to cases where no free disposal takes place in equilibrium, formulates the social planner's program as

$$\max_{\{\hat{T}(\cdot,\cdot),q(\cdot,\cdot),p(\cdot,\cdot)\}} \int_{\theta: c<\hat{T}(q(\sigma,m),p(\sigma,m))} \left[\sigma m S\left(\frac{p(\sigma,m)}{m}\right) - c\right] f(\theta)d\theta \tag{14}$$

subject to

$$(q(\sigma,m), p(\sigma,m)) \in \text{argmax}_{q=\sigma Q\left(\frac{p}{m}\right)} \hat{T}(q,p)$$

and $\hat{T}$ is monotone non-decreasing in both arguments. (q,p) is said to be a strict MOPSD if

1. It is differentiable everywhere.
2. q is monotone increasing in σ, p is monotone increasing[3] in m.
3. $\sigma' > \sigma, m' > m \implies$
   $(q(\sigma',m) - q(\sigma,m))\,(p(\sigma,m') - p(\sigma,m)) > (p(\sigma',m) - p(\sigma,m))\,(q(\sigma,m') - q(\sigma,m))$

(q,p) is staid to be a weak MOPSD if one of the second two conditions may, at some point, hold only weakly.

Proof of Lemma 3. In the forward direction, necessity follows from the fact that a MOPSD is clearly equivalent to a differentiable pricing policy a: by the nature of a MOPSD is isoreward curves in the (σ,m) space may easily be transformed into those in the (q,p) space.

For sufficiency consider some type (σ,m). Suppose that rather than choosing $(q(\sigma,m), p(\sigma,m))$, (σ,m) strictly prefers $\left(\sigma Q\left(\frac{p'}{m}\right), p'\right)$ where $p' > p(\sigma,m)$; in particular consider the point $\left(\sigma Q\left(\frac{p'}{m}\right), p(\sigma,m)\right)$

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[3] Note that, given our assumption of differentiability, these could all be directly stated in their differential forms. However, we prefer to use the more general form as we suspect these results generalize and we believe that the economic and geometric intuitions yielded by the broader forms are greater. Fabinger and Weyl (2011) hope to substantially weaken these technical assumptions.

and let (σ', m') be its inverse under (q, p) which exists as it is a strict MOPSD. Also, let (σ'', m'') be the inverse of $\left(\sigma Q\left(\frac{p'}{m}\right), p'\right)$. It is well-known that the inverse of a strict MOPSD is itself a strict MOPSD so by orientation-preservation (property 3),

$$(m'' - m')(\sigma - \sigma') > (m - m')(\sigma'' - \sigma') \tag{15}$$

while by monotonicity we have that $m' > m''$ and $\sigma > \sigma'$. However we also know that $\left(\sigma Q\left(\frac{p'}{m}\right), p'\right)$ lies on (σ, m) *and* (σ'', m'')'s (downward sloping) demand curves. So either $\sigma > \sigma''$ and $m < m''$ or $\sigma < \sigma''$ and $m > m''$. Suppose the second were the case; then clearly $\sigma'' > \sigma > \sigma'$ and $m > m'' > m'$ so

$$(m - m')(\sigma'' - \sigma') > (m'' - m')(\sigma - \sigma')$$

in contradiction of inequality (15). Thus we must have $\sigma > \sigma''$ and $m < m''$. But then clearly the elasticity demand for (σ'', m'') at p' is great smaller than that of (σ, m). Thus if type (σ'', m'') is locally indifferent to raising q to increase k, type (σ, m) must be able to strictly raise k (which is strictly monotone by construction). Thus imitating (σ'', m'') cannot be the k-maximizing choice for (σ, m). But this argument may be repeated for any point on the frontier $(\hat{\sigma}, m) \neq (\sigma, m)$ (for points to the southeast the argument is analogous but reversed) proving that the k-maximizing point for (σ, m) is (q, p).

For the partial converse, suppose that (q, p) is not a weak MOPSD. Then, by continuity, there exists a neighborhood where either monotonicity in some direction or orientation-preservation in some direction is violated for each pair or triple of relevant points in the neighborhood. This can easily be shown to give rise to the exactly the opposite of the argument above, establishing the local convexity of the objective function of the innovator at a point in the neighborhood and thus establishing that (q, p) is not in fact an optimal choice for her. If $\hat{T}$ is constant over that range, however, the argument fails and $\hat{T}$ may clearly still implement the desired function. □

Note that this implies that over regions where $\hat{T}$ is endogenously optimally flat or near flat it may be optimal to adopt very low prices as anything may be implemented. This is intuitive: if $\hat{T}$ is not optimally increasing over a range there is little incentive to sort, as which isoreward curve innovations are assigned to is irrelevant.

Calculating the Jacobian of the (σ, m) to (q, p) transformation in logs yield

$$\begin{bmatrix} 1 - \epsilon\epsilon_{a_\sigma} & \epsilon_{a_\sigma} \\ -\epsilon\epsilon_{a_m} & 1 + \epsilon_{a_m} \end{bmatrix}$$

the positive definiteness of which yields the conditions in the text.

Of course it is possible to carry forward our analysis through all of the remaining stages, but little changes in consequence. We therefore omit this for brevity, but a full presentation is available on request and will be contained in slides posted at www.glenweyl.com shortly.

5.2 Substantive results with general pricing

We begin be establishing the validity of our change of variables. We want to show that point $(\sigma, m) = \left(kx^{-\frac{\tilde{\epsilon}(k,x)}{1+\tilde{\epsilon}(k,x)}}, kx^{\frac{1}{1+\tilde{\epsilon}(k,x)}}\right)$ lies along an isoreward curve intersecting the 45°, $\sigma = m$ line at $m = \sigma = \sqrt{\sigma m x^{-\frac{1-\tilde{\epsilon}(k,x)}{1+\tilde{\epsilon}(k,x)}}}$ where

$$\tilde{\epsilon}(k, x) \equiv \frac{1}{\overline{\frac{1}{1+\epsilon}}(k, z)} - 1$$

$$\overline{\frac{1}{1+\epsilon}}(k,x) \equiv \int_{z=1}^{x} \frac{1}{z\log(x)\left[1+\epsilon(k,z)\right]}dz$$

where $\epsilon(k,x) \equiv \epsilon\left(a\left(kx^{-\frac{\tilde{\epsilon}(k,x)}{1+\tilde{\epsilon}(k,x)}}, kx^{\frac{1}{1+\tilde{\epsilon}(k,x)}}\right)\right)$. By Lemma 1, along an isoreward curve

$$\frac{\partial \tilde{\sigma}}{\partial \tilde{m}} = -\frac{\epsilon(k,x)}{x}$$

Thus moving along an isoreward curve while adjusting x until one reaches the 45° line makes

$$\log(m)+\int_{l=\log(x)}^{0} \frac{1}{1+\epsilon\left(k,e^{l}\right)}dl = \log(k) \iff \log(m) = \log(k)+\log(x)\overline{\frac{1}{1+\epsilon}}(k,x) \iff m = kx^{\overline{\frac{1}{1+\tilde{\epsilon}(k,x)}}}$$

A similarly derivation applies to σ.

Next we define the notion of a variation of $a(\sigma,m)$ in the direction of another policy $\hat{a}(\sigma,m)$ in keeping with the classical calculus of variations. In particular letting maximized social welfare under policy a be $W(a)$ the first variation of welfare in the direction of $\hat{a}$ is

$$\delta W(\hat{a}) \equiv \lim_{\delta\to 0}\frac{W\left[(1-\delta)a+\delta\hat{a}\right]-W\left(a\right)}{\delta}$$

We now seek to calculate $\delta W(\hat{a})$ for arbitrary $\hat{a}$. To do this we exploit some terminology we now define.

Let $\Delta a = \hat{a}-a$ and generalize $\overline{\cdot}(x,k)$ to potentially run over a range other than x to 1. For example, $\left.\overline{\frac{\Delta a\epsilon'}{(1+\epsilon)^2}}(k,x)\right|_{\tilde{x}}$ denotes a case when the lower bound of integration in the relevant equations is replaced by $\tilde{x}$ and $\log(x)$ is replaced by $\log\left(\frac{x}{\tilde{x}}\right)$ so that $\overline{\frac{\Delta a\epsilon'}{(1+\epsilon)^2}}(k,x) = \left.\overline{\frac{\Delta a\epsilon'}{(1+\epsilon)^2}}(k,x)\right|_{1}$.

First note that, as before, the points lying along the isoreward curve of a point which is, under a, assigned to (k,x) take the form (where we drop the arguments of $\tilde{\sigma}$ and $\tilde{m}$ inside the large expressions and assume $1<\tilde{x}<x$):

$$\tilde{\sigma}(\delta;\tilde{x},k,x) = k\tilde{x}^{-\frac{\tilde{\epsilon}(k,\tilde{x})}{1+\tilde{\epsilon}(k,\tilde{x})}}e^{\int_{z=\log(\tilde{x})}^{\log(x)}\frac{1}{z}\left(\frac{\epsilon(k,z)}{1+\epsilon(k,z)}-\frac{\epsilon([1-\delta]a(\tilde{\sigma},\tilde{m})+\delta\hat{a}(\tilde{\sigma},\tilde{m}))}{1+\epsilon([1-\delta]a(\tilde{\sigma},\tilde{m})+\delta\hat{a}(\tilde{\sigma},\tilde{m}))}\right)dz} \equiv k\tilde{x}^{-\frac{\tilde{\epsilon}(k,\tilde{x})}{1+\tilde{\epsilon}(k,\tilde{x})}}e^{\delta_{\sigma}(\tilde{x};k,x)}$$

$$\tilde{m}(\delta;\tilde{x},k,x) = k\tilde{x}^{\frac{1}{1+\tilde{\epsilon}(k,\tilde{x})}}e^{\int_{z=\log(\tilde{x})}^{\log(x)}\frac{1}{z}\left(\frac{1}{1+\epsilon(k,z)}-\frac{1}{1+\epsilon([1-\delta]a(\tilde{\sigma},\tilde{m})+\delta\hat{a}(\tilde{\sigma},\tilde{m}))}\right)dz} \equiv k\tilde{x}^{\frac{1}{1+\tilde{\epsilon}(k,\tilde{x})}}e^{\delta_{m}(\tilde{x};k,x)}$$

Note that $\lim_{\delta\to 0}\delta_m(\tilde{x};k,x),\delta_\sigma(\tilde{x};k,x) = 0\ \forall(\tilde{x};k,x)$. Furthermore we can obtain

$$d\tilde{\sigma}(\tilde{x},k,x)(\hat{a}) \equiv \lim_{\delta\to 0}\frac{k\tilde{x}^{-\frac{\tilde{\epsilon}(k,\tilde{x})}{1+\tilde{\epsilon}(k,\tilde{x})}}e^{\delta_{\sigma}(\tilde{x};k,x)}-k\tilde{x}^{-\frac{\tilde{\epsilon}(k,\tilde{x})}{1+\tilde{\epsilon}(k,\tilde{x})}}}{\delta} = k\tilde{x}^{-\frac{\tilde{\epsilon}(k,\tilde{x})}{1+\tilde{\epsilon}(k,\tilde{x})}}\lim_{\delta\to 0}\frac{\delta_{\sigma}(\tilde{x};k,x)}{\delta} =$$

$$-k\tilde{x}^{-\frac{\tilde{\epsilon}(k,\tilde{x})}{1+\tilde{\epsilon}(k,\tilde{x})}}\log\left(\frac{x}{\tilde{x}}\right)\left.\overline{\frac{\Delta a\epsilon'}{\left(1+\epsilon\right)^2}}(k,x)\right|_{\tilde{x}}$$

$$d\tilde{m}(\tilde{x},k,x)(\hat{a}) = k\tilde{x}^{\frac{1}{1+\tilde{\epsilon}(k,\tilde{x})}}\log\left(\frac{x}{\tilde{x}}\right)\left.\overline{\frac{\Delta a\epsilon'}{\left(1+\epsilon\right)^2}}(k,x)\right|_{\tilde{x}}$$

We can then calculate that the isoreward curve $\hat{k}$ assigned to (k,x) under the new policy is

$$\hat{k}(k,x;\delta,\hat{a}) = ke^{\int_{z=1}^{\log(x)}\frac{1}{z}\left(\frac{1}{1+\epsilon(k,z)}-\frac{1}{1+\epsilon(\delta\hat{a}(\tilde{\sigma}(z;k,x),\tilde{m}(z;k,x))+[1-\delta]a(\tilde{\sigma}(z;k,x),\tilde{m}(z;k,x)))}\right)dz}$$

Let $dk(k,x)(\hat{a}) \equiv \lim_{\delta\to 0} \frac{\hat{k}(k,x;\delta,\hat{a})-k}{\delta}$. It is straightforward, but tedious, to show that a number of second-order effects drop out because a small move towards $\hat{a}$ only causes a small change in $\tilde{\sigma}$; thus we are left with two effects:

$$\frac{dk(k,x)(\hat{a})}{k} = \log(x)\left(\underbrace{\overline{\frac{\Delta a\epsilon'}{(1+\epsilon)^2}}(k,x)}_{\text{direct sorting}} + \underbrace{\overline{\frac{\log\left(\frac{x}{z}\right)\overline{\frac{\Delta a\epsilon'}{(1+\epsilon)^2}}(k,x)\Big|_z\left(\epsilon_{a_m}-\epsilon_{a_\sigma}\right)\epsilon'}{(1+\epsilon)^2}}(k,x)}_{\text{indirect sorting}}\right)$$

The source of the first effect is both familiar from before and discussed more extensively below: raising a locally causes all innovations with a higher x on the same isoreward curve to have a higher k changing the local direction of the isoreward curve. The second effect is a bit more subtle, but not fundamentally different. To the extent a is not constant, changing a and thus the path of the (k,x) isoreward curve not only directly changes the path of the isoreward curve, but also does so indirectly by changing the set of elasticities "encountered" by the isoreward curve on its way to the 45° line.

We can now return to $\delta W(\hat{a})$. As usual we have two effects: one on the boundary and one on the interior of the integral. If we skip over steps now familiar from the calculation of numerous first-order derivatives of W above, we obtain

$$\delta W(\hat{a})=\int_k\int_{x\geq 1} k\left(T^{\star\prime}(k)\frac{dk(k,x)}{k}\left[S(k,x)k^2x^{\frac{1-\tilde{\epsilon}(k,x)}{1+\tilde{\epsilon}(k,x)}}-T^\star(k)\right]\tilde{f}(k,x,T^\star(k))-\Delta a(k,x)Q(k,x)\epsilon(k,x)x^{\frac{1-\tilde{\epsilon}(k,x)}{1+\tilde{\epsilon}(k,x)}}\int_{c=0}^{T^\star(k)}\tilde{f}(k,x,c)\right)dx+\ldots dk=$$

where ... represents the corresponding opposite term for $x<1$. Dropping arguments where possible

$$\int_k\int_{x\geq 1}k^2\left(\frac{T^{\star\prime}}{k}\log(x)\overline{\frac{\left[\log\left(\frac{x}{z}\right)\overline{\frac{\Delta a\epsilon'}{(1+\epsilon)^2}}\Big|_z\left(\epsilon_{a_m}-\epsilon_{a_\sigma}\right)+\Delta a\right]\epsilon'}{(1+\epsilon)^2}}\left[Sk^2x^{\frac{1-\tilde{\epsilon}}{1+\tilde{\epsilon}}}-T^\star(k)\right]\tilde{f}\left(x|k,T^\star\right)\tilde{f}\left(T^\star|k\right)-\Delta aQ\epsilon x^{\frac{1-\tilde{\epsilon}}{1+\tilde{\epsilon}}}\tilde{f}\left(x|k,c<T^\star\right)\tilde{F}\left(T^\star|k\right)\right)\tilde{f}(k)dx+\ldots dk\propto$$

$$\int_k k^4\int_{x\geq 1}\left(\frac{T^{\star\prime}}{k}\log(x)\overline{\frac{\left[\log\left(\frac{x}{z}\right)\overline{\frac{\Delta a\epsilon'}{(1+\epsilon)^2}}\Big|_z\left(\epsilon_{a_m}-\epsilon_{a_\sigma}\right)+\Delta a\right]\epsilon'}{(1+\epsilon)^2}}\left(Sx^{\frac{1-\tilde{\epsilon}}{1+\tilde{\epsilon}}}-E_{\tilde{f},x}\left[Sx^{\frac{1-\tilde{\epsilon}}{1+\tilde{\epsilon}}}\right]\right)\eta(T^\star(k)|k)-\Delta aQ\epsilon x^{\frac{1-\tilde{\epsilon}}{1+\tilde{\epsilon}}}E_{\tilde{f},x}\left[Sx^{\frac{1-\tilde{\epsilon}}{1+\tilde{\epsilon}}}\right]\tilde{f}(\hat{x}|\hat{k},T^\star)\frac{E\left[\tilde{f}(\hat{x}|k,c)\Big|c<T^\star(k),k\right]}{\tilde{f}(x|c=T^\star(k),k)}\right)\tilde{f}(k)dx+\ldots \tag{16}$$

Clearly the resemblance to our earlier analysis is beginning to emerge.

First-order conditions for maximization over all differentiable schedules a require that $\delta W(\hat{a})$ be 0 for all differentiable $\hat{a}$. It is well known that this condition[4] is equivalent to this condition holding for the class of differentiable functions

$$\hat{a}(k,x;\hat{k},\hat{x},\eta) = a(k,x) + \frac{e^{-\frac{(k-\hat{k})^2+(x-\hat{x})^2}{\eta^2}}}{2\pi\eta}$$

for all choices of $(\hat{k},\hat{x})$ and all $\eta<\overline{\eta}$ for an arbitrarily small η, as such functions form a basis for the set of all differentiable functions. If we allow $\eta\to 0$ we obtain a point mass difference between a and $\hat{a}$ and the notion of a *perturbation* of W at $(\hat{k},\hat{x})$:

$$W'(\hat{k},\hat{x}) \equiv \lim_{\eta\to 0}\frac{W\left(\hat{a}\left(\hat{k},\hat{x},\eta\right)\right)}{\eta}$$

As the proof of our first result in this section shows, these perturbations can be computed as limits of formula (16) when Δa converges to 0 everywhere by $(\hat{k},\hat{x})$ and to ∞ at that point.

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[4] Note that π in this condition is the geometric constant, not a variable for profits.

Proof of Proposition 7. As usual with Dirac-convergent weighting functions, the integral converges to the value of the density at the limit mass point. The value of the second term of (16) is easy to evaluate at this mass point so we focus on the first term and begin by analyzing:

$$\log(x)\frac{\overline{\left[\log\left(\frac{x}{z}\right)\overline{\frac{\Delta a\epsilon'}{(1+\epsilon)^2}}\Big|_z(\epsilon_{a_m}-\epsilon_{a_\sigma})+\Delta a\right]\epsilon'}}{(1+\epsilon)^2}=\int_{z=1}^{x}\frac{1}{z}\frac{\epsilon'(k,z)}{[1+\epsilon(k,z)]^2}\left(\Delta a(k,z)+[\epsilon_{a_m}(k,z)-\epsilon_{a_\sigma}(k,z)]\int_{\alpha=z}^{x}\frac{1}{\alpha}\frac{\epsilon'(k,\alpha)\Delta a(k,\alpha)}{[1+\epsilon(k,\alpha)]^2}d\alpha\right)dz$$

Evaluating this in the limit as Δa becomes a point mass of 1 on $(\hat{k},\hat{x})$ yields

$$\frac{1_{x\geq\hat{x}}}{\hat{x}}\frac{\epsilon'(\hat{k},\hat{x})}{\left[1+\epsilon(\hat{k},\hat{x})\right]^2}+\int_{z=1}^{x}\frac{1_{z<\hat{x}<x}\epsilon'(\hat{k},\hat{x})}{\left[1+\epsilon(\hat{k},\hat{x})\right]^2}\left[\epsilon_{a_m}(\hat{k},z)-\epsilon_{a_\sigma}(\hat{k},z)\right]dz$$

$$\frac{1_{x\geq\hat{x}}}{\hat{x}}\frac{\epsilon'(\hat{k},\hat{x})}{\left[1+\epsilon(\hat{k},\hat{x})\right]^2}\left[1+\log(\hat{x})\overline{\frac{\epsilon'\left(\epsilon_{a_m}-\epsilon_{a_\sigma}\right)}{(1+\epsilon)^2}}(\hat{k},\hat{x})\right]$$

So by equation (16) for $\hat{x}>1$, $W'(\hat{k},\hat{x})\propto$

$$\int_{x=\hat{x}}^{\infty}\frac{{T^{\star}}'\epsilon'(\hat{k},\hat{x})}{\hat{k}\hat{x}\left[1+\epsilon(\hat{k},\hat{x})\right]^2}\left[1+\log(\hat{x})\overline{\frac{\epsilon'(\epsilon_{a_m}-\epsilon_{a_\sigma})}{(1+\epsilon)^2}}(\hat{k},\hat{x})\right]\left(Sx^{\frac{1-\tilde{\epsilon}}{1+\tilde{\epsilon}}}-E_{\tilde{f},x}\left[Sx^{\frac{1-\tilde{\epsilon}}{1+\tilde{\epsilon}}}\right]\right)\eta\left(T^{\star}(k)|k\right)-Q\epsilon\hat{x}^{\frac{1-\tilde{\epsilon}}{1+\tilde{\epsilon}}}E_{\tilde{f},x}\left[Sx^{\frac{1-\tilde{\epsilon}}{1+\tilde{\epsilon}}}\right]\tilde{f}(\hat{x}|\hat{k},T^{\star})\frac{E\left[\tilde{f}(\hat{x}|k,c)\middle|c<T^{\star}(k),k\right]}{\tilde{f}(x|c=T^{\star}(k),k)}=$$

$$\frac{{T^{\star}}'}{\hat{k}}\frac{\epsilon'(\hat{k},\hat{x})}{\hat{x}\left[1+\epsilon(\hat{k},\hat{x})\right]^2}\left[1+\log(\hat{x})\overline{\frac{\epsilon'\left(\epsilon_{a_m}-\epsilon_{a_\sigma}\right)}{(1+\epsilon)^2}}(\hat{k},\hat{x})\right]\left(\frac{E_{\tilde{f},x>\hat{x}}\left[Sx^{\frac{1-\tilde{\epsilon}}{1+\tilde{\epsilon}}}\right]}{E_{\tilde{f},x}\left[Sx^{\frac{1-\tilde{\epsilon}}{1+\tilde{\epsilon}}}\right]}-1\right)\eta\left(T^{\star}(k)|k\right)-Q\epsilon\hat{x}^{\frac{1-\tilde{\epsilon}}{1+\tilde{\epsilon}}}H\left(\hat{x}|\hat{k},T^{\star}\right)\frac{E\left[\tilde{f}\left(\hat{x}|\,k,c\right)\middle|\,c<T^{\star}(k),k\right]}{\tilde{f}\left(x|\,c=T^{\star}(k),k\right)}\tag{17}$$

as in the text. For $\hat{x}<1$ the reasoning is analogous and thus omitted. ☐

Proof of Proposition 8. Lowering $\frac{1}{1+\epsilon}$ uniformly by one unit corresponds to $\Delta a=\frac{[1+\epsilon]^2}{\epsilon'}$. Plugging this into expression (16) yields

$$\delta W\left(a+\frac{(1+\epsilon)^2}{\epsilon'}\right)=\int_k k^4\int_{x\geq1}\left(\frac{{T^{\star}}'}{k}\log(x)\left[1+\overline{\log\left(\frac{x}{z}\right)(\epsilon_{a_m}-\epsilon_{a_\sigma})}\right]\left(Sx^{\frac{1-\tilde{\epsilon}}{1+\tilde{\epsilon}}}-E_{\tilde{f},x}\left[Sx^{\frac{1-\tilde{\epsilon}}{1+\tilde{\epsilon}}}\right]\right)\eta\left(T^{\star}(k)|k,x\right)-\frac{Q\epsilon[1+\epsilon]^2}{\epsilon'}x^{\frac{1-\tilde{\epsilon}}{1+\tilde{\epsilon}}}E_{\tilde{f},x}\left[Sx^{\frac{1-\tilde{\epsilon}}{1+\tilde{\epsilon}}}\right]\right)\tilde{f}(k,x)dx+\ldots dk$$

The corresponding term for $x<1$ is essentially identical so we obtain

$$\delta W\left(a+\frac{(1+\epsilon)^2}{\epsilon'}\right)=\int_k k^4\int_{x}\left(\frac{{T^{\star}}'}{k}\log(x)\left[1+\overline{\log\left(\frac{x}{z}\right)(\epsilon_{a_m}-\epsilon_{a_\sigma})}\right]\left(Sx^{\frac{1-\tilde{\epsilon}}{1+\tilde{\epsilon}}}-E_{\tilde{f},x}\left[Sx^{\frac{1-\tilde{\epsilon}}{1+\tilde{\epsilon}}}\right]\right)\eta\left(T^{\star}(k)|k,x\right)-\frac{Q\epsilon[1+\epsilon]^2}{\epsilon'}x^{\frac{1-\tilde{\epsilon}}{1+\tilde{\epsilon}}}E_{\tilde{f},x}\left[Sx^{\frac{1-\tilde{\epsilon}}{1+\tilde{\epsilon}}}\right]\right)\tilde{f}(k,x)dxdk\propto$$

$$E_{k,\tilde{f}}\left[k^4\left(\frac{\eta{T^{\star}}'}{k}\mathrm{Cov}_{x,\tilde{f}}\left[\frac{\epsilon'}{(1+\epsilon)^2}\left(1+\overline{\log\left(\frac{x}{z}\right)(\epsilon_{a_m}-\epsilon_{a_\sigma})}\right)\log(x),Sx^{\frac{1-\tilde{\epsilon}}{1+\tilde{\epsilon}}}\right]\right)-E_x\left[Q\epsilon x^{\frac{1-\tilde{\epsilon}}{1+\tilde{\epsilon}}}\middle|k,c<T^{\star}(k)\right]E_{\tilde{f},x}\left[Sx^{\frac{1-\tilde{\epsilon}}{1+\tilde{\epsilon}}}\right]\right]$$

☐

Proof of Theorem 4. At globally ex-post efficient prices $\epsilon=\epsilon_{a_m}=\epsilon_{a_\sigma}=0$ and $S=1$ for all innovations. Thus expression (17) becomes

$$\frac{{T^{\star}}'}{k}\epsilon'(0)\left(\frac{E_{\tilde{f},x>\hat{x}}\left[x\right]}{E_{\tilde{f},x}\left[x\right]}-1\right)\eta\left(T^{\star}(k)|k,x\right)$$

which is strictly positive whenever ${T^{\star}}'\neq0$. By similar tricks at global monopoly pricing

$$\frac{E_{\tilde{f},x>\hat{x}}\left[Sx^{\frac{1-\tilde{\epsilon}}{1+\tilde{\epsilon}}}\right]}{E_{\tilde{f},x}\left[Sx^{\frac{1-\tilde{\epsilon}}{1+\tilde{\epsilon}}}\right]}-1=0$$

as $\tilde{\epsilon} = 1$ everywhere while the second term is

$$-Q\epsilon H(\hat{x}|k) < 0$$

A similar logic holds for $x < 1$. □

Proof of Theorem 5. If a is sufficiently, uniformly close to 1 or 0 then, because $\frac{\epsilon'}{(1+\epsilon)^2}$ is approximately constant, moving towards monopoly pricing is equivalent to uniformly increasing $\frac{1}{1+\epsilon}$. Furthermore all the simplifications from the proof of Theorem 4 apply and the first-order derivative from Proposition 4 simplify to those in the proofs of Theorem 2 and 3 above and the results therefore follow by the same reasoning as there. □

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